Qualification Exam: Quantum Mechanics

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1 Undergraduate level

Problem 1. *1983-Fall-QM-U-1* ID:QM-U-2

Consider two spin 1/2 particles interacting with one another and with an external uniform magnetic field $\vec{B}$ directed along the z-axis. The Hamiltonian is given by

$$H = -A\vec{S}_1 \cdot \vec{S}_2 - \mu_B(g_1\vec{S}_1 + g_2\vec{S}_2) \cdot \vec{B}$$

where μ_B is the Bohr magneton, g_1 and g_2 are the g-factors, and A is a constant.

1. In the large field limit, what are the eigenvectors and eigenvalues of H in the "spin-space" – i.e. in the basis of eigenstates of S_{1z} and S_{2z}?
2. In the limit when $|\vec{B}| \to 0$, what are the eigenvectors and eigenvalues of H in the same basis?
3. In the Intermediate regime, what are the eigenvectors and eigenvalues of H in the spin space? Show that you obtain the results of the previous two parts in the appropriate limits.

Problem 2. *1983-Fall-QM-U-2* ID:QM-U-20

1. Show that, for an arbitrary normalized function $|\psi\rangle$, $\langle\psi|H|\psi\rangle > E_0$, where E_0 is the lowest eigenvalue of H.
2. A particle of mass m moves in a potential

$$V(x) = \begin{cases} \frac{1}{2}kx^2, & x \leq 0 \\ +\infty, & x < 0 \end{cases} \tag{1}$$

Find the trial state of the lowest energy among those parameterized by σ

$$\psi(x) = Axe^{-\frac{x^2}{2\sigma^2}}.$$

What does the first part tell you about E_0? (Give your answers in terms of k, m, and $\omega = \sqrt{k/m}$).

Problem 3. *1983-Fall-QM-U-3* ID:QM-U-44

Consider two identical particles of spin zero, each having a mass m, that are constrained to rotate in a plane with separation r. Bearing in mind that the wave-function $\psi(\theta)$ must be symmetric with respect to the interchange of these bosons, determine the allowed energy levels of this system. (Give the answer in terms of m, r, and an integer n.)

Problem 4. *1983-Spring-QM-U-1* ID:QM-U-54

A spinless particle of mass m moves non-relativistically in one dimension in the potential

$$V(x) = -V_0, \qquad -d/2 \leq x \leq d/2$$
$$V(x) = 0, \qquad \text{elsewhere}$$

This particle is incident with energy E on the potential well from $x = -\infty$, moving toward $x = +\infty$.

1. What is the probability that the particle will, sooner or later, reach $x = 100d$?
2. What is the most likely time interval between when the particle passes $x = -100d$, and when the particle arrives at $x = 100d$?
3. Compare your answer to the previous part to the corresponding answer from classical mechanics.

Problem 5. *1983-Spring-QM-U-2* ID:QM-U-72

A spinless particle of mass m moves non-relativistically in one dimension in the potential well

$$V(\vec{r}) = \begin{cases} -V_0 & |\vec{r}| \leq a = 1\text{Å} = 10^{-10}m \\ 0 & \text{elsewhere} \end{cases} .$$

1. The potential has just one bound state. From this fact, derive "upper and lower bounds on V_0 (for fixed a).
2. Given that the particle is in its bound state, find the probability that it is in the classically forbidden region.
3. Given that the particle is in its bound state, find the probability that its momentum is between p and $p + dp$, where dp is very small.

Problem 6. *1983-Spring-QM-U-3* ID:QM-U-95

An electron (mass m_e, intrinsic spin $\frac{\hbar}{2}$) moves non-relativistically in 3 dimensions in the potential

$$V(\vec{r}) = \frac{1}{2} m_e \omega^2 |\vec{r}|^2$$

1. Find a complete set of commuting observables and describe their eigenfunctions and eigenvalues.
2. Show that the total angular momentum J is conserved.
3. The energy of the electron is $\frac{5}{2}\hbar\omega$. A measurement of J is performed. What are the possible results?
4. List, in the basis of part first part, all the wavefunctions corresponding to each possible eigenvalue of J in the third part.

5. What is the degeneracy of the ground state of two non-interacting electrons in this potential? What are the corresponding wave functions?

Problem 7. *1984-Fall-QM-U-1* ID:QM-U-117

Let us apply Bohr's ideas to a nonrelativistic electron moving in a constant magnetic field $\vec{B}$. The electron's orbit is a circle of radius r in the xy plane, and $\vec{B}$ points along the z axis. The angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is to be quantized just as in Bohr's theory of the hydrogen atom, where $\vec{p}$ is the canonical momentum. Now, however,

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}, \qquad q = -e,$$

where $m\vec{v}$ is the mechanical momentum and $\vec{A}$ is the vector potential.

1. Show that we can choose $\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B}$.
2. Using the fact that the centripetal force is the force due to the magnetic field, obtain the allowed values of r_n. I.e , obtain r_n in terms of $\hbar$, c, e, B, and the quantum number n.
3. Determine the allowed energies E_n. How does your result compare with the exact result, $\epsilon_n = (n + 1/2)\,\hbar\omega_c$, where ω_c is the cyclotron frequency?

Problem 8. *1984-Fall-QM-U-2* ID:QM-U-137

Let us perform a proper quantum-mechanical calculation for the problem of a nonrelativistic electron moving in a constant uniform magnetic field $\vec{B}$ directed along the z axis. The classical Hamiltonian is

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2,$$

where m is the electron's mass, $q = -e$ is the electron's charge, $\vec{p} = m\vec{v}$ is the electron's mechanical momentum, and $\vec{A}$ is the vector potential. It is convenient to choose the Landau gauge:

$$\vec{A} = Bx\hat{y},$$

where $\hat{y}$ is the unit vector in the y direction. Following Landau, let us look for a solution of the form

$$\psi(\vec{r}) = \phi(x)e^{i(k_y y + k_z z)}.$$

1. Show that, if $k_z = 0$, $\phi(x)$ satisfies the Schrödinger equation for a one-dimensional harmonic oscillator.
2. What are the angular frequency ω and the equilibrium position x_0 for this effective harmonic oscillator, in terms of e, B, m, c, and $p_y = \hbar k_y$?
3. If $k_z \neq 0$, what are the allowed energies $E_n(k_z)$?

Problem 9. *1984-Fall-QM-U-3* ID:QM-U-164

Consider a particle of mass m in the 1-dimensional potential

$$V(x) = \begin{cases} \infty, & x \le 0, & \text{region I} \\ 0, & 0 < x \le a, & \text{region II} \\ V_0, & a < x, & \text{region III} \end{cases}$$

1. Write down the general solution to the time independent Schrodinger equation in each of the above three regions.

2. Derive an equation which, at least formally, determines the energy eigenvalues.

3. If $V_0 a^2 = 4\pi^2\hbar^2/m$, how many bound levels does the potential have?

Problem 10. *1984-Spring-QM-U-1* ID:QM-U-185

Consider an electron moving in a deformable medium (in one dimension). The coordinate of the electron is x, and the deformation of the medium is X. The classical Hamiltonian is modeled by

$$H_{cl} = \frac{p^2}{2m} + \frac{P^2}{2M} + \frac{1}{2}KX^2 + ApX,$$

where M and K are parameters describing the medium (which is thus equivalent to a harmonic oscillator with mass M and force constant K). After quantization,

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} + \frac{1}{2}KX^2 + A\left(-i\hbar\frac{\partial}{\partial x}\right)X.$$

Consider solutions of the form

$$\Psi_{kn}(x, X) = ce^{ikx}\phi_n(X),$$

where c is a normalization constant.

1. Find the energy eigenvalues $E_n(k)$ when $A = 0$.

2. Find the energy eigenvalues when $A \neq 0$.

3. Find the effective mass m^* of the electron, when $A \neq 0$. (The electron has been renormalized and is a model "polaron". The effective mass is defined by $\hbar^2 k/m^* = dE_n(k)/dk$.)

Problem 11. *1984-Spring-QM-U-2* ID:QM-U-211

The Schrödinger equation for a simple harmonic oscillator is

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right)\Psi_n = \epsilon_n \Psi_n.$$

Show that if Ψ_n is a solution then so are

$$\Phi_a \equiv \left(\frac{d}{dx} + x\right)\Psi_n \qquad \text{and} \qquad \Phi_b \equiv \left(-\frac{d}{dx} + x\right)\Psi_n$$

Find the eigenvalues of Φ_a and Φ_b in terms of ϵ_n. By consider1ng $\Psi_0 = e^{-x^2/2}$ find what ϵ_n is.

Problem 12. *1984-Spring-QM-U-3* ID:QM-U-224

1. For a one-dimensional single-particle system, prove that any two nondegenerate eigenfunctions ψ_E and $\psi_{E'}$ of $H = \frac{p^2}{2m} + V(x)$ must be orthogonal. (You may assume that ψ_E and $\psi_{E'}$ go to zero exponentially as $x \to \pm\infty$. You must prove that the energy eigenvalues E and E' are real, if that is required by your proof. Hint: Consider the time-independent Schrödinger equation for ψ_E and $\psi_{E'}$. The potential $V(x)$ is real.)

2. Prove that $\frac{d}{dt}\langle\Psi|x|\Psi\rangle = \frac{1}{m}\langle\Psi|p_x|\Psi\rangle$ for a single particle three dimensional system, where the only condition imposed on Ψ is that it satisfies the time-dependent Schrödinger equation. (This is part of Ehrenfest's theorem. For simplicity assume that Ψ goes to zero exponentially as $r \to \infty$.)

3. Consider an infinite well of width $2a$,

$$V(x) = \begin{cases} \infty, & \text{for } |x| \geq a \\ 0, & \text{for } -a < x < a \end{cases}.$$

At time $t = 0$ the wavefunction of a particle of mass m in this well is

$$\Psi(x,0) = \begin{cases} N\sin(\pi x/a), & \text{for } -a \leq x \leq 0 \\ 0, & \text{for } x < -a \text{ and } x > 0 \end{cases},$$

where N is a constant. At a later time t what is the probability that a measurement of the energy will yield the value

$$E = \frac{4\pi^2\hbar^2}{8ma^2} \quad ?$$

Note:

$$\int \sin(nx)\sin(mx)dx = \frac{\sin(x(n-m))}{n-m} + \frac{\sin(x(n+m))}{n+m}, \qquad n \neq m$$

$$\int \sin^2(y)ydy = \frac{1}{2}y - \frac{1}{4}\sin(2y)$$

Problem 13. *1985-Fall-QM-U-1* ID:QM-U-260

The wave function of a particle of mass m trapped in an infinite square well potential, symmetric about the origin and of width 2a,

$$V(x) = \begin{cases} \infty, & \text{for } |x| \geq a \\ 0, & \text{for } -a < x < a \end{cases}$$

is found to be:

$$\Psi(x) = C\left[\cos(\pi x/2a) + \sin(3\pi x)/a + \frac{i}{4}\cos(3\pi x/2a)\right]$$

inside the well and $\Psi(x) = 0$ outside.

1. Evaluate the coefficient C.
2. If a measurement of the total energy of the particle is made, what are the possible results of such a measurement and what is the probability of obtaining each value?
3. What is the mass current at $x = a/2$?

Problem 14. *1985-Fall-QM-U-2* ID:QM-U-286

A particle of mass m moves under the influence of an attractive central force $\vec{F} = k\vec{r}$. Apply the assumptions of the Bohr Model to this system to find an expression for the allowed, quantum mechanical energies.

Discuss very briefly any significant difference between the lowest energy state of this system in this model and that which would result from a solution of the appropriate Schrödinger equation for this force.

Problem 15. *1985-Fall-QM-U-3* ID:QM-U-296

3. Consider the $1D$ symmetric potential $V(x)$ given by:

$$V(x) = \begin{cases} V_0, & \text{for } |x| < a \\ 0, & \text{for } a < |x| < b \\ \infty, & \text{for } b < |x| \end{cases},$$

where V_0 and $a < b$ are constants.

1. Sketch the approximate character of the two lowest energy solutions to the time-independent Schrödinger equation for this potential. (Call them Ψ_1 and Ψ_2 and the corresponding energies E_1 and E_2 and assume V_0 is greater than E_1 and E_2.)
2. A particular solution of the time-dependent Schrödinger equation for this potential can be constructed by superimposing

$$\Psi_1 e^{iE_1 t/\hbar} \qquad \text{and} \qquad \Psi_2 e^{iE_2 t/\hbar}.$$

Construct a wave packet ψ which at time $t = 0$ is (almost) entirely in the left-hand well. Describe the motion of this wave packet as a function of time.

Problem 16. *1985-Spring-QM-U-1* ID:QM-U-322

The wave for the two lowest lying states of the one-dimensional harmonic oscillator are $\Psi_0(x) = A_0 e^{-x^2/2a^2}$ and $\Psi_1(x) = A_1 x e^{-x^2/2a^2}$, where a is the corresponding classical amplitude.

1. Determine the constants A_0 and A_1 by normalizing the wave functions.
2. Calculate the ground state expectation values of x and x^2.
3. Calculate the ground state expectation values of p and p^2.
4. Assume that the uncertainty in the position for a harmonic oscillator is $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and that the uncertainty in momentum is $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$. Show that these uncertainties are consistent with the Heisenberg uncertainty principle.

Problem 17. *1985-Spring-QM-U-2* ID:QM-U-338

1. Show that the frequency of revolution of an electron in its circular orbit in the Bohr model of the atom is $\nu = mZ^2e^4/4\epsilon_0^2 n^3 h^3$.
2. Show that when n is very large, the frequency of revolution equals the radiated frequency calculated from

$$\nu = \frac{m_e e^4 Z^2}{8\epsilon^2 h^3}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

   for a transition from $n_2 = n + 1$ to $n_1 = n$.
3. Give at least two examples where the Bohr model fails.

Problem 18. *1985-Spring-QM-U-3.jpg* ID:QM-U-355

A particle of mass m and charge $-e$ in the potential of a massive point nucleus of charge Ze, has the ground state wave function

$$\Psi_n = \pi^{-1/2}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0},$$

where a_0 is the Bohr radius, given by

$$a_0 = \frac{4\pi\hbar^2\epsilon_0}{me^2}$$

for the potential of the form

$$V(r) = \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}.$$

For an electron $a_0 = 5.29 \times 10^{-11}$ meters.

A real nucleus is not pointlike but approximately spherical with radius r_0 given by

$$r_0 = 2 \times 10^{-15} A^{1/3} \quad \text{meters. (A is the atomic number.)}$$

1. A negative μ-meson (charge $-e$, mass 207 electron masses) is captured in an orbit around a helium nucleus. The atom thus formed has one electron and one μ-meson; suppose these particles are both in their lowest energy states. Give reasons why the electron wave function can be approximated as

$$\Psi_e = \pi^{-1/2} a_e^{-3/2} e^{-r/a_e}$$

while the μ-meson wave function is approximately

$$\Psi_\mu = \pi^{-1/2} 2^{3/2} a_\mu^{-3/2} e^{-2r/a_\mu}$$

Here a_e and a_μ are the Bohr radii for the electron and the μ-meson respectively.

2. When a μ-meson is captured into an orbit around a nucleus of charge Z it sometime reacts with one of the protons: $\mu + p \to n + \nu$. The rate at which this process takes place depends on the nucleus. For small Z the rate is proportional to Z^4. Give reasons why you might expect the exponent to have the particular value 4.

3. For large Z the power law changes. Estimate a value of Z beyond which you would not expect the original power law to hold.

