Qualification Exam: Mathematical Methods

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1 Mathematical Methods I

Problem 1. ID:MM-1-2

Solve the differential equation

$$\frac{dy}{dx} + y = \sin x.$$

Problem 2. ID:MM-1-7

Find the solution of

$$y' = \frac{y^2 - x^2 + xy}{x^2}$$

Problem 3. ID:MM-1-13

Find the general solution of

$$x^2y'' + 5xy' + 4y = 0$$

Problem 4. ID:MM-1-18

Verify that

$$\nabla^2\psi(r,\theta,\varphi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]\psi(r,\theta,\varphi) = 0$$

is separable in spherical coordinates. The functions f, g and h are functions only of the variables indicated; k^2 is a constant.

Problem 5. ID:MM-1-23

Note that $y = x$ would be a solution of

$$(1-x)y'' + xy' - y = (1-x)^3$$

if the right side were zero. Use this fact to obtain the general solution of the equation as given.

Problem 6. ID:MM-1-28

Find the general solution of

$$x^2y'' + 3xy' + y = 9x^2 + 4x + 1$$

Problem 7. ID:MM-1-33

Find the general solution of differential equation

$$y'' + 4y' + 4y = 3x^2 + x$$

Problem 8. ID:MM-1-38

Use power series to find the general solution of

$$x(1-x)y'' + 4y' + 2y = 0$$

Problem 9. ID:MM-1-43

Find the analytic function

$$f(z) = u(x,y) + iv(x,y)$$

if $v(x,y) = 6x^2y^2 - x^4 - y^4$ and $u(x,y) = e^{-x}\sin y$

Problem 10. ID:MM-1-48

Using $f(re^{i\theta}) = R(r,\theta)e^{i\Phi(r,\theta)}$, in which $R(r,\theta)$ and $\Phi(r,\theta)$ are differentiable real functions of r and θ, show that the Cauchy-Riemann conditions in polar coordinates become $\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Phi}{\partial \theta}$, and $\frac{1}{r}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Phi}{\partial r}$

Problem 11. ID:MM-1-53

Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x,y) + iv(x,y)$. We label the real part, $u(x,y)$, the velocity potential and the imaginary part, $v(x,y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic,

1. Show that $\frac{df}{dz} = \mathbf{V}_x - i\mathbf{V}_y$;
2. Show that $\nabla \cdot \mathbf{V} = 0$ (no source or sinks);
3. Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

Problem 12. ID:MM-1-58

Show that the integral

$$I = \int_{(0,0)}^{(1,1)} \bar{z}dz$$

depends on the path, by evaluating the integral for two different paths $C_1 + C_2$ and $C_3 + C_4$, indicated in page 51 of the lecture note.

Problem 13. ID:MM-1-63

Develop the first three nonzero terms of the Laurent Expansion of

$$f(z) = (e^z - 1)^{-1}$$

about the origin.

Problem 14. ID:MM-1-68

Determine the nature of singularities of each of the following functions and evaluate the residues ($a > 0$).

1. $\frac{z^2}{(z^2+a^2)^2}$
2. $\frac{\sin\frac{1}{z}}{z^2+a^2}$
3. $\frac{ze^{iz}}{z^2+a^2}$
4. $\frac{z^{-k}}{z+1}, \qquad 0 < k < 1$

Hint. For the point at infinity, use the transformation $w = 1/z$ for $|z| \to 0$. For the residue, transform $f(z)dz$ into $g(w)dw$ and look at the behavior of $g(w)$.

Problem 15. ID:MM-1-73

Evaluate the integral

$$I = \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2}, \qquad a > 1$$

Problem 16. ID:MM-1-78

Evaluate the integral

$$I = \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$$

Problem 17. ID:MM-1-83

Expand the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 \le x < \pi \end{cases}$$

in a sine-cosine Fourier series. Use this Fourier expansion to show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Problem 18. ID:MM-1-88

Expand the function

$$f(x) = x^2, \qquad -\pi \le x \le \pi$$

in a sine-cosine Fourier series. Then evaluate the series you get at $x = \pi$. (this will yield the Riemann Zeta function)

Problem 19. ID:MM-1-93

Obtain the Fourier sine and cosine transform of the following functions

1. $\frac{\sin wx}{x^2+1}$, $w > 0$
2. $\frac{\cos wx}{x^2+1}$, $w > 0$
3. $\frac{x}{x^2+1}$

Problem 20. ID:MM-1-98

Obtain the inverse Fourier transformation of the function

$$\Phi(\vec{k}) = -\frac{4\pi g^2}{k^2 + m^2}$$

where m and g are constants and $\vec{k}$ is a vector in three dimensions.(Hint: use spherical coordinates, and contour integral)

Problem 21. ID:MM-1-103

Compute the following inverse Laplace transformation,

1.
$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right], \qquad a^2 \neq b^2$$

2.
$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right], \qquad a^2 \neq b^2$$

Problem 22. ID:MM-1-108

Use the expression for the transform of a second derivative to obtain the Laplace transform of the function

$$f(x) = \cos kx + \sin kx$$

Problem 23. ID:MM-1-113

Determine the leading behavior of the following integrals as $x \to +\infty$.