Problem 19. *1987-Fall-QM-U-1.jpg* ID:QM-U-395

Consider a system of angular momentum $l = 1$. The basis of its state space is given by $\{|+\rangle, |0\rangle, |-\rangle\}$, which are the eigenstates of the z-component of the angular momentum operator L_x. Let the Hamiltonian for this system in this basis be

$$\hat{H} = \hbar\omega \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω is a real constant.

1. Find the stationary states of the system and their energies.

2. At time $t = 0$, the system is in the state

$$|\psi(0)\rangle = \frac{1}{3^{1/2}} \left(|-\rangle + |0\rangle - |-\rangle\right).$$

Find the state vector $|\psi(t)\rangle$ at time t.

3. At time t the value of L_z is measured, find the probabilities of the various possible results.

Problem 20. *1987-Fall-QM-U-2* ID:QM-U-422

Consider a system of two non-identical spin 1/2 particles with spins $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$. They are placed in a uniform magnetic field $\vec{B}$ parallel to the z-axis. Assume the gyromagnetic ratio of the two particles are the same so that the Hamiltonian $\hat{H}_0$ of the system can be written as .

$$\hat{H}_0 = \omega\left(\hat{S}_{1z} + \hat{S}_{2z}\right).$$

1. Find the possible energies of the system and their degrees of degeneracy. Draw the energy diagram.

2. We now take coupling of the spine into account by adding the Hamiltonian

$$\hat{W} = a\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2,$$

where a is a real, positive constant. Assume that $a\hbar^2 \ll \hbar\omega$ so that $\hat{W}$ can be treated like a perturbation. Find the eigenvalues to first order in $\hat{W}$ and eigenstates to zeroth order in $\hat{W}$. Draw the energy diagram.

Problem 21. *1987-Fall-QM-U-3* ID:QM-U-442

A one dimensional quantum mechanical system consists of two particles, each of mass m. Both particles are subject to the attractive external harmonic potential

$$V_{ext} = \frac{1}{2}kx^2.$$

In addition they interact via the repulsive potential

$$V_{int} = -\frac{1}{2}k\lambda(x_1 - x_2)^2, \qquad 0 < \lambda < 1/2$$

1. Suppose the two particles are not identical. Find the energy levels of the two particle system.

2. Suppose the two particles are identical spin 0 bosons. What are the allowed energy levels of the two-particle system?

3. Suppose that the two particles are identical spin 1/2 fermions. What are the allowed energy levels of the two-particle system? Indicate which levels are singlet states ($S_{tot} = 0$) and which levels are triplet states ($S_{tot} = 1$). $\hat{\vec{S}}_{tot} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$.

Problem 22. *1988-Fall-QM-U-1* ID:QM-U-465

The time dependent wave equation of a rigid rotor is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2I}\frac{\partial^2\Psi}{\partial\phi^2},$$

where $0 \le \phi \le 2\pi$ and I is a constant (the moment of inertia).

1. Separate variables to obtain an expression for the energy eigenfunctions $u(\phi)$. Solve for the $u(\phi)$. From the boundary condition that $u(\phi) = u(\phi+2\pi)$ obtain a general expression for the energy eigenvalues and eigenfunctions for the system. Is there any degeneracy of the energy levels?

2. b. At $t = 0$ the wavefunction for the rotor is

$$\Psi(\phi, 0) = A\sin^2(\phi), \qquad A = \text{const}.$$

   (a) Derive an explicit expression for the wavefunction at a later time.

   (b) Calculate $\langle E\rangle$, the expectation value of the energy. How does this quantity depend on time?

   (c) Calculate the probability that s measurement of the energy will yield the ground state value obtained in the first part. How does this probability depend on time?

Problem 23. *1988-Fall-QM-U-2* ID:QM-U-493

A particle moves in a three-dimensional harmonic-oscillator potential

$$V(r) = \frac{1}{2}kr^2,$$

where k is the spring constant.

1. Determine the ground-state wavefunction in Cartesian coordinates using the method of separation of variables.

2. Find also the ground-state energy in terms of the frequency ω of the classical oscillations.

3. Estimate the ground state energy using the uncertainty principle and compare it with the answer to the previous part.

Problem 24. *1988-Fall-QM-U-3.jpg* ID:QM-U-512

Consider a one dimensional infinite-wall potential $V = \infty$ for $x > L$ and $x < 0$, and $V = 0$ for $0 \le x \le L$.

1. Find the eigenvalues and the corresponding wavefunctions.

2. Two identical spin 1/2 fermions are in this potential well. Denoting $|\uparrow\rangle$ and $|\downarrow\rangle$for spin up and down, respectively, write the ground-state wavefunction (including the spin part) and energy when the two particles are in a triplet spin state.

3. Repeat the previous part when the fermions are in a singlet spin state.

Problem 25. *1989-Fall-QM-U-1.jpg* ID:QM-U-528

Consider two nucleons, each of mass m, described by the non-relativistic Schrödinger equation; thus

$$\left[-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)+V(r)\right]\psi=E\psi,$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of nucleon 1 and nucleon 2 respectively, and $|\vec{r}_1-\vec{r}_2|=r$ is the distance between nucleons. If $V(r)$ is an attractive square well of depth V_0 and radius b, i.e..

$$V=\begin{cases}-V_0, & 0<r<b\\ 0, & b\le r<\infty\end{cases}$$

and if the system has two bound $l=0$ (s-wave) states with the upper state infinitesimally below 0MeV, then what value must V_0 have.

Problem 26. *1989-Fall-QM-U-2.jpg* ID:QM-U-544

A particle of mass m moves in a one-dimensional infinite square well potential

$$V(x)=\begin{cases}\infty, & x<0\\ 0, & 0<x<a\\ \infty, & a<x\end{cases}.$$

Let the energy eigenstates be labeled $|n\rangle$, $n=1,2,\ldots$; $E_m>E_n$ if $m>n$.

1. Find the possible energies E_n and normalized wavefunctions $\langle x|n\rangle=U_n(x)$.

2. At $t=0$, the system is in the state $|2\rangle$ (the first excited state). A measurement is made and determines that the particle has position between $x_0-\epsilon/2$ and $x_0+\epsilon/2$, where ϵ is small compared to a. What is the wavefunction just after the measurement?

3. At some time $t>0$ following the measurement of the previous part, a measurement is made of the energy. What are the <u>possible</u> results and the probabilities of obtaining each result?

4. Do the probabilities found in the previous part depend on time?

5. In the limit as ϵ is made very small is the distribution of probabilities consistent with the Heisenberg uncertainty principle. DISCUSS.

Problem 27. *1989-Fall-QM-U-3.jpg* ID:QM-U-570

Consider a three-dimensional oscillator with mass m, charge q, and Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_1^2\left(x^2+y^2\right) + \frac{1}{2}m\omega_2^2 z^2, \qquad \text{where } \omega_1 > \omega_2.$$

1. Write down an expression for the energy levels. What is the energy and degeneracy of the ground state and of the first excited state?

2. Does H commute with each of the following operators:

   (a) $\vec{p}$ (linear momentum operator);

   (b) L^2;

   (c) L_z?

   You do not have to actually evaluate the commutator in each case, but you must give the reasoning behind your answer.

The oscillator is now placed in an electric field of magnitude $\mathcal{E}$ and in the z-direction; this produces the perturbation $H' = -q\mathcal{E}z$.

3. Using perturbation theory, what is the first-order correction to the energy of the ground state and of the first excited state due to this perturbation?

4. What is the second-order correction to the energy of the ground state? If the second-order correction is nonzero, be sure to indicate whether it is positive or negative,

Note: For a, one-dimensional harmonic oscillator in the x-direction the x-operator can be written in terms of the raising and lowering operators.

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right).$$

Problem 28. *1989-Spring-QM-U-1.jpg* ID:QM-U-609

A spin-1/2 particle with mass m is constrained to move along the x-axis in a potential given by

$$V = V_0\delta(x)S_x$$

where $\delta(x)$ is the Dirac delta function and S_x is the x-component of the spin-operator.

1. Let ψ_L, be the wavefunction describing the spin-1/2 particle in the region $x<0$ and let ψ_R be the wavefunction describing the particle in the region $x>0$. What conditions relate ψ_L and ψ_R across $x=0$?

2. If the incident wave is polarized with spin up along the z-direction i.e. $\psi_I = e^{ikx}\begin{pmatrix}1\\0\end{pmatrix}$ find the transmitted and reflected waves.

3. Calculate the probability that the particle has spin up in the transmitted wave.

Problem 29. *1989-Spring-QM-U-2.jpg* ID:QM-U-630

Consider a particle of mass μ that is constrained to move on a sphere of radius a and which is described by a Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2\mu a^2}\left(\mathbf{L}^2 + 4L_z^2\right)$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital angular momentum operator.

1. Derive an expression for the energy levels of the particle. In particular what are the energies and degeneracies of the three lowest energy levels?

Consider now adding a potential of the form $\mathcal{H}_i = 2\epsilon \sin\theta\cos\theta$ to the above $\mathcal{H}_0$.

2. Calculate to <u>second order</u> in the perturbation the corrected energy of the <u>ground state</u>.
3. Calculate to <u>first order</u> in the perturbation the corrected energy of the second excited state.

Problem 30. *1989-Spring-QM-U-3.jpg* ID:QM-U-653

A spin-1/2 particle's state space has a basis $|+\rangle$, $|-\rangle$. On this basis the matrix representations of the spin operators are

$$\hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The particle is in a uniform magnetic field in the $+x$-direction, so the Hamiltonian for the particle is $\mathcal{H} = \omega S_x$, where $\omega = -\gamma B$. At $t = 0$ the wavefunction of the particle is

$$|\psi_0\rangle = \frac{1}{\sqrt{10}}\left[3|+\rangle + |-\rangle\right].$$

1. At $t = 0$, S_z, is measured. What are the possible results of this measurement, and what is the probability of each being obtained?
2. Instead of measuring S_z, at $t = 0$, it is measured at some later time t. What are the possible results of this measurement, and what is the probability of each being, obtained? Is S_z, a constant of the motion?

Problem 31. *1990-Fall-QM-U-1.jpg* ID:QM-U-683

Consider a one-dimensional harmonic oscillator described by the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

in which the operators x and p can be written in terms of the raising and lowering operators a and $a^\dagger$,

$$x = \alpha\left(a + a^\dagger\right), \qquad p = \beta\left(a^\dagger - a\right),$$

where α and β are complex numbers.

1. Write down the energy eigenvalues for this system.
2. From the average kinetic and potential energies, and the operator definitions determine α and β.
3. Using operator techniques, determine the matrix elements of the operator xp.
4. If the result of a measurement of $\mathcal{H}$ gives $\hbar\omega/2$, what are the possible results of a measurement of p, and what is the probability of each value being obtained?
5. A state is prepared of the form $\left(2a^\dagger a^\dagger + a^\dagger + 1\right)|0\rangle$ where $|0\rangle$ represents the ground state. What are the possible results of a measurement of $\mathcal{H}$ and what is the probability of each value being obtained?

Problem 32. *1990-Fall-QM-U-2* ID:QM-U-712

A single electron (charge $-e$ and mass m_e,) interacts via a potential $Ze^2/4\pi\epsilon_0 r$ with a nucleus of charge $+Ze$ and radius $2\times 10^{-15}Z^{1/3}$ meters. The ground state wavefunction has the form

$$\Psi_0 = \pi^{-1/2}\left(Z/a_0\right)^{3/2} e^{-Zr/a_0}.$$

1. Using the Bohr atom model, express a_0 in terms of fundamental constants and give its approximate numerical value.
2. A negatively charged μ-meson (charge $-e$ and mass $m_\mu = 207m_e$,) is captured in an orbit about a helium nucleus. This new atom has one electron and one μ-meson. Suppose they are in their lowest energy states. Give approximate expressions for the
   (a) electron-wavefunction
   (b) μ-meson wavefunction.
3. When a μ-meson is captured by an atom, it is possible for the μ-meson to react with one of the protons via a very short ranged interaction resulting in $\mu + p \to n + \nu$. The rate of this reaction depends on the Z of the nucleus and behaves as a power of Z, i.e. Z^α. Assume the nucleus contains Z independent, distinguishable protons, and that the nuclear size is negligible, i.e. take nuclear wave function to be

   $$\psi_N(\vec{r}_1, \vec{r}_2, \dots \vec{r}_Z) = \delta(\vec{r}_1)\delta(\vec{r}_2)\dots\delta(\vec{r}_Z).$$

   Calculate α.
4. For large Z the functional dependence changes. Estimate the value of Z beyond which you would not expect to observe the Z^α power law.

Problem 33. *1990-Fall-QM-U-3* ID:QM-U-741

Consider two distinguishable spin-1/2 Fermions of mass m which are restricted to one dimension and have an interaction of the form

$$V(x_1 - x_2) = -V_0 \delta(x_1 - x_2) \left[\frac{1}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2 + \frac{1}{2} \right],$$

where $\delta(x)$ is a delta function $\vec{S}_1$ and $\vec{S}_2$ are spin operators, and V_0 is a positive number with dimensions of energy times length.

1. Determine the eigenfunctions and energy eigenvalues for the bound states.
2. Discuss how the results of the first part are changed if the particles are indistinguishable.

Problem 34. *1990-Spring-QM-U-2* ID:QM-U-799

A beam of neutrons of $\vec{P} = \hbar k \hat{z}$ moves along the z-axis and impinges upon a crystal. The crystal consists of planes of atoms parallel to the $x - y$ plane. The interaction between a neutron and the crystal is given $V(\vec{r})$. The exact wavefunction for the neutron is given by:

$$\psi(\vec{r}) = e^{ikz} + \frac{e^{ikr}}{r} \int \frac{d^3 r'}{4\pi} \frac{2mV(\vec{r'})}{\hbar^2} \psi(\vec{r'}) e^{ik\hat{r}\cdot\vec{r'}}$$

1. Write the Born Approximation, elastic scattering amplitude $f_k(\vec{k}')$ for scattering an incoming neutron of momentum $\vec{k} = \hbar k \hat{z}$ to an outgoing neutron of momentum $\vec{k}'$. (note: $|\vec{k}| = |\vec{k}'|$.)
2. Taking account of the translational invariance of the potential

   $$V(\vec{r}) = V(\vec{r} + a\hat{z})$$

   where a is the distance between crystal planes, show that only a discrete set of scattering angles are possible. Derive these angles (Bragg angles).

Problem 35. *1990-Spring-QM-U-3.jpg* ID:QM-U-818

A non-relativistic particle of mass m is bound in a finite one-dimensional potential well of width a and depth $|V_0|$:

$$V(x) = \begin{cases} -|V_0|, & \text{for } |x| < a/2 \\ 0, & \text{for } |x| > a/2 \end{cases}.$$

1. If the binding energy is W, what is the asymptotic form of the wavefunction? From your analysis, what length scale characterizes the range of the wavefunction?

2. From the uncertainty principle estimate the binding energy W. When is your estimate valid?

Now consider that a second identical potential well is placed a distance d from the first well (the distance between the centers of the wells is $d+a$). The non-zero value of the wavefunction outside of the original potential well at the second well is responsible for an additional binding energy. (This leads to quantum particle exchange which is responsible for covalent bonds and nuclear forces.)

3. From qualitative arguments give the values of this binding energy in the limits $d \gg a$ and $d \ll a$.

4. Using first-order perturbation theory estimate the effective binding energy as a function of d, $W(d)$. Assume $d \gg a$.

Problem 36. *1991-Fall-QM-U-1.jpg* ID:QM-U-846

Suppose that a system in an energy eigenstate $|0\rangle$ at $t = 0$ is acted upon by an external perturbation $H'(t)$.

1. At time t, we define the state of the system as $\psi(t) = \sum_j a_j(t)|j\rangle$, where $a_0(0) = 1$, and all other $a_j(0)$ are zero as per the initial condition. Show that to first order in H',

$$\frac{\partial}{\partial t} a_j(t) = \frac{E_j^0}{i\hbar} a_j(t) + \frac{\langle j|H'(t)|0\rangle}{i\hbar} a_0(t).$$

Here E_j^0 is the unperturbed energy of state $|j\rangle$, and we assume that $H'(t)$ is non-resonant, so that $\psi(t)$ remains nearly equal to $|0\rangle$ at all t.

2. Calculate the time-dependent dipole moment μ induced in a one dimensional harmonic oscillator initially in the ground state by an electric field $\mathcal{E} = \mathcal{E}_0 \cos \omega t$ turned on at $t = 0$.

3. What is the resonance condition?

Hints:

1. $\hat{\mu} = e\hat{x}$, where e is the electric charge and $\hat{x}$ is the position operator.

2. $H' = -\hat{\mu}\mathcal{E}$.

3. $\hat{x} = \sqrt{\frac{2\hbar}{m\omega_0}}(\hat{a} + \hat{a}^\dagger)$; $\hat{a}|0\rangle = 0$; $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$; $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$; $H|n\rangle = \hbar\omega_0(n + 1/2)|n\rangle$.

4. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

Problem 37. *1991-Fall-QM-U-2.jpg* ID:QM-U-877

1. Consider a particle in a spherically symmetric potential of the form, $U(r) = U_0 r^\alpha$. Demonstrate the Virial theorem for this system by considering,

$$\langle\psi|[H, rp_r]|\psi\rangle$$

where $|\psi\rangle$ is an eigenstate of H and p_r is the radial component of the momentum operator. By evaluating the above expression find a relationship between $\langle T\rangle$ and $\langle V\rangle$, the expectation values of kinetic and potential energy, respectively.

The hydrogenic atom has a Hamiltonian

$$H = T + V, \qquad \text{with} \qquad V = -\frac{Ze^2}{r}$$

The ground state has energy

$$E_0 = -\frac{Ze^2}{2a}$$

and eigenfunction

$$\psi_0(\vec{r}) = 2\left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} Y_{00}(\vec{r})$$

where Y_{lm} is a normalized spherical harmonic and a is the Bohr radius, $a = \hbar^2\mu e^2$.

2. For the ground state of this system calculate the expectation value $\langle T\rangle$ of the kinetic energy and $\langle V\rangle$ of the potential energy, using the Virial theorem.

3. Any region of space in which the kinetic energy T would be negative is forbidden for classical motion. For a hydrogenic atom in the ground state

   (a) find the classically forbidden region (in terms of the Bohr radius a)

   (b) calculate the probability of finding the electron in this region. How does this probability depend on the nuclear charge Z?

Note that in spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}$$

and

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Problem 38. *1991-Fall-QM-U-3.jpg* ID:QM-U-926

The following statements are made about the wave function $\psi(\mathbf{r}, t)$ of a particle. Which of them are correct, and which ones are incorrect? Explain!

1. The wave function can always be written in the form $\psi(\mathbf{r}, t) = f(t)\phi(\mathbf{r})$.

2. In quantum mechanics, given the Hamiltonian $\mathcal{H}$, the knowledge of the state $\psi(\mathbf{r}, t)$ of an *isolated* system at $t = 0$ is sufficient to determine the state $\psi(\mathbf{r}, t)$ of the system at any time $t > 0$.

3. The wave function is always an eigenfunction of the Hamiltonian.

4. Every linear combination of eigenfunctions of the Hamiltonian is another eigenfunction of the Hamiltonian.

5. If the wave function is not an eigenfunction of an observable $\mathbf{A}$, then it is possible that a measurement of $\mathbf{A}$ gives a value that is *not* an eigenvalue of $\mathbf{A}$.

For the following give the coordinate space representation of the most general wave function describing the following situation. Ignore spin-degrees of freedom in all parts where spin is not mentioned explicitly.

6. A particle in three dimensions with fixed momentum vector $\mathbf{p}$.

7. A particle in one dimension with definite position x_0.

8. A particle in one dimension that is confined to a region $-a < x < a$.

9. A particle in three dimensions with fixed orbital angular momentum l, but undetermined z-component of the angular momentum, m.

10. A spin 1/2 particle in three dimensions with fixed z component of the spin, $s_z = \hbar/2$.

Problem 39. *1991-Spring-QM-U-1.jpg* ID:QM-U-959

A potential in one dimension is attractive and of the form,

$$-V_0 e^{-Br^2}$$

1. Show, using a trial variational wavefunction of the form, $e^{-\alpha r^2}$, that this system has at least one bound state. (Note: You are not required to find the optimum α.)

2. Using the same approach for the three-dimensional case (where now r is the magnitude of the position vector, $|\vec{r}|$) find if there is a range of B for which a bound state exists. Hint: a graphical approach to solving the polynomial equation is acceptable.

$$\int_{-\infty}^{\infty} e^{-\gamma x^2} dx = \sqrt{\pi/\gamma}$$

$$\int_{-\infty}^{\infty} x^{2N} e^{-\gamma x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2N-1)}{2^{N+1}\gamma^N} \sqrt{\pi/\gamma}$$

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

Problem 40. *1991-Spring-QM-U-2.jpg* ID:QM-U-982

A particle of mass m moves in one dimension in a potential $V(x) = \frac{\hbar^2}{2m}\lambda\delta(x)$, with $\delta(x)$ the Dirac delta function, and λ a parameter which determines the potential strength.

1. If $\lambda < 0$, how many bound states exist, and what is the ground state energy?

2. What is the transmission probability for the potential for $\lambda < 0$ and $\lambda > 0$?

Note that $\delta(x)$ has the properties, $\delta(x) = 0$ if $x \neq 0$, and $\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1$ for $\epsilon > 0$.

Problem 41. *1991-Spring-QM-U-3.jpg* ID:QM-U-995

Consider an atomic system that consists of a single point nucleus and two electrons. We consider initially electrons with no mutual interaction, so that the two-electron eigenstate can be constructed of standard one-electron hydrogenic eigenstates. The one-electron eigenstates can be written as product states of the form $\psi_{nlm}(r_i)\sigma^{\pm}(s_i)$, where σ^+ and σ^- are the two possible spin states. We will restrict our attention to the 1s and 2s orbital states, ψ_{100} and ψ_{200}.

1. Find the ground-state noninteracting two-electron wavefunction, constructed o f orbital and spin functions $\psi(r_1)$, $\psi(r_2)$, $\sigma(s_1)$, $\sigma(s_2)$ as defined above, which has the proper symmetry with respect to exchange of the two electrons.

2. Find a set of eigenstates that represent the first excited states for the system, again with proper exchange symmetry. Look for products of two-electron spin states and two-electron orbital states (the latter constructed of 1s and 2s states only). What is the degeneracy?

3. Show that the spin states of the previous part are eigenstates of S^2, where $\vec{S}$ is the total spin operator, $\vec{S} = \vec{S}_1 + \vec{S}_2$. What are the eigenvalues? Use the properties, $S_{iz}\sigma^{\pm} = \pm\frac{1}{2}\sigma^{\pm}(s_i)$, $S_{ix}\sigma^+(s_i) = \frac{1}{2}\sigma^-(S_i)$, $S_{ix}\sigma^-(s_i) = \frac{1}{2}\sigma^+(S_i)$, , $S_{iy}\sigma^+(s_i) = -\frac{i}{2}\sigma^-(S_i)$, $S_{iy}\sigma^-(s_i) = \frac{i}{2}\sigma^+(S_i)$.

4. Now add as a perturbation an interaction between electrons of the form, $\lambda\vec{S}_1\cdot\vec{S}_2$. Find the change in energy of the first excited states, and the final degeneracy.

Problem 42. *1992-Fall-QM-U-1.jpg* ID:QM-U-1012

The Hamiltonian for N spinless, non-interacting particles in a one-dimensional harmonic oscillator is given by ($\hbar = \omega = m = 1$)

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i=1}^{N}x_i^2.$$

1. Find the ground state energy if the N particles are bosons.

2. Find the ground state energy if the N particles are fermions.

3. For the boson ground state with its wavefunction denoted by

$$\Psi_0 = e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2},$$

show that

$$-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}(Q\Psi_0) = -(N-1)\Psi + 2Q\Psi_0 + Q\left[-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\Psi_0\right],$$

where

$$Q = \sum_{i=1}^{N}(x_i - x_{cm})^2, \qquad x_{cm} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

4. Show that the wavefunction

$$\Psi' = (Q - C)\Psi_0,$$

where C is a constant to be determined, is an exact N-particle excited state. What must be C? What is the energy of this state?

Problem 43. *1992-Fall-QM-U-2.jpg* ID:QM-U-1046

Consider a delta function potential in one dimension

$$V(x) = \alpha\delta(x),$$

where α has the dimension of energy times length. A particle of mass m and momentum p is incident from the left. Find the transmission coefficient and the phase shift of the transmitted wave relative to the original incident wave.

Problem 44. *1992-Fall-QM-U-3.jpg* ID:QM-U-1056

A box containing a particle is divided into a right and left compartment by a thin partition. If the particle is known to be on the right or left side with certainty, the state is represented by the normalized position eigenket $|R\rangle$ or $|L\rangle$, respectively. The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian

$$H = \epsilon\left(|L\rangle\langle R| + |R\rangle\langle L|\right)$$

where ϵ is a real number with the dimension of energy.

1. Find the normalized energy eigenkets. What are the corresponding energy eigenvalues?

2. Suppose at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

3. If we have instead

$$H = \epsilon|L\rangle\langle R|,$$

show that the probability conservation is violated by repeating the previous part with this Hamiltonian.

Problem 45. *1992-Spring-QM-U-1.jpg* ID:QM-U-1078

An electron with charge $q = -e$ and mass m moves with velocity $\vec{v}$ in a constant magnetic field $\vec{B}$. $\vec{B}$ points along the z-axis, and $v_z = 0$. Let $\vec{P} = m\vec{v}$ (mechanical momentum) and $\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}$ (canonical or conjugate momentum). (If you wish to use MKS units, delete the speed of light c.)

1. Show that $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$ is an acceptable choice for the vector potential $\vec{A}$. (This is called the symmetric gauge.).

2. Draw a picture showing the electron's classical orbit in the xy-plane. At a given point in this orbit, show the directions of $\vec{r}$ (with origin at center of motion), $\vec{v}$, $\vec{B}$, and $\vec{A}$.

3. In quantum mechanics, what is the angular momentum operator $\hat{\vec{L}}$ in terms of $\vec{r}$ and $\vec{\nabla}$? Then which is the classical angular momentum: $\vec{r} \times \vec{P}$ or $\vec{r} \times \vec{p}$? Explain.

4. Now let us generalize Bohr's postulate: $mvr = h\hbar$. Write down the generalization $L = n\hbar$ for an electron moving in a magnetic field B, which reduces to Bohr's postulate as $B \to 0$.

5. Write down another equation which states that the magnetic force on the electron is equal to the required centripetal force.

6. Combine these two equations to find the allowed radii r_n and velocities v_n of the quantized orbits.

7. Find the classical angular frequency of rotation ω (cyclotron frequency) in terms of e, B, m, and c.

8. Determine the allowed kinetic energies E_n in terms of ω.

[This problem is relevant to electrons describing quantized orbits in a metal, at strong magnetic fields and low temperatures, and to the quantum Hall effect.]

Problem 46. *1992-Spring-QM-U-2.jpg* ID:QM-U-1104

1. Using the uncertainty principle, estimate the ground state energy E of a single negative pion bound to a nucleus of charge Ze. You will also need to use the variational principle, which says that the ground state minimizes the total energy. Give your answer in terms of $\hbar$, m_π, e, and Z (plus the Coulomb's law constant $k = 1/4\pi\epsilon_0$ if you are using MKS units).

   [Hint: Take the state of the pion to be characterized by sn average distance r from the nucleus, and a corresponding average momentum p.]

2. Similarly estimate the value of Z at which relativity begins to have an appreciable effect, in the sense that the average velocity v is roughly a tenth the speed of light. Use the numerical values of the constants given below to obtain a number for Z.

[The short-lived pionic atom has been observed. In this problem, the only relevant properties of the pion π^- are its mass m_π and its charge $-e$.]

$$m_\pi \approx 140MeV/c^2, \quad e \approx 1.6\times 10^{-19}C, \quad k \approx 9\times 10^9 Nm^2/C^2, \quad \hbar \approx 10^{-34}m^2kg/s$$

Problem 47. *1992-Spring-QM-U-3.jpg* ID:QM-U-1121

Consider an electron with charge $-e$ and effective mass m bound to an ionized impurity atom with charge $+e$, within a thin semiconducting layer having dielectric constant K (or permittivity $\epsilon = K\epsilon_0$). The interaction between electron and ion is ke^2/Kr (with $k = 1$ in CGS units).

This problem is equivalent to a hydrogen atom in two dimensions, except that $m \to m^*$ and $e^2 \to e^2/K$. The Laplacian in two dimensions is

$$\nabla_2^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

1. Write down the time-independent Schrödinger equation for the wavefunction $\psi(r,\theta)$.
2. Look for a ground state wavefunction having the form $\psi(r,\theta) = Ae^{-r/a}?$. Determine a and the energy E in terms of m^*, K, and the other constants.
3. Let a_0 and E_0 represent the values for $m^* = m$ and $K = 1$ i.e., for a free electron. Then what are the values for an electron orbiting a phosphorus impurity in silicon, for which $m^* = 0.2m$ and $K = 12$? I.e., how much bigger is the electron's orbit, and how much smaller is the binding energy?
4. Use the numerical values of the constants to find E_0 in eV and a_0 in Å. Is an electron more or less tightly bound in 2 dimensions than in 3? Recall that $1Rydberg = 13.6eV$ and Bohr radius is 0.529Å.

$$1\text{Å} = 10^{-10}m, \quad e \approx 1.6\times10^{-19}C, \quad m \approx 9\times10^{-31}kg \approx 5\times10^5 eV/c^2, \quad c = 3\times10^8 m/s$$
$$k \approx 9\times10^9 Nm^2/C^2 \approx 5\times10^{38} eV\cdot\text{Å}/C^2, \quad \hbar \approx 10^{-34}m^2kg/s \approx 4.14\times10^{-15}eV\cdot s$$

Problem 48. *1993-Fall-QM-U-1.jpg* ID:QM-U-1152

Consider a particle in an 1D infinite square well:

$$V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & x < 0 \text{ or } x > L \end{cases} .$$

This particle is prepared in a state such that

$$\psi(x) = \text{constant}, \quad 0 < x < L.$$

What is the probability that it will be found in a particular eigenstate $\psi_n(x)$ of the Hamiltonian? For what values of n is the probability zero? Why?

Problem 49. *1993-Fall-QM-U-2.jpg* ID:QM-U-1169

Use the uncertainty relation to estimate (aside from numerical factors) the ground state energy for each of the following systems:

1. a particle of mass m in a box of length L.
2. a harmonic oscillator of classical frequency ω.
3. a particle of mass m sitting on a table under the influence of gravity (acceleration of gravity g).