1. $I_1(x) = \int_0^{\pi/2} e^{-x\tan t} dt$
2. $I_2(x) = \int_{-1}^{1} e^{-x\sin^4 t} dt$

Problem 24. ID:MM-1-118
Find the leading behavior of the integral

$$I(x) = \int_0^\infty e^{xt - e^{2t}} dt$$

as $x \to +\infty$.

Problem 25. ID:MM-1-123
Show first three terms of the asymptotic expansion of function

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt, \qquad (x > 0)$$

as $x \to 0+$ for the case $a = -\frac{1}{2}$. (Hint: first integral by part, and then use $\int_x^\infty = \int_0^\infty - \int_0^x$ for the rest of integral)

Problem 26. ID:MM-1-128
Find the leading behavior of the integral

$$I(x) = \int_0^\infty \cos(xt^2 - t) dt$$

as $x \to +\infty$.

Problem 27. ID:MM-1-133
Find the leading behavior of the integral

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin t}\, e^{-x \sin^4 t} dt$$

as $x \to +\infty$.

Problem 28. ID:MM-1-138
Evaluate the integral,

$$I(k) = \int_{-\infty}^\infty e^{ikt} (1 + t^2)^{-k} dt,$$

for $k \to \infty$.

Problem 29. ID:MM-1-143
Find a set of Gamma matrices in $d = 2$ which satisfy the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} 1, \qquad \mu, \nu = 0, ..., d - 1 \tag{1}$$

Then construct a set of matrices in $d = 4$ using the $d = 2$ Gamma matrices and verify that they satisfy the above equation. (you can use mathematica to multiply the matrices if you need)

Problem 30. ID:MM-1-148

Use the method of steepest descents to get the leading behavior of the integral

$$I(n) = \int_0^\pi \cos(nt) e^{ia\cos t} dt$$

where n is an integer, and $n \to \infty$.

Problem 31. ID:MM-1-153

The linear vibrations of three atoms of a CO_2 molecule satisfy the following equations,

$$\begin{cases} m_1\ddot{x}_1 = (x_2 - x_1)k_1 + (x_3 - x_1)k_2 \\ m_2\ddot{x}_2 = (x_3 - x_2)k_1 + (x_1 - x_2)k_1 \\ m_1\ddot{x}_3 = (x_2 - x_3)k_1 + (x_1 - x_3)k_2 \end{cases}$$

where m_1 and m_2 are the masses of oxygen and carbon atoms respectively. x_1 and x_3 are the positions of two oxygen atoms. x_2 is the position of carbon atom. Find the normal modes and normal frequencies for the linear vibrations.

Problem 32. ID:MM-1-158

Show that the Green's function for a Hermitian differential operator L satisfies the symmetry relation

$$G(\mathbf{x}, \mathbf{x}') = [G(\mathbf{x}', \mathbf{x})]^*$$

Hint: you can begin with differential equations for $G(\mathbf{x}, \mathbf{x}')$ and $G(\mathbf{x}, \mathbf{x}'')$.

Problem 33. ID:MM-1-163

Show if the following operators are Hermitian or not:

1. $i\vec{\nabla}$
2. $i\frac{\partial}{\partial\theta}$
3. $i\frac{\partial}{\partial\phi}$

where θ and ϕ are azimuthal and polar coordinates for the spherical coordinates (r, θ, ϕ).

Problem 34. ID:MM-1-168

A unitary matric U can be written as $U = e^{i\theta T}$, where θ is a real parameter and T is a matric.

1. Show that T is Hermitian.
2. If $\texttt{det}U = 1$, show that T is traceless, ie $\texttt{Tr}T = 0$

Problem 35. ID:MM-1-173

1. For a finite and abelian group $G = \{f_1, f_2, \cdots, f_n\}$, show that
$$(f_1 f_2 \cdots f_n)^2 = 1$$
2. If $f^2 = 1$ for any group element $f \in G$, show that group G is abelian.

Problem 36. ID:MM-1-178

The translation operator $T(a)$ in one dimension is defined by
$$T(a)\psi(x) = \psi(x + a) \tag{2}$$

1. Show that the set of $T(a)$ for all $a \in R$ forms an abelian group.
2. Show that $T(a) = e^{a\frac{d}{dx}}$ satisfy Eq.(2), so it is a representation of the translation group.