Problem 50. *1993-Fall-QM-U-3.jpg* ID:QM-U-1183

1. Write down the time-independent Schrödinger equation for the one-dimensional harmonic oscillator.
2. Express the Hamiltonian H in terms of $\hbar\omega$ and a dimensionless variable $\hat{x} = \alpha x$. What is α?
3. (c) The annihilation operator a can be written

$$a = \frac{1}{\sqrt{2}}\left(\hat{x} + i\hat{p}\right), \qquad \hat{p} \equiv p/\hbar\alpha,$$

   where p is the momentum operator. Write a and $a^\dagger$ in terms of $\hat{x}$ and $d/d\hat{x}$. Also write H in terms of a and $a^\dagger$.
4. Solve for the lowest energy eigenfunction ϕ_0 using the fact that the annihilation operator destroys it; i.e., $a\phi_0 = 0$. Do not bother to normalize ϕ_0.
5. Compute ϕ_1 and ϕ_2 by operating on ϕ_0 with the creation operator $a^\dagger$. (The same undetermined normalization constant C that appears in ϕ_0 will appear in ϕ_1 and ϕ_2.
6. If the wavefunction is $\psi = A\hat{x}^2 e^{-\hat{x}^2/2}$, then what is the expectation value of the energy? A is again a normalization constant.
7. What will be the possible results of a measurement of the energy? With what probabilities?

Note: $a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}$.

Problem 51. *1993-Spring-QM-U-1.jpg* ID:QM-U-1210

1. Consider an electron moving in one dimension within a solid. If its effective mass varies with position,

$$m = m(x),$$

   the appropriate Schrödinger equation turns out to be

$$-\frac{\partial}{\partial x}\frac{\hbar}{2m(x)}\frac{\partial}{\partial x}\psi + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

(This can be obtained from a variational principle. $V(x)$ and $m(x)$ are real.) Show that one still obtains the usual equation of continuity

$$\frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0,$$

where P is the probability density and j is the probability current density:

$$j = \frac{\hbar}{2im(x)} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) = \frac{\hbar}{m(x)} \Im \left(\psi^* \frac{\partial \psi}{\partial x} \right).$$

2. It is consistent with the above expression for j to require that ψ and

$$\frac{1}{m} \frac{\partial \psi}{\partial x}$$

be continuous in the steady state (<u>not</u> ψ and $\partial\psi/\partial x$). Suppose that an electron is incident on an interface between two solids with effective masses m_1 and m_2, and constant potentials V_1 and V_2, respectively. If the incident and transmitted waves are respectively

$$Ae^{ikx} \qquad \text{and} \qquad Ce^{ik'x},$$

find the transmission probability — i.e., the ratio of the transmitted current to the incident current.

Problem 52. *1993-Spring-QM-U-2* ID:QM-U-1244

A spinless particle with charge e and mass m moves in a uniform magnetic field $\vec{B}$ which points in the $\hat{z}$ direction: $\vec{B} = B\hat{z}$. Let us choose the gauge $\vec{A}(\vec{r}) = \frac{1}{2}\left(\vec{B} \times \vec{r}\right)$, which satisfies the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$. The Hamiltonian operator is

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - e\vec{A}\right)^2 \qquad \text{[MKS units]}$$

1. Show that
$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{e}{2m}B\hat{L}_z + \frac{e^2B^2}{8m}\left(x^2 + y^2\right),$$
where $\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$.

2. Show that $\hat{H}$ commutes with $\hat{L}_z$.

3. Let ψ be a simultaneous eigen ket of $\hat{H}$ and $\hat{L}_z$, so that $\hat{H}\psi = E\psi$ and $\hat{L}_z\psi = m_l\hbar\psi$. Show that ψ can be written in the form $\psi(\vec{r}) = u(\rho, \phi)e^{ikz}$, and obtain an equation for $u(\rho, \phi)$. Here ρ, ϕ, and z are cylindrical coordinates, so that $\rho^2 = x^2 + y^2$.

4. Relate the allowed energies E for a charged particle in a magnetic field to the allowed energies E' for a two dimensional harmonic oscillator.

5. For special case $m_l = 0$ and $k = 0$, look for a solution of the form

$$u(\rho) = Ae^{-\rho^2/a^2}.$$

Determine a, and obtain the energy E in terms of the cyclotron frequency $\omega_c = eB/m$.

[Hint: The Laplacian in the cylindrical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.$$

]

Problem 53. *1993-Spring-QM-U-3* ID:QM-U-1247

Consider a particle of mass M placed in a 1 dimensional harmonic oscillator potential

$$V(x) = \frac{M\omega^2 x^2}{2}.$$

Suppose that at $t = 0$ the particle is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right),$$

where $|0\rangle and |1\rangle$ are respectively the ground state and the first excited state of the oscillator.

Let us introduce dimentionless coordinates

$$\tilde{x} \equiv \sqrt{\frac{M\omega}{\hbar}}x, \qquad \tilde{p} \equiv \sqrt{\frac{1}{M\omega\hbar}}p.$$

1. Calculate $\langle\tilde{x}\rangle$ and $\langle\tilde{p}\rangle$ for this state $|\psi(0)\rangle$.
2. Later, we find that $\langle\tilde{x}\rangle$ has the same magnitude as the value found above, but the opposite sign. Calculate the times when this occurs.
3. Calculate the maximum value taken on by $\langle\tilde{p}\rangle$ as a function of time, and also calculate at which time this value occurs.

[Hint: You may want to use creation and annihilation operators:

$$a^\dagger = \frac{1}{\sqrt{2}}(\tilde{x}-i\tilde{p}), \qquad a = \frac{1}{\sqrt{2}}(\tilde{x}+i\tilde{p}), \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad a|n\rangle = \sqrt{n}|n-1\rangle.$$

]

Problem 54. *1994-Fall-QM-U-1* ID:QM-U-1250

A particle of mass m bounces elastically between two infinite parallel plane walls separated by a distance a, i.e. in a potential

$$V(x) = \begin{cases} 0, & \text{if } |x| < a/2; \\ \infty, & \text{if } |x| \geq a/2. \end{cases}$$

The energies and normalized wave functions are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

and

$$\phi_n = \begin{cases} \sqrt{\frac{a}{2}} \sin\left(\frac{n\pi x}{a}\right), & \text{if } n \text{ even;} \\ \sqrt{\frac{a}{2}} \cos\left(\frac{n\pi x}{a}\right), & \text{if } n \text{ odd;} \end{cases}$$

where n are positive integers. At time $t = 0$ the state of the particle is given by

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \left[|\phi_1\rangle + |\phi_2\rangle\right].$$

1. Find the state of the particle at time t.
2. Find the mean value of the energy and its root-mean square deviation $\Delta H = \left(\langle H^2\rangle - \langle H\rangle^2\right)^{1/2}$ at time t.
3. Calculate the mean value of the position at time t.
4. For the time interval over which the wave packet evolves appreciably, verify the time-energy uncertainty relation.

You will need

$$\int dy y \sin(y)\cos^2(y) = -\frac{1}{3}\left[y\cos^3(y) - \sin(y) + \frac{1}{3}\sin^3(y)\right].$$

Problem 55. *1994-Fall-QM-U-2* ID:QM-U-1253

Consider a one dimensional bound state problem with an asymmetric harmonic oscillator potential

$$V(x) = \begin{cases} \frac{1}{2}(\omega_1 x)^2, & \text{if } x \geq 0; \\ \frac{1}{2}(\omega_2 x)^2, & \text{if } x < 0; \end{cases}$$

The dimensionless Hamiltonian is just

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x).$$

1. Consider the trial wavefunction given by

$$\Psi(x) = \begin{cases} e^{-\frac{1}{2}\omega_1 x^2}, & \text{if } x \geq 0; \\ e^{-\frac{1}{2}\omega_2 x^2}, & \text{if } x < 0. \end{cases}$$

Is this an eigenfunction of $\hat{H}$? Give a reason for your answer.

2. Compute the energy expectation value of this wavefunction.

3. If $\omega_2 = 5\omega_1$, find *one* exact, analytic wavefunction of the Hamiltonian $\hat{H}$. Give the analytic form of the wavefunction, state its energy and sketch the wavefunction.

Problem 56. *1994-Fall-QM-U-3* ID:QM-U-1256

1. A one-dimensional harmonic oscillator of mass m and potential energy $\frac{1}{2}m\omega^2x^2$, so the Hamiltonian is
$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$$
   (a) What are the energies of the lowest two energy eigenstates?
   (b) Sketch the wavefunctions $\phi_n(x)$ for the lowest two energy eigenstates.

2. In a one-dimensional two-particle system each particle (mass m) moves in identical external harmonic oscillator potentials. In addition, the two particles interact with each other via a repulsive potential $V_{int} = -\frac{1}{2}m\omega^2\lambda(x_1 - x_2)^2$, with $0 < \lambda < 1/2$. The Hamiltonian is therefore
$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + \frac{1}{2}m\omega^2x_1^2 + \frac{1}{2}m\omega^2x_2^2 - \frac{1}{2}m\omega^2\lambda(x_1 - x_2)^2,$$
where x_1 and x_2 are the position operators for the two particles.
   (a) Rewrite the Hamiltonian in terms of the position operator X_{cm} for the center of mass and the relative position operator x, where
$$X_{cm} = \frac{x_1 + x_2}{2}, \qquad x = x_1 - x_2.$$
   Show that the Hamiltonian can be written as the sum of two harmonic oscillator Hamiltonians.
   (**Hint:** The kinetic energy operator for the two particles in these new variables is
$$-\frac{\hbar^2}{2M}\frac{d^2}{dX_{cm}^2} - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2},$$
   where $M = 2m$ is the total mass and $\mu = m/2$ is the reduced mass.)
   (b) Suppose the two particles are identical spin 0 bosons. What is the energy of the ground state of the system? Express your answer in terms of $\hbar$, ω, and λ.
   (c) Suppose the two particles are identical spin 1/2 fermions. Ignore the magnetic interaction of the two spins. What is the energy of the lowest energy singlet state (total $S = 0$)? Express your answer in terms of $\hbar$, ω, and λ.
   (d) Suppose the two particles are identical spin 1/2 fermions. Ignore the magnetic interaction of the two spins. What is the energy of the lowest energy triplet state (total $S = 1$)? Express your answer in terms of $\hbar$, ω, and λ.

(e) On the basis of your results for fermions which spin state is lower in energy, the singlet or the triplet?

Problem 57. *1994-Spring-QM-U-1* ID:QM-U-1259

An electron is attracted to a plane surface by a potential

$$V(\vec{r}) = \begin{cases} \infty, & \text{if } z < 0; \\ -\alpha e^2/z, & \text{if } z > 0. \end{cases}$$

($\alpha > 0$, the surface is in the $x - y$ plane at $z = 0$.) Calculate

1. The ground-state energy of the electron.

2. Its most likely distance from the surface.

3. Its average distance from the surface.

Hint: You may need $\int_0^\infty x^n e^{-x} dx = n!$.

Problem 58. *1994-Spring-QM-U-2* ID:QM-U-1262

The Hilbert space describing a two-state system has orthonormal basis states $|a\rangle$ and $|b\rangle$. The Hamiltonian operator $\hat{H}$ of the system acts on these basis states as follows:

$$\hat{H}|a\rangle = \epsilon(|a\rangle - |b\rangle)$$
$$\hat{H}|b\rangle = \epsilon(-|a\rangle + |b\rangle),$$

where ϵ is a constant.

Another operator $\hat{P}$ acts on the basis states as follows:

$$\hat{P}|a\rangle = p|a\rangle$$
$$\hat{P}|b\rangle = -p|b\rangle,$$

where p is a constant.

At time $t = 0$ the wavefunction of the system is $|\psi(0)\rangle = |a\rangle$.

1. If the observable for which $\hat{P}$ is the operator measured at $t = 0$, what are the possible results of the measurement and what is the probability that each will be obtained?

2. If the measurement of $\hat{P}$ is not made at $t = 0$, but instead at $t = \pi\hbar/(2\epsilon)$, what are the possible results of the measurement and what is the probability that each will be obtained?

Problem 59. *1994-Spring-QM-U-3* ID:QM-U-1265

Consider an anisotropic harmonic oscillator described by the potential

$$V(x,y,z) = \frac{1}{2}m\omega_1^2(x^2+y^2) + \frac{1}{2}m\omega_2^2 z^2.$$

1. Find the stationary states using rectangular coordinates. What are the degeneracies od the states, assuming that ω_1/ω_2 is irrational?
2. Can the stationary states be eigenstates of $\hat{L}^2$? Of $\hat{L}_z$? Explain in each case!

Problem 60. *1995-Fall-QM-U-1* ID:QM-U-1268

Consider a simple plane rotator (rotating in a plane about a fixed axis perpendicular to the plane) with (angular) coordinate ϕ and conjugate momentum p_ϕ. Assume a Hamiltonian $\hat{H} = Ap_\phi^2$, where A is positive constant.

1. Calculate the energies and normalized wavefunctions of this system. State the degeneracy of each level. (Note: $\phi + 2\pi$ is equivalent to ϕ.)

Now consider two distinguishable plane rotators with coordinates ϕ_1 and ϕ_2 that are coupled according to the Hamiltonian

$$\hat{H} = A\left(p_{\phi_1}^2 + p_{\phi_2}^2\right) - B\cos^2(\phi_1 - \phi_2),$$

where A and B are positive constants.

2. First consider the uncoupled case $B = 0$. Solve for the energies and wavefunctions of all states of this system. Discuss the degeneracy of the levels.
3. Consider $B \ll A\hbar^2$, in other words treat the second term in $\hat{H}$ as a perturbation. Calculate the energy correction to the ground state to the first order in $\frac{B}{A\hbar^2}$.

Problem 61. *1995-Fall-QM-U-2* ID:QM-U-1271

A two-level system has an orthonormal set of basis states $|1\rangle$ and $|2\rangle$, where

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this basis the Hamiltonian operator $\hat{H}$ has matrix representation

$$\hat{H} = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$$

where α is a real positive constant. Let the eigenvectors of $\hat{H}$ be denoted $|I\rangle$ and $|II\rangle$.

1. If the energy of the system is measured, what are the possible results?

In the basis $|1\rangle$, $|2\rangle$ an observable $\hat{A}$ has matrix representation

$$\hat{A} = \begin{bmatrix} a & 0 \\ 0 & 2a \end{bmatrix}$$

where a is a real, positive constant.

2. Is A a constant of the motion? Give the reasoning behind your answer.

3. At time $t = 0$ the system has state vector $|\phi(0)\rangle = |1\rangle$ At a later time t_1, what are the possible results of a measurement of A and what is the probability of each being obtained?

4. What is $\langle A\rangle$ at time t_1?

In the basis $|1\rangle$, $|2\rangle$ an observable $\hat{B}$ has matrix representation

$$\hat{B} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$

where b is a real, positive constant.

5. Is B a constant of the motion? Give the reasoning behind your answer.

6. At time $t = 0$ the system has state vector $|\phi(0)\rangle = |1\rangle$ At a later time t_1, what are the possible results of a measurement of B and what is the probability of each being obtained?

7. What is $\langle B\rangle$ at time t_1?

Problem 62. *1995-Fall-QM-U-3* ID:QM-U-1274

A particle with mass m moves in an attractive spherically symmetric potential

$$V(r) = -\frac{\hbar^2 C}{2m}\delta(r - a)$$

where $C > 0$ and $a > 0$. The analytic form of the wave function corresponding to a fixed angular momentum is, in spherical coordinates, $R_{nl}(r)Y_{lm}(\theta, \phi)$.

1. Write down the time-independent Schrodinger equation for $u_{nl}(r) = rR_{nl}(r)$.

2. What is the normalization condition for the $u_{nl}(r)$?

3. What are the boundary conditions for the $u_{nl}(r)$ at $r = 0$ and $r \to \infty$?

4. What are the conditions on the continuity of $u_{nl}(r)$ and $du_{nl}(r)/dr$?

5. Derive a **simple** transcendental equation for determining the energies of the bound state(s) for $l = 0$.

6. How does the number of bound states depend on the strength C of the potential? Give a graphical argument.

Problem 63. *1995-Spring-QM-U-1* ID:QM-U-1277

A particle in one dimension is in its ground state in a box with sides at $x = -L$ and $x = L$.

1. What is the energy and the wavefunction of the ground state?

2. The particle is in the ground state, and the walls at $x = \pm L$ are suddenly moved outward to $x = \pm 5L$. Calculate the probability that the particle will be found in each of the eigenstates of the expanded box.

Problem 64. *1995-Spring-QM-U-2* ID:QM-U-1280

A wavefunction in spherical coordinates describing a spinless particle of mass M is

$$\psi(r) = A\frac{e^{-\alpha r} - e^{-\beta r}}{r},$$

where A, α, and β are positive constants with $\beta > \alpha$. This particle is in a spherically symmetric potential which satisfies $V(r \to \infty) = 0$, and the wavefunction given above is known to be an eigenstate.

1. What are the expectation values of $\hat{L}_z$, and $\hat{L}^2$ for this state?

2. What is the energy of this state?

3. Calculate the potential which produced this wavefunction.

Hint: $\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$.

Problem 65. *1995-Spring-QM-U-3* ID:QM-U-1283

A one-dimensional harmonic oscillator of mass m and classical frequency ω is in its ground state.

1. Write down the normalized coordinate space wavefunctions for the ground and first excited states.

2. The oscillator is subjected to a perturbation of the form:

$$W = \mathcal{E}a\delta(x-a),$$

   where $a = \sqrt{\frac{\hbar}{m\omega}}$ and $\mathcal{E}$ is a small energy (small compared to the oscillator's energy). Calculate the 1st order correction to the oscillator's ground state energy.

3. Calculate the probability that the particle in the ground state of the perturbed Hamiltonian would be found in the 1st excited state of the unperturbated oscillator.

Hint:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Problem 66. *1996-Fall-QM-U-1* ID:QM-U-1286

Consider a hydrogen atom whose wave function at $t = 0$ is the following superposition of energy eigenstates $\psi_{nlm}(\vec{r})$:

$$\Psi(\vec{r}, t = 0) = A\left(2\psi_{100}(\vec{r}) - 3\psi_{210}(\vec{r}) + \psi_{322}(\vec{r})\right)$$

where A is a constant.

1. Is this wave function an eigenfunction of the parity operator $\hat{P}$, where $\hat{P}\psi(\vec{r}) = \psi(-\vec{r})$? Explain.
2. What is the probability of finding the system in the energy ground state at $t = 0$? Does this probability depend on time, so it is different for $t \neq 0$ from what it is at $t = 0$? Explain.
3. What is the expectation value $\langle E\rangle$ of the energy of the electron? Express your result in eV. Select the zero of energy to be such that the ground state energy is -13.6eV.
4. At $t = 0$ what are the possible outcomes of the measurement of the z-component of angular momentum of the electron and what are the probabilities of each result being obtained?
5. At $t = 0$ what is the expectation value $\langle L_x\rangle$, where $\vec{L}$ is the angular momentum operator? (Recall that $L_x = \frac{1}{2}(L_+ + L_-)$ and $L_y = \frac{1}{2i}(L_+ - L_-)$, where L_+ and L_- are the angular momentum raising and lowering operators.) Give the reasoning behind your answer.

Problem 67. *1996-Fall-QM-U-2* ID:QM-U-1289

A particle is constrained to be in a spherical box of radius a:

$$V(\vec{r}) = \begin{cases} 0, & \text{if } |\vec{r}| \leq a; \\ \infty, & \text{if } |\vec{r}| > a. \end{cases}$$

1. Find the analytic conditions that detemiine the energy levels of the system. Make a sketch of the energy levels labelling the quantum numbers.
2. Find an analytic formula for the energy levels in the limit $\sqrt{2mE/\hbar^2}a \gg l$, where l is the usual angular momentum quantum number.

3. Compare the result of the previous question with the prediction from the Wilson-Sommerfeld quantization.

Problem 68. *1996-Fall-QM-U-3* ID:QM-U-1292

Consider two spin 1/2 particles interacting through a magnetic dipole-dipole interaction,

$$V = A\frac{(\hat{\sigma}_1 \cdot \hat{\sigma}_2)\, r^2 - (\hat{\sigma}_1 \cdot \vec{r})\,(\hat{\sigma}_2 \cdot \vec{r})}{r^5}$$

If the two spins are at a fixed distance d apart and if at $t = 0$ one spin is parallel to $\vec{r}$ and the other one antiparallel to $\vec{r}$, calculate the time after which the parallel spin is antiparallel and the antiparallel spin parallel.

Problem 69. *1996-Spring-QM-U-1* ID:QM-U-1295

A particle of mass M moves in one dimension in an infinite square well:

$$V(x) = \begin{cases} \infty, & \text{if } x \le 0; \\ 0, & \text{if } 0 < x < L; \\ \infty, & \text{if } x \ge 0 \end{cases}$$

Let ψ_1 and ψ_2 denote the wavefunctions of the ground and first excited states. At $t = 0$, the particle is in the state

$$\psi(t = 0) = \frac{1}{\sqrt{2}}\,(\psi_1 + \psi_2)\,.$$

Calculate: $\langle x\rangle(t)$, $\langle p\rangle(t)$, $\langle H\rangle$ and sketch a graph of their behavior versus time.

Problem 70. *1996-Spring-QM-U-2* ID:QM-U-1298

Two particles, each of mass m, interact with each other through a restoring (spring) force with spring constant k_1. They are confined to move along the x-axis. In addition, each particle is attracted to the coordinate origin by a similar restoring force, but with a different spring constant k_2.

Find the energy levels of this system.

Problem 71. *1996-Spring-QM-U-3.jpg* ID:QM-U-1301

Consider a particle of mass μ that is constrained to move on a sphere of radius a and whose Hamiltonian is

$$\hat{H} = \frac{1}{2\mu a^2}\left(\hat{\vec{L}}^2 + 2\hat{L}_z\right),$$

where $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$ is the angular momentum operator.

1. Derive an expression for the energy levels of the particle. In particular, what are the energies and degeneracies of the lowest three levels?

2. At $t = 0$ the particle has normalized wavefunction

$$\psi(\theta, \phi) = \sqrt{\frac{3}{16\pi}}\,(1 + \sin\theta\cos\phi)\,.$$

(a) At $t = 0$ what are the possible outcomes of a measurement of $\vec{L}^2$ and what is the probability of each? What is $\langle \hat{\vec{L}}^2 \rangle$?

(b) At $t = 0$ what are the possible outcomes of a measurement of L_z and what is the probability of each? What is $\langle \hat{L}_z \rangle$?

Note: Take the $l = 0$ and $l = 1$ spherical harmonics Y_l^m to be

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta, \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin(\theta) e^{\pm i\phi}.$$

Problem 72. *1997-Fall-QM-U-1* ID:QM-U-1304

Consider two particles of mass m in a one dimensional space with interaction potential $V(x_1 - x_2)$, where x_1, and $x_2 < x_1$, are the coordinates of the particles. The interaction potential has a short range attractive part and a hard core repulsion.

$$V(x) = \begin{cases} 0, & \text{for } x > a \\ -V_0, & \text{for } 0 < x < a \\ \infty, & \text{for } x < 0 \end{cases},$$

where $V_0 > 0$ and $a > 0$.

1. Find the Schrödinger equation for this system.

2. What will be the difference between the wavefunctions for two Bose-particles and two Fermi-particles?

3. Find the value of V_0 at which the first bound state appears.

Hint: Use variables $\frac{x_1+x_2}{2}$, and $x_1 - x_2$.

Problem 73. *1997-Fall-QM-U-2* ID:QM-U-1307

Consider two spin 1/2 particles interacting with one another and with an external uniform magnetic field $\vec{B}$ directed along the z-axis. The Hamiltonian is given by

$$\hat{H} = -A\vec{S}_1 \cdot \vec{S}_2 - \mu_B \left(g_1\vec{S}_1 + g_2\vec{S}_2 \right) \cdot \vec{B},$$

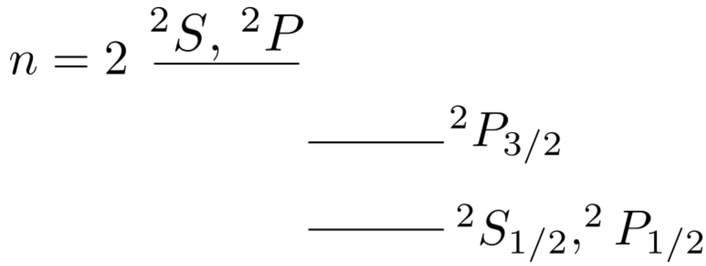
where μ_B is the Bohr magneton, g_1, and g_2 are the g-factors, and A is a constant. (Note: Do not assume $g_1 = g_2$.)

1. In the large-field limit, what are the eigenvectors and eigenvalues of $\hat{H}$ in the basis of eigenstates of S_{iz} and S_{2z}? (Note: Treat the A term as a perturbation and calculate the energy to the first order and the eigenvectors to zero order in this perturbation.)

2. In the limit where $B \to 0$, what are the eigenvectors and eigenvectors of $\hat{H}$ in the same basis as used in the previous question? (Note: Treat the μ_B term as a perturbation and calculate the energy to first order and the eigenvectors to zero order in this perturbation.)

Problem 74. *1997-Fall-QM-U-3* ID:QM-U-1310

The $n = 2$ hydrogenic levels are split by one fine structure interaction into states $^2P_{3/2}$, $^2S_{1/2}$, and $^2P_{1/2}$



An electric field E along the z direction is turned on with interaction $\hat{H}_E = -eEz$ much weaker than the fine structure splitting.

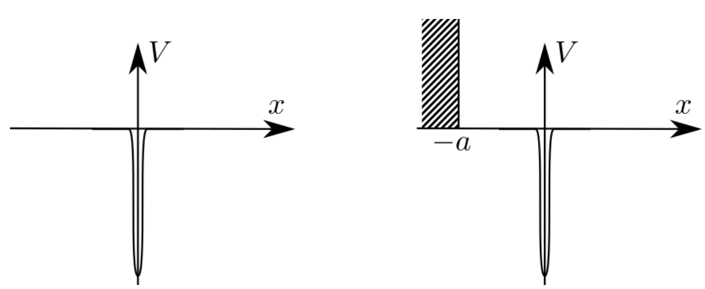
1. What are the allowed electric and magnetic transitions and selection rules?
2. Obtain the shifts in energy levels due to the electric field to first order perturbation theory in terms of the non-vanishing matrix elements of $\hat{H}_E$.

   (Label but do not evaluate the matrix elements.)

Problem 75. *1997-Spring-QM-U-1* ID:QM-U-1313

Consider a particle of mass m in one dimension, subject to the potential $V(x) = -V_0\delta(x)$ where $V_0 > 0$. See the left panel on the figure.

1. Find the possible bound states and the condition for their existence.
2. Find the possible bound states and the condition for their existence when the particle cannot penetrate the region $x \leq -a$, where $a > 0$ (see the right panel on the figure).
3. Give a qualitative interpretation of the results obtained in for the previous question.



Problem 76. *1997-Spring-QM-U-2* ID:QM-U-1316

A two-dimensional isotropic harmonic oscillator has Hamiltonian

$$\hat{H}_0 = \hat{H}_0^x + \hat{H}_0^y,$$

where

$$\hat{H}_0^x = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2, \qquad \hat{H}_0^y = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2.$$

Denote the energy eigenkets of $\hat{H}_0^x$ as $|n_x\rangle$ and the energy eigenkets of $\hat{H}_0^y$ as $|n_y\rangle$.

1. What are the energy and degeneracy of the ground state and of the first excited state of the oscillator?

The perturbation $\hat{V} = gxy$, where g is a positive, real constant, is now added, so that the Hamiltonian becomes

$$\hat{H} = \hat{H}_0 + \hat{V}$$

2. Calculate the ground state energy to second order in V.
3. Calculate the ground state time-independent wavefunction to first order in V (Express your answer in terms of products of $|n_x\rangle$ and $|n_y\rangle$.)
4. Calculate the energy of the first excited state to first order in V.

Note:

$$\langle n_x'|x|n_x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n_x+1}\,\delta_{n_x',n_{x+1}} + \sqrt{n_x}\,\delta_{n_x',n_x-1}\right).$$

Problem 77. *1997-Spring-QM-U-3* ID:QM-U-1319

Consider the two operators $\hat{a}$ and $\hat{N}$, where:

$$\hat{N} = \hat{a}^\dagger\hat{a}, \qquad \{\hat{a}, \hat{a}^\dagger\} = \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 1.$$

Assume $\hat{N}$ has a set of orthonormal eigenstates $|c\rangle$, with non-degenerate eigenvalues c, given by:

$$\hat{N}|c\rangle = c|c\rangle.$$

1. Find $\hat{a}|c\rangle$ and $\hat{a}^\dagger|c\rangle$.
2. Specify all of the possible eigenvalues c.
3. Now assume that the additional constraint $\hat{a}^2 = 0$ also holds. What are the eigenvalues c in this case?

Problem 78. *1998-Fall-QM-U-1* ID:QM-U-1322

Consider a spin 1/2 particle with magnetic moment $\vec{\mu} = \gamma\vec{S}$, where γ is the gyromagnetic ratio. At $t = 0$, we measure the observable S_y and find the eigenvalue $+\hbar/2$. Immediately after the measurement, we apply a time-dependent magnetic field $\vec{B}(t) = B(t)\hat{z}$, such that

$$B(t) = \begin{cases} B_0 t/T, & \text{if } 0 < t < T; \\ 0, & \text{if } t > T; \end{cases}$$

where B_0 and T are constants. At a time $T = \tau > T$, we measure S_z. What results can we find, and with what probabilities?

Problem 79. *1998-Fall-QM-U-2* ID:QM-U-1325

An equilateral right-triangular region is defined by the three sides: (i) $y = 0$; (ii) $x = a$; and (iii) $x = y$. Inside this triangular region, $V(x, y) = 0$ Outside this triangular region, $V(x, y) = \infty$.

Consider the two dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x, y) + V(x, y)\psi(x, y) = E\psi(x, y),$$

where the potential $V(x, y)$ is defined above. Note that the boundary conditions of this eigenvalue problem exclude separable solutions of the form $\psi(x, y) = X(x)Y(y)$.

1. However, show that there exist linear combinations of two separable solutions (of the same energy) that do satisfy this Schrödinger equation and its boundary conditions.

2. Using the method suggested above, find the ground-state and the first-excited-state energies and wavefunctions. (Normalization is not required.)

Problem 80. *1998-Fall-QM-U-3* ID:QM-U-1328

Consider a hydrogen atom governed by the Hamiltonian:

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}.$$

If one includes the spin of the electron but ignores any other perturbations, then the $n = 2$ excited energy level is eight-fold degenerate: three $l = 1$ states and one $l = 0$ state, with electron spin up or down for each.

Consider now the following four perturbations $\hat{H}'$:

1. $\beta\hat{L}_z/\hbar$;

2. $\beta\hat{\vec{L}} \cdot \hat{\vec{S}}/\hbar^2$;

3. $\beta\hat{L}^2/\hbar^2$;

where $\hat{\vec{L}}$ and $\hat{\vec{S}}$ are the electronic orbital and spin angular momentum operators, respectively, and $\beta > 0$ is a parameter with units of energy that describes the strength of the perturbation.

For each of these perturbations, calculate all of the shifted $n = 2$ energy levels to first order in β. Show how the eight-fold degeneracy is broken, and give the remaining degeneracy of each level. You may express any of your answers, if appropriate, in terms of integrals involving the hydrogen-atom radial wavefunctions; you do **not** need to evaluate these integrals.