Problem 37. ID:MM-1-183

Show that the Spin 1 representation of $SU(2)$ algebra is equivalent to its adjoint representation.

Problem 38. ID:MM-1-188

Define
$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$
and let symplectic group $Sp(2n)$ consist of all matrices satisfying
$$M^t J M = J$$

1. Show that Lie algebra $sp(2n)$ is the algebra of matrices of the form
$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$
where B and C are symmetric and A is arbitrary $n \times n$ matrices.
2. Show that $sp(2n)$ has $n(2n + 1)$ independent parameters.

Problem 39. ID:MM-1-193

Use Schäfli's integral representation to show that Legendre polynomials can be written as
$$P_n(z) = \frac{1}{\pi}\int_0^\pi (z + \sqrt{z^2 - 1}\cos\phi)^n d\phi$$

Problem 40. ID:MM-1-198

A_{ij} is a tensor with the anti-symmetric index, ie. $A_{ij} = -A_{ji}$. Show that

$$\frac{\partial}{\partial x^i} A_{jk} + \frac{\partial}{\partial x^j} A_{ki} + \frac{\partial}{\partial x^k} A_{ij}$$

is still a tensor.

Problem 41. ID:MM-1-203

Show that $D_l V_r = \frac{\partial V_r}{\partial x^l} - \Gamma^j_{rl} V_j$, where Γ^j_{rl} is the Christoffel symbol, is a tensor under coordinate transformation. .

Problem 42. ID:MM-1-208

Operating in spherical polar coordinates, show that

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = -(n+1)\frac{P_{n+1}(\cos\theta)}{r^{n+2}}$$

This is the key step in the mathematical argument that the derivative of one multipole leads to next higher multipole.

Problem 43. ID:MM-1-213

Find the general solution of

$$x^2 y' + y^2 = xyy'$$

Problem 44. ID:MM-1-218

Find the general solution of the differential equation

$$xy'' + \frac{3}{x} y = 1 + x^2$$

in real form (no i's in answer)

Problem 45. ID:MM-1-223

Consider the differential equation

$$\frac{dy}{dx} = e^{y/x}$$

suppose $y(1) = 0$. Give a series expression for $y(x)$ which is valid for x near 1. Neglect terms of order $(x-1)^4$.

Problem 46. ID:MM-1-228

Find the general solution for the second order differential equation

$$y'' + 6y' + 9y = 100e^{2x}$$

Problem 47. ID:MM-1-233

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4x + 5)^2}$$

Problem 48. ID:MM-1-238

Do the integral by contour integration

$$I = \int_{0}^{\infty} \frac{x^2 dx}{(a^2 + x^2)^3}$$

Problem 49. ID:MM-1-243

Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)^2 (x^2 + 2x + 2)}$$

Problem 50. ID:MM-1-248

A quantum mechanical calculation of a transition probability leads to the function $f(t, w) = 2(1 - \cos wt)/w^2$. Compute the integral

$$\int_{-\infty}^{\infty} f(t, w) dw$$

Problem 51. ID:MM-1-253

Evaluate

$$I = \int_{0}^{2\pi} \frac{\sin 3\theta}{5 - 3\cos\theta} d\theta$$

Problem 52. ID:MM-1-258

Do the integral by contour integration

$$I = \int_{0}^{\infty} \frac{\sin x}{x(a^2 + x^2)} dx$$

Problem 53. ID:MM-1-263

Solve the first order differential equation

$$xy^2 dy + (2y^3 - x^3) dx = 0$$

Problem 54. ID:MM-1-268

Note that $y = x^2$ would be a solution of

$$x^2(x-3)y'' - x^2y' + 6y = (x-3)^2$$

if the right side were zero. Use this fact to obtain the general solution of the equation as given.

Problem 55. ID:MM-1-273

Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}$$

Problem 56. ID:MM-1-278

Evaluate the integral

$$I = \int_0^{\infty} \frac{dx}{(x^2+1)(4x^2+1)^2}$$

Problem 57. ID:MM-1-283

Use Liouville's theorem to prove that any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_m z^m$$

with $a_m \neq 0$ and integral $m \geq 0$, has m roots.

Problem 58. ID:MM-1-288

Expand the periodic function

$$f(x) = x^4, \qquad -\pi \leq x \leq \pi,$$

in a sine-cosine Fourier series. Use this Fourier expansion to show that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

(Hint: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

Problem 59. ID:MM-1-293

Find the Fourier transform of the wave function for a 2p electron in hydrogen

$$\Psi(r,\theta,\phi) = \frac{1}{\sqrt{32\pi a_0^5}} r\cos\theta e^{-r/2a_0} \tag{3}$$

where $a_0 = \textit{radius}$ of first Bohr orbit, $\theta \in [0,\pi]$, and $\phi \in [0, 2\pi]$.