Problem 81. *1998-Spring-QM-U-1* ID:QM-U-1331

Consider a particle of mass m in a one dimensional potential well, $U = 0$ for $0 \le x \le L$, with infinite barriers at $x = 0$ and $x = L$.

1. What are the wave functions and energy levels for this system?

2. Consider a perturbation potential of the form $V(x) = V_0 x^{-2}$ .Find the correction, to first order, to the ground state energy due to this potential. Assume that dimension-less integrals converge but you need not evaluate them. Express your result in terms of the dimensionless integral(s).

3. The perturbation is changed to $V(x) = V_0 x^{-\alpha}$ with $1 \le \alpha < 3$. For what values of α will perturbation theory fail to work in the limit when $L \to \infty$, i.e., when the first order correction to the energy becomes large in comparison with the zeroth-order energy in the limit when $L \to \infty$, even when V_0 is small? Once again you can assume that all integrals converge for any finite L.

Problem 82. *1998-Spring-QM-U-2* ID:QM-U-1334

1. A point particle of mass m is initially in the ground state of an infinite one-dimensional square-well potential. (That is, the potential $V(x) = 0$ for $0 < x < a$, and $V(x) = \infty$ for $x > a$ or $x < 0$.) At $t = 0$ the right wall of the potential well is moved suddenly to the right by a distance a so the well width is doubled. Find the probability that the particle will be found in the ground state of the new potential well.

2. Next, both walls of the potential well are suddenly moved apart symmetrically so that the well width is again doubled. What are the appropriate ground state wave functions to use in this case? Will the probability for the particle to remain in the ground state be greater in case previous case on in the current case, or will it be the same? You need not carry out a precise calculation to answer this question, but you must explain your qualitative reasoning convincingly.

3. For which of the two previous cases will the probability of observing the particle in the first excited state be non-zero? Give a reason for your answer but no calculation need be done.

4. If the wall(s) is (are) moved adiabatically (i.e., very slowly) rather than suddenly, in the two cases, then what will be the probabilities for observing the particle in the ground state and the first excited state?

Problem 83. *1998-Spring-QM-U-3* ID:QM-U-1337

Let $\hat{\vec{S}}$ be the spin operator for a spin one-half particle. The operator $\hat{\vec{S}}$ has components $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. The spin space of the particle has basis $\{|\alpha\rangle, |\beta\rangle\}$ where the orthonormal kets $|\alpha\rangle$ and $|\beta\rangle$ satisfy the equations

$$\begin{aligned} \hat{S}_z|\alpha\rangle &= +(\hbar/2)|\alpha\rangle, \qquad \hat{S}^+|\alpha\rangle = 0, \qquad \hat{S}^-|\alpha\rangle = \hbar|\beta\rangle, \\ \hat{S}_z|\beta\rangle &= -(\hbar/2)|\beta\rangle, \qquad \hat{S}^+|\beta\rangle = \hbar|\alpha\rangle, \qquad \hat{S}^-|\beta\rangle = 0 \end{aligned} \tag{2}$$

where $\hat{S}^+ = \hat{S}_x + i\hat{S}_y$ and $\hat{S}^- = \hat{S}_x - \hat{S}_y$. The particle is in an external magnetic field in the $+z$-direction, so the Hamiltonian operator is $\hat{H} = A\hat{S}_z$ where A is a constant. At $t = 0$ the spin wavefunction for the particle is $|\phi\rangle = \frac{1}{\sqrt{5}}(2|\alpha\rangle + |\beta\rangle)$.

1. At $t = 0$ a measurement is made of the x-component of spin, S_x, What are the possible results of this measurement and what is the probability of each? What is the expectation value $\langle\hat{S}_x\rangle$?

2. The measurement of S_x is not made at $t = 0$. Instead, we wait until a later time $t = \pi/A$ to measure S_x. What are the possible results of this measurement and what is the probability of each? What is the expectation value $\langle\hat{S}_x\rangle$?

Problem 84. *1999-Fall-QM-U-1* ID:QM-U-1340

Consider the Hamiltonian of two interacting oscillators having the same spring constant k;

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{1}{2}kx_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}kx_2^2 + vx_1x_2,$$

with $v < k$.

1. Find the exact energy levels.

2. Write down the wave function of the ground state in coordinate space.

3. Find the expectation value of $x_1^2 + x_2^2$ for any given state of the two oscillators.

Hint: Under an appropriate coordinate-transformation, the Hamiltonian is separable.

Problem 85. *1999-Fall-QM-U-2* ID:QM-U-1343

The state vector of a particle is constrained in the space spanned by $\{|1\rangle, |2\rangle\}$. Its Hamiltonian is

$$\hat{H} = \hbar\omega\{|1\rangle\langle 1| - \sqrt{3}|1\rangle\langle 2| - \sqrt{3}|2\rangle\langle 1| + 3|2\rangle\langle 2|\} = \hbar\omega \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}.$$

1. Find the eigenenergies and eigenvectors of the particle.

2. At time $t = 0$, the particle is in the state $|1\rangle$. Find its state vector at time t and the probability that the particle remains in the state $|1\rangle$.

Problem 86. *1999-Fall-QM-U-3* ID:QM-U-1346

Two spin-1/2 particles are separated by a distance $\vec{a} = a\hat{z}$ and interact only through the magnetic dipole energy

$$\hat{H} = \frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{a^3} - 3\frac{(\vec{\mu}_1 \cdot \vec{a})(\vec{\mu}_2 \cdot \vec{a})}{a^5},$$

where $\vec{\mu}_i$ is the magnetic moment of particle i. The system of two spins consists of eigenstates of the total spin $\hat{\vec{S}}^2$ and $\hat{S}_z$ operators.

1. Write the Hamiltonian in terms of spin operators.

2. Write the Hamiltonian in terms of $\hat{\vec{S}}^2$ and $\hat{S}_z$.

3. Give the eigenenergies for all states.

Problem 87. *1999-Spring-QM-U-1* ID:QM-U-1349

A particle of mass m is bound in a modified one-dimensional square well defined by the potential energy function

$$V(x) = \begin{cases} \infty, & \text{if } x < 0; \\ 0, & \text{if } 0 < x < a; \\ V_0, & \text{if } a < x. \end{cases}$$

1. What is the form of the bound-state solutions of the energy eigenvalue equation in each of the regions defined above?

2. What conditions must be satisfied by these solutions at $x = \infty$, $x = 0$, and $x = a$?

3. Find the equations satisfied by the energy eigenvalues and show that the depth of the well V_0 must satisfy
$$V_0 > \frac{\pi^2\hbar^2}{8ma^2}.$$

4. The above model potential is used to describe the attraction between a nucleus of radius 5 fm and a neutron of mass 940 MeV/c^2. Calculate how deep (answer in MeV) the well must be in order to bind the extra neutron ($\hbar c = 197$ MeV·fm).

Problem 88. *1999-Spring-QM-U-2* ID:QM-U-1352

An electron of mass m and momentum $\hbar k$: is incident on a one-dimensional spin-dependent δ-potential $\gamma\delta(z)\sigma_x$, where σ_x is the Pauli spin matrix. The initial spin of the electron is polarized along the incident direction, which is taken to be the z-axis.

1. Find the reflected and transmitted waves.
2. Find the transmission coefficient for the electron to remain polarized along the incident direction (spin nonflip).
3. Find the transmission coefficient for the electron to reverse its spin, i.e., pointing opposite to the incident direction (spin flip).

Note:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 89. *1999-Spring-QM-U.jpg* ID:QM-U-1355

Consider a particle of mass m that is constrained to move on a sphere of radius a and whose Hamiltonian is:

$$\hat{H} = \frac{1}{2ma^2}\left(\hat{L}^2 + 2\hat{L}_z\right),$$

where $\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}}$ is the angular momentum operator.

1. Derive an expression for the energy levels of the particle. In particular, what are the energies and degeneracies of the lowest three levels?

At $t = 0$ the particle has normalized wave function

$$\psi(\theta, \phi) = \frac{1}{\sqrt{12\pi}}\left(1 + \sqrt{\frac{9}{2}}\cos\theta - i\sqrt{\frac{3}{2}}\sin\theta\sin\phi\right).$$

2. Denoting the eigenkets of $\hat{L}^2$ and $\hat{L}_z$ by $|lm\rangle$, what is the state vector of the particle at $t = 0$?
3. Find the state vector of the particle at time t.
4. Using $\hat{L}_x = \left(\hat{L}_+ + \hat{L}_-\right)/2$ and $\hat{L}_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}|l, m \pm 1\rangle$, find the eigenvalues and eigenkets of $\hat{L}_x$, for $l = 0$ and $l = 1$.
5. At time t what are the possible outcomes of a measurement of $\hat{L}_x$ and what is the probability of each? What is $\langle L_x \rangle$?

Note: The $l = 0$ and $l = 1$ spherical harmonics (Y_l^m) are

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \qquad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}.$$

Problem 90. *2000-Fall-QM-U-1* ID:QM-U-1358

A box containing a particle is divided into a right and left compartments by a thin partition. If the particle is known to be on the right and left with certainty, the state is represented by the normalized position eigenket $|R\rangle$ and $|L\rangle$, respectively. The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian

$$\hat{H} = \epsilon\left(|L\rangle\langle R| + |R\rangle\langle L|\right),$$

where ϵ is a real number with the dimension of energy.

1. Taking the energy of $|R\rangle$ and $|L\rangle$ to be zero, find the normalized energy eigenkets. What are the corresponding eigenvalues?

2. Suppose at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particles on the left side as a function of time?

3. If we have instead

$$\hat{H} = \epsilon|L\rangle\langle R|,$$

show that the probability conservation is violated if at $t = 0$ the particle is on the right side with certainty. **Hint:** Use the expansion of the time evolution operator.

Problem 91. *2000-Fall-QM-U-2* ID:QM-U-1361

A particle of charge q and mass m is constrained to move in a circle of radius b. Along the axis runs an extremely long solenoid of radius $a < b$, carrying a total magnetic flux Φ. The vector potential outside of the solenoid is then

$$\vec{A} = \frac{\Phi}{2\pi r}\hat{\phi}, \qquad r > a,$$

where $\hat{\phi}$ is a unit vector in the azimuthal direction. Find the eigenenergies of the charged particle. Note that the gradient operator for a particle moving in a circle of radius b is

$$\nabla = \frac{1}{b}\frac{d}{d\phi}\hat{\phi}.$$

Problem 92. *2000-Fall-QM-U-3* ID:QM-U-1364

A spin-1/2 particle with mass m and energy E moves from $-\infty$ in the positive x direction towards the potential:

$$V(x) = \begin{cases} V_0\sigma_z, & \text{if } x > 0; \\ 0, & \text{if } x \le 0. \end{cases}$$

If it is in an eigenstate of σ_x with eigenvalue $+1$ and $E > V_0 > 0$, find the transmitted and reflected waves.

Note, that the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 93. *2000-Spring-QM-U-1* ID:QM-U-1367

onsider two electrons bound to a nucleus with charge Ze by Coulomb interaction. Neglect the Coulomb repulsion between the two electrons.

1. What are the ground state energy and wave function for the electrons? Take into account spin variables.

2. Consider that a weak potential exists between the two electrons of the form

$$V(\vec{r_1} - \vec{r_2}) = \alpha\delta^3(\vec{r_1} - \vec{r_2})\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2,$$

   where α is a constant and $\hat{\vec{s}}_j$ is the spin operator for electron j. Use first-order perturbation theory to estimate how this potential alters the ground state energy.

Hint: In terms of the Bohr radius $a_0 = \frac{\hbar^2}{mZe^2}$ the ground state energy and wave function of a single electron are:

$$E = -\frac{Ze^2}{2a_0}, \qquad \Psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}.$$

Problem 94. *2000-Spring-QM-U-2.jpg* ID:QM-U-1370

1. Consider a particle moving in one—dimensional harmonic potential

$$V(x) = \frac{m\Omega^2}{2} x^2.$$

   Let us assume that initially the particle is in the ground state. Then assume that it suddenly receives a "kick", i.e., momentum p in the x-direction. The "kick" received by the particle means that its wavefunction must be multiplied by the factor $e^{-ipx/\hbar}$. Find the probability that the particle stays in the ground state after the "kick" in terms of Ω and m.

2. The situation described relates to the Mossbauer effect which occurs when the nucleus in a solid emits or absorbs an X-ray photon without recoil. Estimate the probability that an ^{57}Fe nucleus in a solid with vibration frequency $\Omega = 8 \cdot 10^{13} \mathrm{s}^{-1}$ emits or absorbs a resonant 14.4eV photon without recoil.

Hint: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Problem 95. *2000-Spring-QM-U-3* ID:QM-U-1373

A particle of mass m is confined to a box, in one dimension, between $-a < x < a$, and the box has walls of infinite potential. An attractive delta-function potential $V(x) = -\alpha\delta(x)$ is at the center of the box.

1. Derive the transcendent equation defining the eigenvalues and wave functions of the bound states that have negative energies.
2. Find the value of α for which the lowest eigenvalue is zero.
3. What are the eigenvalues and eigenfunctions of the odd-parity states?

Problem 96. *2001-Fall-QM-U-1* ID:QM-U-1376

A spinless particle of mass m moves non-relativistically in one dimension in the potential well

$$V(x) = \begin{cases} \infty, & \text{if } x \leq 0; \\ -V_0, & \text{if } 0 < x \leq a; \\ 0, & \text{if } a < x; \end{cases}$$

where V_0 is a positive constant.

1. The potential has just one bound state. From this fact, derive the upper and lower bounds on V_0 for fixed a.
2. Given that the particle is in its bound state, find the probability that it is in the classically forbidden region. Express your results in a closed form.

Problem 97. *2001-Fall-QM-U-2* ID:QM-U-1379

On the one hand, free neutrinos of electron and muon types are mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ with masses m_1 and m_2, respectively. A free neutrino with momentum p thus satisfies the eigenvalue equation

$$\hat{H}|\nu_i\rangle = E_i|\nu_i\rangle, \qquad E_i = \sqrt{m_i^2c^4 + p^2c^2}, \qquad \text{with } i = 1, 2.$$

On the other hand, the neutrinos accompanying the electron and muon, that are produced in weak interactions, are the neutrino weak interaction eigenstates $|\nu_e\rangle$ and $|\nu_\mu\rangle$. The mass and weak eigenstates are related by

$$\begin{aligned} |\nu_1\rangle &= \cos\theta|\nu_e\rangle + \sin\theta|\nu_\mu\rangle \\ |\nu_2\rangle &= -\sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle, \end{aligned}$$

where θ is a mixing angle. Suppose that at time $t = 0$ a muon neutrino is produced with fixed momentum p, i.e., $\psi(t = 0) = |\nu_\mu\rangle$, find at time t the probability $P_e(t)$ that the ν_μ will have transformed into ν_e. Express your result in terms of p, θ and $\delta m^2 = m_1^2 - m_2^2$, using the assumption that the neutrino masses are very small compared to the ratios of their momenta to the velocity of light.

Problem 98. *2001-Fall-QM-U-3* ID:QM-U-1382

Let $j, m\rangle$ be the normalized eigenstate of angular momentum operator $\hat{\vec{J}}$ and its z-component $\hat{J}_z$, i.e.,

$$\hat{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle, \qquad \hat{J}_z|j,m\rangle = m\hbar|j,m\rangle.$$

An angular-momentum eigenstate $|j,j\rangle$ is rotated by an infinitesimal angle ϵ about the $x$axis. Find the probability through terms of order ϵ^2 for the new state to be found in the original state.

Problem 99. *2001-Spring-QM-U-1* ID:QM-U-1385

Give the order-of-magnitude estimates for the following quantities.

1. The kinetic energy of a nucleon in a typical nucleus.
2. The huge magnetic field (in Gauss) required to produce a Zeeman splitting in atomic hydrogen comparable to the Coulomb binding energy of the ground state.
3. The quantum number n of the harmonic oscillator energy eigenstate that contributes most to the wave function of a classical one-dimensional oscillator with mass $m = 1$ gram, period $T = 1$ sec, amplitude $x_0 = 1$ cm.

Note:

$$\begin{aligned}
&\hbar = 1.1 \times 10^{-27}\ \text{erg}\cdot\text{sec},\\
&\mu_B = 9.3 \times 10^{-21}\ \text{erg/G},\\
&m_p = 1.67 \times 10^{-24}\ \text{g},\\
&1\ \text{G} = 10^{-4}\ \text{T},\\
&1\ \text{erg} = 10^{-7}\ \text{J},\\
&1\ \text{eV} = 1.6 \times 10^{-19}\ \text{J}.
\end{aligned}$$

Problem 100. *2001-Spring-QM-U-2* ID:QM-U-1388

The spin operator in an arbitrary direction can be written as

$$\hat{\sigma}(\theta,\phi) = \hat{\sigma}_x \sin\theta\cos\phi + \hat{\sigma}_y \sin\theta\sin\phi + \hat{\sigma}_z \cos\theta,$$

where the Pauli spin matrices are given by

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Find the eigenvectors and eigenvalues for the operator $\hat{\sigma}(\theta,\phi)$.

2. Choose $\phi = 0$ and find for an entangled state of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\begin{array}{c}1\\0\end{array}\right\rangle_1\left|\begin{array}{c}0\\1\end{array}\right\rangle_2 - \left|\begin{array}{c}0\\1\end{array}\right\rangle_1\left|\begin{array}{c}1\\0\end{array}\right\rangle_2\right)$$

the probability of detecting particle 1 in spin-up with respect to an angle θ_1 and at the same time particle 2 in spin-up with respect to an angle θ_2.

Problem 101. *2001-Spring-QM-U.jpg* ID:QM-U-1391

A particle with charge e and mass m is confined to move counterclockwisely on the circumference of a circle of radius r. The only term in the Hamiltonian is the kinetic energy, so the eigenfunctions and eigenvalues are

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}}e^{in\phi}, \qquad E_n = \frac{\hbar^2 n^2}{2mr^2}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

where ϕ is the angle around the circle. An electric field E is imposed in the plane of the circle and points in $\phi = 0$ direction.

1. Evaluate the first—order correction to the energy levels.
2. Evaluate the second-order correction to the energies of excited states.
3. Evaluate the second-order correction to the energy of the ground state.

Problem 102. *by Morten Eskildsen* ID:QM-U-1394

Consider two operators $\hat{A}$ and $\hat{B}$.

1. Prove the following commutator relation:

$$\left[\hat{A}^2, \hat{B}\right] = \hat{A}\left[\hat{A}, \hat{B}\right] + \left[\hat{A}, \hat{B}\right]\hat{A}$$

2. Use the above result and the commutator relation $[\hat{x}, \hat{p}] = i\hbar$ to show that:

$$\left[\hat{p}^2, \hat{x}^2\right] = -2i\hbar(\hat{p}\hat{x} + \hat{x}\hat{p})$$

Problem 103. *by Morten Eskildsen* ID:QM-U-1397

Consider a particle with mass m, which is confined to an infinitely deep, one-dimensional square potential well of width a. For this problem you should work with a coordinate system which is centered in the middle of the well, ie.

$$V(x) = \begin{cases} 0 & |x| < a/2 \\ \infty & |x| \geq a/2 \end{cases}$$

1. Find the two lowest energy eigenstates and their corresponding energies.

At time $t = 0$ the particle is in the state

$$\psi(x) = A\left(\psi_1(x) + i\psi_2(x)\right),$$

where $\psi_1(x)$ and $\psi_2(x)$ denote respectively the normalized ground state and first excited state that you found above. Assume that we have choosen our phases in the "usual" fashion such that both ψ_1 and ψ_2 are real and positive for $x > 0$.

2. Normalize the wavefunction $\psi(x)$.
3. Find the expectation value of the energy corresponding to $\psi(x)$.
4. What is the probability of finding the particle in the rigth half of the well at $t = 0$?
5. Find the time-dependent wavefunction $\Psi(x,t)$ and $|\Psi(x,t)|^2$. You may express your results in terms of ψ_1, ψ_2, E_1 and E_2.
6. At which times is the probability of finding the particle in the rigth half of the well at a maximum, and what is that probability?

Problem 104. *by Morten Eskildsen* ID:QM-U-1400

Consider a free particle in one dimension, ie. a system with a Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2m}.$$

At time $t = 0$ the particle is represented by the (normalized) wavefunction

$$\psi(x) = (a\sqrt{\pi})^{-1/2}\,\exp\left(-x^2/2a^2 + ik_0x\right).$$

At $t = 0$ answer the following questions:

1. Calculate the expectation value of the position, x, and its square, x^2. Hint: If you re-use these results, you will *not need to evaluate any new integrals* before you reach question (f)!
2. Is $\psi(x)$ an eigenfunction to $\hat{H}$? Justify your answer.
3. Calculate the expectation value of the momentum, p, and its square, p^2. Note that $\psi(x)$ is *not* a stationary state.
4. Calculate the expectation value of the operator $\hat{p}\hat{x} + \hat{x}\hat{p}$ and of the Hamilton operator $\hat{H}$.

In the following questions you will determine how $\langle\hat{x}^2\rangle$ evolves with time.

5. Using the results from Problem I, show that

$$\frac{d\langle\hat{x}^2\rangle}{dt} = \frac{1}{m}\,\langle\hat{p}\hat{x} + \hat{x}\hat{p}\rangle.$$

6. Show that
$$\frac{d^2\langle\hat{x}^2\rangle}{dt^2} = \frac{4}{m}\langle\hat{H}\rangle,$$
and use this result to find
$$\langle\hat{x}^2\rangle = \left(\frac{\hbar}{m}\right)^2\left(\frac{1}{2a^2}+k_0^2\right)t^2 + \frac{a^2}{2}.$$

Problem 105. *by Morten Eskildsen* ID:QM-U-1403

A particle in a harmonic oscillator potential,
$$V(x) = \frac{1}{2}m\omega^2 x^2,$$
starts out ($t = 0$) in the state $A\left(4\left|2\right\rangle + 3\left|3\right\rangle\right)$. You may solve the following problem using either the Dirac notation ($\left|n\right\rangle = \psi_n(x)$) or the function notation as you please.

1. If you measured the energy of this particle, what values might you get, and with what probabilities?
2. Find the expectation value $\langle x\rangle$, as a function of time.
3. Find the expectation value $\langle p\rangle$, as a function of time.
4. Check that Ehrenfest's theorem,
$$\frac{d\left\langle p\right\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle,$$
holds for this state.

Problem 106. *by Morten Eskildsen* ID:QM-U-1406

The objective of this problem is to calculate the probability that an electron in the ground state of hydrogen will be found inside the nucleus.

1. What is the electron wave function, for the ground state of hydrogen?
2. Assume that the wavefuntion found above is correct all the way down to $r = 0$. Calculate the probability of finding the electron inside a sphere of radius b.
3. Use $b \approx 10^{-15}$ m as the size of the nucleus, and evaluate a numerical estimate of the probability found in question (b). Roughly speaking, this represents the "fraction of its time that the electron spends inside the nucleus." Note: You may have to use a series expansion of your answer to question (b) to calculate the probability, depending on the working precision of your calculator.

Problem 107. *by Morten Eskildsen* ID:QM-U-1409

Consider a spin-1 particle. In this case the spin operators become 3×3 matrices:

$$\boldsymbol{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \boldsymbol{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenspinors for $\boldsymbol{S}_x$ and $\boldsymbol{S}_z$ are:

$$\chi^{(x)}_{\pm 1} = \frac{1}{2} \begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix} \qquad \chi^{(x)}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \chi_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here we have used the usual convention that if a spinor has no superscript, it is understood to be an eigenspinor of $\boldsymbol{S}_z$.

The particle is initially in the state $\chi = \frac{1}{\sqrt{2}} (\chi_1 + \chi_0)$.

1. What are the possible outcomes of a measurement of S_x?
2. Calculate the probability of measuring each of the possibilities found in first question.
3. Calculate the expectation value $\langle S_y \rangle$. **Hint**: You don't need to find the eigenspinors for $\boldsymbol{S}_y$ to answer this question.
4. Suppose a measurement of the spin along the x-direction yields the result, $S_x = 0$. What is the probability that a subsequent measurement of the spin along the z-direction would yield $S_z = -\hbar$?
5. Find the eigenspinors for $\boldsymbol{S}_y$. Answer questions (a) and (b) but this time for S_y.

Problem 108. *by Morten Eskildsen* ID:QM-U-1412

Consider the element ^{50}Sn (Tin).

1. What is the electron configuration for Sn? Give your answer in the notation of Eq. (5.33) from Griffiths. Only electrons in excess of those in ^{36}Kr need to be specified.
2. What are the total spin (S), total orbital (L) and grand total (J) angular momentum quantum numbers for the ground state of Sn?
3. Give the angular momentum configuration found above in the $^{2S+1}L_J$-notation, and indentify lower-Z elements with the same configuration.

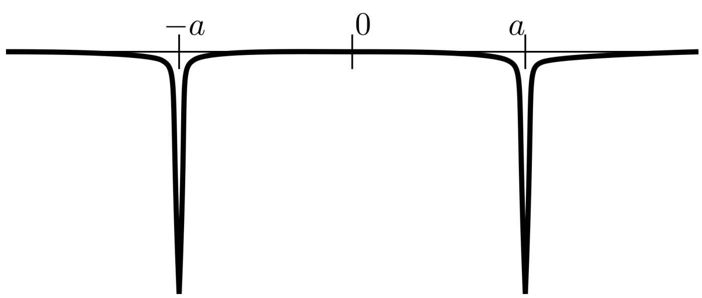
Problem 109. *by Morten Eskildsen* ID:QM-U-1415

Consider a delta-function potential well, $V(x) = -\alpha\delta(x)$. Using the known wave-function and energy for the bound state of a particle with mass m for this potential, answer the following questions:

1. What is the expectation value of the potential energy, $\langle \hat{V} \rangle$.
2. What is the expectation value of the kinetic energy, $\langle \hat{K} \rangle$, where

$$\hat{K} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}.$$

For the remainder of this problem consider a potential consisting of two delta-function wells situated at $x = \pm a$: $V(x) = -\alpha\left(\delta(x+a) + \delta(x-a)\right)$.



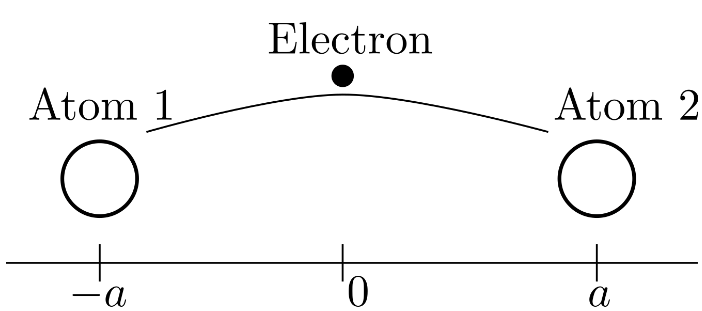
3. Show that this system have bound states with energies which satisfy the condition:

$$\frac{\hbar^2\kappa}{m\alpha} - 1 = \pm e^{-2\kappa a} \quad \text{where} \quad E = -\frac{\hbar^2\kappa^2}{2m}.$$

4. Show that if $a \leq \hbar^2/(2m\alpha)$ the system only has one bound state. Is this state even or odd? Note: You should be able to answer this even if you have not completed question (c).

Problem 110. *by Morten Eskildsen* ID:QM-U-1418

Consider a simple model for a molecule, in which a single electron can move between two identical atoms separated by a distance, $2a$.



Each atom have one bound state and the Hilbert space thus consists of two states, $\{|1\rangle, |2\rangle\}$, corresponding to the electron being at atom 1 or 2 respectively. In this basis the position operator, $\hat{X}$, is given by

$$\boldsymbol{X} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \quad \text{where} \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume that the Hamiltonian for the electron, $\hat{H}$, is given by:

$$\boldsymbol{H} = \begin{pmatrix} E_0 & -\varepsilon \\ -\varepsilon & E_0 \end{pmatrix}$$

1. Is the Hamiltonian operator hermitian? Justify your answer!
2. Is it possible to find common eigenstates for $\hat{H}$ and $\hat{X}$? Justify!
3. Based on the answer to question (b), are $|1\rangle$ and $|2\rangle$ stationary states?
4. Find eigenvalues and eigenstates for $\hat{H}$.

Assume that the electron starts in state $|2\rangle$ at $t = 0$.

5. Find the electron wavefunction at time, t.
6. Find the expectation value for the electron position, $\langle \hat{X} \rangle$, as a function of time.

Problem 111. *by Morten Eskildsen* ID:QM-U-1421

An electron is in the ground state of tritium, for which the nucleus is the isotope of hydrogen with one proton and two neutrons (^{3}H). Assume that a nuclear reaction (β-decay: $n \to p + e + \nu_e$) instantaneously changes the nucleus into ^{3}He, which consists of two protons and one neutron. The energy released in the reaction greatly exceeds the He binding energy and the electron generated by the decay therefore escapes the atom. The final state thus consists of a single electron and the ^{3}He nucleus.

1. What is the ground state wave function for the electron before the decay?
2. What is the ground state wave function for the electron after the decay?
3. What is the probability that the electron is in the ground state of the new (^{3}He) atom after the decay?

Problem 112. *by Morten Eskildsen* ID:QM-U-1424

Assume that the state of a particle is described by the normalized wave function

$$\psi(r,\theta,\phi) = f(r)\,\sin^2\theta\,\left(\cos^2\phi - \sin^2\phi\right),$$

expressed in spherical coordinates. Here $f(r)$ denotes a function of the radial variable r.

1. Is ψ an eigenfunction of L^2? If so what is the eigenvalue?
2. Is ψ an eigenfunction of L_z? If so what is the eigenvalue?
3. Is ψ an eigenfunction of L_z^2? If so what is the eigenvalue?
4. Calculate the standard deviations σ_{L^2}, σ_{L_z} and $\sigma_{L_z^2}$.

Problem 113. *by Morten Eskildsen* ID:QM-U-1427

In this problem you will derive the Clebsch-Gordan coefficients for the combination of two angular momenta with $s_1 = s_2 = 1$, noting that this of course applies to either two spins, two orbital angular momenta or a combination of the two.