Problem 60. ID:MM-1-298

Obtain the inverse Laplace transforms of the following functions

1. $\frac{1}{s^2(s^2+w_1^2)}$
2. $\frac{1}{(s+w_1)^2+w_2^2}$
3. $\frac{1}{(s^2-w_1^2)^2}$

where $w_1, w_2 > 0$.

Problem 61. ID:MM-1-303

Find the Laplace transform of the function

$$f(x) = \frac{1-e^{-x}}{x}$$

Problem 62. ID:MM-1-308

Find the leading behavior of the integral

$$I(x) = \int_0^\infty e^{xt-t^5/5} dt$$

as $x \to +\infty$.

Problem 63. ID:MM-1-313

Use integration by part to show at least the first two terms of the integral

$$\int_0^x t^{-1/2} e^{-t} dt, \qquad x \to +\infty$$

Problem 64. ID:MM-1-318

Evaluate $I(x) = \int_0^1 e^{ixt^3} dt$ approximately for large positive x.

Problem 65. ID:MM-1-323

When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t - nt) dt.$$

Find the leading behavior of $J_n(n)$ as $n \to +\infty$.

Problem 66. ID:MM-1-328

Suppose the matrices **A** and **B** are Hermitian and the matrices **C** and **D** are unitary. Prove that

1. $C^{-1}AC$ is Hermitian
2. $C^{-1}DC$ is unitary
3. $i(AB - BA)$ is Hermitian

Problem 67. ID:MM-1-333

Given

$$M = \begin{pmatrix} 1 & 0 & 2i \\ i & -3 & 0 \\ 1 & 0 & i \end{pmatrix}$$

Find the Hermitian matrices A and B to decompose M as

$$M = A + iB$$

Problem 68. ID:MM-1-338

Define

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and let symplectic group $Sp(2n)$ consist of all matrices satisfying

$$A^t J A = J$$

1. Show that Lie algebra $sp(2n)$ is the algebra of matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

   where B and C are symmetric and A is arbitrary $n \times n$ matrices.

2. Show that $sp(2n)$ has $n(2n+1)$ independent parameters.

Problem 69. ID:MM-1-343

Find the leading behavior of the integral

$$I = \int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}}\, e^{-x \sin^2 t} dt$$

as $x \to +\infty$.

Problem 70. ID:MM-1-348

Suppose the matrices **A** and **B** are Hermitian and the matrices **C** and **D** are unitary. Prove that

1. $C^{-1}AC$ is Hermitian
2. $C^{-1}DC$ is unitary
3. $i(AB - BA)$ is Hermitian

Problem 71. ID:MM-1-353

Compute the Laplace transform of the function

$$f(t) = t\, e^{-\gamma t} \cos[w(t-a)],$$

where a is real, and $\gamma > 0$.

Problem 72. ID:MM-1-358

Find the Fourier transform of the wave function

$$\Psi(x,y,z) = \frac{1}{\sqrt{32\pi a_0^5}}\, x\, e^{-r/2a_0} \tag{4}$$

where $a_0 = radius$ of first Bohr orbit, and $r = \sqrt{x^2+y^2+z^2}$.

Problem 73. ID:MM-1-363

Explain the procedure of Gram-Schmidt orthogonalization, from which you always can construct a set of orthogonal eigenvectors.

Problem 74. ID:MM-1-368

Find the general solution of

$$x^2y'' + 5xy' + 4y = x^3 + 1$$

Problem 75. ID:MM-1-373

Do the integral by contour integration

$$I = \int_0^\infty \frac{\sin x}{x(a^2+x^2)} dx.$$

Problem 76. ID:MM-1-378

Evaluate the first two non-zero terms of the integral

$$I(k) = \int_0^\infty \frac{1}{(1+t^2)^2} e^{-kt} dt, \qquad k \to +\infty$$

Problem 77. ID:MM-1-383

Find the Fourier transform of the Yukawa potential

$$V(x, y, z) = -g^2 \frac{e^{-mr}}{r},$$

with $m > 0$, and the spherical coordinate $r = \sqrt{x^2 + y^2 + z^2}$.

Problem 78. ID:MM-1-388

Which of the following are groups? If it does not form a group, please identify the group conditions it violates.

1. All real numbers (group multiplication= ordinary multiplication)
2. All real numbers (group multiplication = addition)
3. All Complex numbers except zero (group multiplication = ordinary multiplication)
4. All positive rational numbers (product of **a** and **b** is a/b)

2 Mathematical Methods II