Total angular momentum states should be written as a single ket $|s\,m\rangle$ and the product of two single-particle momenta as a double ket $|s_1\,m_1\rangle|s_2\,m_2\rangle$. You will hence have to determine the coefficients, $C^{s_1\,s_2\,s}_{m_1\,m_2\,m}$, such that

$$|s\,m\rangle = \sum_{m_1+m_2=m} C^{1\,1\,s}_{m_1\,m_2\,m}|1\,m_1\rangle|1\,m_2\rangle.$$

1. List all the possible combinations of total angular momentum numbers (i.e. pairs of s and m) one can obtain by combining to two angular momenta with $s_1 = s_2 = 1$.
2. Assume that $|2\,2\rangle = |1\,1\rangle|1\,1\rangle$, and use successive applications of $S_- = S_-^{(1)} + S_-^{(2)}$ to generate the $|2\,1\rangle$ and $|2\,0\rangle$ states. Do not forget to normalize! List the Clebsch-Gordan coefficients obtained in this way.
3. Are the $|2\,m\rangle$ states obtained in question (b) symmetric or antisymmetric? Justify your answer.
4. Argue that $|1\,1\rangle = A\,(\alpha\,|1\,1\rangle|1\,0\rangle + \beta\,|1\,0\rangle|1\,1\rangle)$. Hint: How many possible ways can you get $m_1 + m_2 = 1$?
5. Use the requirement that $\langle 2\,1|1\,1\rangle = 0$ to determine α and β. Apply S_- to find $|1\,0\rangle$ and list the obtained Clebsch-Gordan coefficients. Are the $|1\,m\rangle$ states symmetric or antisymmetric?
6. Use the approach from questions (d) and (e) to find $|0\,0\rangle$.

7. Use $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$ to show that $S^2 |2\,2\rangle = 2(2+1)\hbar^2 |2\,2\rangle$ and thus prove the assumption in question (b).

Problem 114. *by Morten Eskildsen* ID:QM-U-1430
Consider a a particle with mass m in a one-dimensional infinite square well potential:

$$V(x) = \begin{cases} 0 & |x| < a \\ \infty & |x| \geq a \end{cases}$$

The initial wavefunction for the system is $\Psi(x,0) = A\,x(a^2 - x^2)$.

1. Determine the constant A.
2. Specify the solutions to the time-independent Schrödinger equation and their corresponding energies. Note: You do not have to work this out in detail, but you must explain how you obtained your answers.
3. What is the probability of finding the particle in respective the ground state and the first excited state?
4. Calculate the expectation value of the energy.
5. Comment on the answers to questions (c) and (d). Are the results as you would expect?

Problem 115. *by Morten Eskildsen* ID:QM-U-1433
Consider a two-state system with stationary states ψ_1 and ψ_2 and corresponding energies $E_1 \neq E_2$. Assume that the system initially is in the state $\Psi(x,0) = 1/\sqrt{2}\,(\psi_1 + \psi_2)$.

1. Find $\Psi(x,t)$ and calculate the inner product $\langle \Psi(x,0) | \Psi(x,t) \rangle$.
2. Find the first time T at which $\Psi(x,T)$ is orthogonal to the initial state.
3. Calculate σ_H for the state $\Psi(x,t)$.
4. Use your answers to the previous questions to define a Δt which satisfied the time-energy uncertainty principle.

Problem 116. *by Morten Eskildsen* ID:QM-U-1436
In this problem you will consider the two-dimensional harmonic oscillator to explore the difference between quantum mechanical angular momentum in 2 and 3 dimensions. Specifically consider a spinless particle in the potential $V = \frac{1}{2}m\omega^2(x^2 + y^2)$.

1. Show that the Hamiltonian can be written as

$$\hat{H} = \hbar\omega\left(a_+^{(x)}a_-^{(x)} + a_+^{(y)}a_-^{(y)} + 1\right),$$

where $a_\pm^{(x)}$ and $a_\pm^{(y)}$ are the raising and lowering operators for the oscillator along the x- and y-axis respectively. Note that the motion along the two axes is decoupled and therefore $\left[a_\pm^{(x)}, a_\pm^{(y)}\right] = 0$.

If we describe the states for the two-dimensional harmonic oscillator by independent quantum numbers for the motion along the x- and y-axis, show that $|n_x\, n_y\rangle$ is an eigenstate to the hamiltonian with the eigenvalue $E_{n_x\, n_y} = (n_x + n_y + 1)\hbar\omega$.

2. Show that the angular momentum $L = xp_y - yp_x$ expressed in terms of raising and lowering operators can be written as

$$\hat{L} = i\hbar\left(a_-^{(x)}a_+^{(y)} - a_+^{(x)}a_-^{(y)}\right).$$

Does $\hat{L}$ commute with the hamiltonian? You may argue your answer without calculating.

Are the states $|n_x\, n_y\rangle$ eigenstates for $\hat{L}$?

For the remainder of this problem we consider the subspace associated with the energy $E = 2\hbar\omega$, ie. the states $|\alpha\rangle = |1\,0\rangle$ and $|\beta\rangle = |0\,1\rangle$. These two states are orthogonal, $\langle\alpha|\beta\rangle = \langle 1\,0|0\,1\rangle = 0$.

3. Show that $\hat{L}|1\,0\rangle = i\hbar|0\,1\rangle$ and $\hat{L}|0\,1\rangle = -i\hbar|1\,0\rangle$. Use these results to express $\hat{L}$ for the $E = 2\hbar\omega$ subspace as a 2×2 matrix.

What are the eigenvalues of this matrix?

4. Consider the operators $\hat{M} = \hbar\left(a_+^{(x)}a_-^{(x)} - a_+^{(y)}a_-^{(y)}\right)$ and $\hat{N} = \hbar\left(a_-^{(x)}a_+^{(y)} + a_+^{(x)}a_-^{(y)}\right)$ which both represent conserved quantities (commutes with $\hat{H}$). Express $\hat{M}$ and $\hat{N}$ as 2×2 matrices for the $E = 2\hbar\omega$ subspace.

What does the matrices for $\hat{L}$, $\hat{M}$ and $\hat{N}$ remind you of? Do these operators commute?

5. Consider your answer to question (c). How is this different from angular momentum in 3 dimensions? Can you give an explanation for this difference?

Problem 117. *by Morten Eskildsen* ID:QM-U-1439

This problem concerns the electron in a Coulomb potential due to a charge $|e|$ at the origin (hydrogen atom). Consider the case with $l = n - 1$.

1. Show that the radial wave function takes the form

$$R_{n(n-1)} = N_n\, r^{n-1}\, e^{-r/na},$$

and determine the normalization constant N_n by direct integration.

2. Calculate $\langle r \rangle$ and $\langle r^2 \rangle$ for states of the form $\psi_{n(n-1)m}$.

3. Show that the "uncertainty" in r is given by $\sigma_r = \langle r \rangle / \sqrt{2n+1}$ for such states.

4. Explain in words what the above result imply for the classical limit ($n \gg 1$)?

Problem 118. *by Morten Eskildsen* ID:QM-U-1442

Muonium ($\mu^+ e^-$) can be considered a light "isotope" of hydrogen, in which the proton has been replaced by a positive muon μ^+, a particle which is which is very similar to a positron e^+, except that it has a mass $m_\mu \simeq 207 m_e$ and that it is unstable, with a lifetime of about 2.2×10^{-6} s. Muonium was first observed in 1960 and attracted a great deal of interest because it contains only leptons and thus presents a particularly suitable system to verify the predictions of quantum electrodynamics.

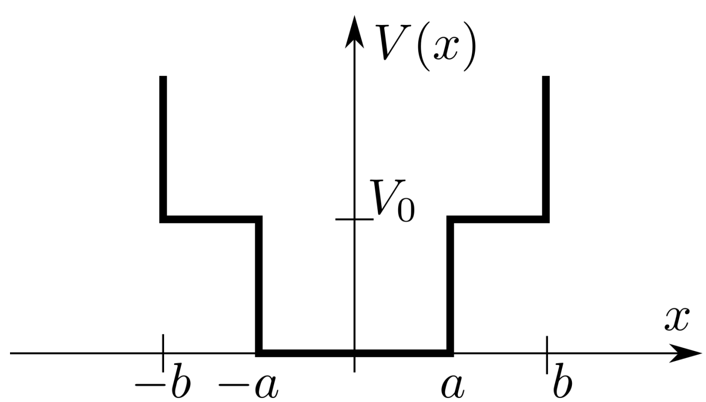
1. Find the binding energy of muonium.

2. Find the separation (to three significant digits) in wavelength between the first Lyman line ($n = 2 \to n = 1$) for hydrogen and muonium.

Problem 119. *by Morten Eskildsen* ID:QM-U-1445

Consider a particle with mass m in a one-dimensional potential given by:

$$V(x) = \begin{cases} 0 & |x| < a \\ V_0 & a < |x| < b \\ \infty & |x| \geq b \end{cases}$$



Here we will consider states with energy $E < V_0$.

1. Sketch the wave function for the ground state and first excited state for the potential in question.

2. Is the ground state energy for the potential above greater or smaller than for the infinite square well of width $2a$? How about the finite square well with width $2a$ and depth V_0?
   Do not attempt to calculate the answers but provide qualitative arguments.

3. Find separate expressions for the wave function for in the three regions of the potential above: $\psi_{\mathrm{I}}(-b < x < -a)$, $\psi_{\mathrm{II}}(-a < x < a)$, and $\psi_{\mathrm{III}}(a < x < b)$.

4. Show that the boundary condition for $x = b$ yields a wave function for the rightmost region $\psi_{\mathrm{III}}(a < x < b) \propto \sinh[\kappa(x-b)]$ with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$.

5. For the even wave functions show that the $x = a$ boundary conditions yields a transcendental equation for the allowed energies

$$\tan ka = \frac{\kappa}{k}\coth[\kappa(b-a)],$$

where $k = \sqrt{2mE}/\hbar$.

6. Discuss the transcendental equation in the limits $b \to a$ and $b \to \infty$. Do you obtain the expected results?

7. Does the transcendental equation confirm your answers to question (b)? Provide a careful justification for your answer.

Problem 120. *by Morten Eskildsen* ID:QM-U-1448

Neutrinos are uncharged, relativistic particles produced in processes such as

$$\begin{aligned} p &\to n + e^+ + \nu_e \\ \pi^+ &\to \mu^+ + \nu_\mu, \end{aligned}$$

describing respectively a proton decaying to a neutron, a positron and an electron neutrino (ν_e), and a pion decaying to a muon and a muon neutrino (ν_μ). The quantum states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are eigenstates to the Hamiltonian describing their creation in processes such as the ones listed above (weak interaction). However, when neutrinos propagate in free space, the only Hamiltonian of relevance is that due to the (relativistic) energy of the particles. The eigenstates of this Hamiltonian are referred to as the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, with energies E_1 and E_2 ($E_1 \neq E_2$). Either of the two bases can be used to express any general state in this system. Assume that the relation between the bases is given by

$$\begin{aligned} |\nu_1\rangle &= \cos\tfrac{\theta}{2}\,|\nu_e\rangle + \sin\tfrac{\theta}{2}\,|\nu_\mu\rangle \\ |\nu_2\rangle &= \sin\tfrac{\theta}{2}\,|\nu_e\rangle - \cos\tfrac{\theta}{2}\,|\nu_\mu\rangle, \end{aligned}$$

where the θ is generally referred to as the mixing angle.

1. Find expressions for $|\nu_e\rangle$ and $|\nu_\mu\rangle$ in terms of the mass eigenstates and the mixing angle.

Assume that an electron neutrino is created at $t = 0$.

1. Provide an expression for the neutrino state $|\psi(t)\rangle$ as it propagates through free space towards a detector. Give this in terms of the mass eigenstates and energies, and the mixing angle.

2. Show that the probability that a measurement at time t would detect a muon neutrino is given by

$$\mathcal{P}_{\nu_e\to\nu_\mu} = |\langle\nu_\mu|\psi(t)\rangle|^2 = \sin^2\theta\,\sin^2\left(\frac{(E_1-E_2)t}{2\hbar}\right).$$

3. The process $p \to n + e^+ + \nu_e$ is part of the thermonuclear reaction chain in the sun. However, only about half of the electron neutrinos arrive at Earth while the other half having "oscillated" to muon neutrinos. Use this information provide a lower bound on the mixing angle (on the interval 0 to 90°. The actual value is $\theta \simeq 69°$. Is this consistent with your answer?

Problem 121. *by Morten Eskildsen* ID:QM-U-1451

The (time independent) momentum space wave function in three dimensions is defined by

$$\Phi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int e^{-i(\mathbf{p}\cdot\mathbf{r})/\hbar}\,\psi(\mathbf{r})\,d^3\mathbf{r}.$$

Some useful integrals are given at the bottom of the page.

1. Show that the momentum space wave function for the ground state of hydrogen is given by

$$\Phi(\mathbf{p}) = \frac{1}{\pi}\left(\frac{2a}{\hbar}\right)^{3/2}\frac{1}{[1+(ap/\hbar)^2]^2}.$$

   Hint: Use spherical coordinates, setting the polar axis along the direction of $\mathbf{p}$. Be careful not to confuse the momentum space wave function (Φ) with the azimuthal angle (ϕ).

2. Show that $\Phi(\mathbf{p})$ is normalized by evaluating the integral

$$\int |\Phi(\mathbf{p})|^2\,d^3\mathbf{p}.$$

3. Use $\Phi(\mathbf{p})$ to calculate $\langle p^2\rangle$, for the ground state of hydrogen. Recall that in momentum space, $\hat{p} = p$. You should get a very simple result!

4. Use the result from above to calculate the expectation value for the kinetic energy. Express you answer in terms of hydrogen ground state energy, E_1. Does your result agree with the virial theorem for hydrogen, $\langle T\rangle = -E_n$?

Useful integrals:

$$\begin{aligned}\int_0^\infty x\,e^{-\alpha x}\sin\beta x\,dx &= \frac{2\alpha\beta}{[\alpha^2+\beta^2]^2}\\ \int_0^\infty \frac{x^m}{[x^2+\alpha^2]^4}\,dx &= \frac{\pi\alpha^{m-7}}{32}, \qquad \text{for } m = 2 \text{ or } 4\end{aligned}$$

Problem 122. *by Morten Eskildsen* ID:QM-U-1454

Quantum dots are nanoparticles characterized by a specific color that depends on the particle diameter, d. These are often made from semiconductor materials such as CdSe, CdS or PbS.

Assume that an electron in a quantum dots can be described by a spherical symmetric three-dimensional harmonic oscillator potential:

$$V(r) = \tfrac{1}{2}m\omega^2 r^2.$$

1. Show that separation of variables in cartesian coordinates turns this problem into three independent one-dimensional oscillators. Write down separate time-independent Schrödinger equations for the x-, y-, z-axis motion, but do not solve these.

2. Exploit your knowledge of the 1D harmonic oscillator to express the energy for the 3D case in terms of three independent quantum numbers, n_x, n_y, and n_z, and explain what these quantum numbers represent. Make a table showing the total energy, values of the quantum numbers and the degeneracy, for the three lowest energy levels of the 3D harmonic oscillator.

The angular momentum operators $\hat{L}_z$ and $\hat{L}^2$ can be expressed in terms of the harmonic oscillator raising and lowering operators as follows:

$$\begin{aligned}
\hat{L}_z &= i\hbar\left[a_-^{(x)}a_+^{(y)} - a_+^{(x)}a_-^{(y)}\right] \\
\hat{L}^2 &= -\hbar^2\left[(a_-^{(y)}a_+^{(z)} - a_+^{(y)}a_-^{(z)})^2 + (a_-^{(z)}a_+^{(x)} - a_+^{(z)}a_-^{(x)})^2 + (a_-^{(x)}a_+^{(y)} - a_+^{(x)}a_-^{(y)})^2\right]
\end{aligned}$$

3. Consider an electron state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\,0\rangle + i|0\,1\,0\rangle\right)$, where the numbers in the kets represent n_x, n_y, and n_z. Show that $|\psi\rangle$ is an eigenstate for $\hat{L}_z$ and $\hat{L}^2$ and find the respective eigenvalues.

Alternatively, the wave functions for the 3D harmonic oscillator can be expressed in spherical coordinates. In this case eigenstates to the Hamiltonian are expressed by a different set of quantum numbers: n_r, ℓ and m. Here, n_r describes the radial motion of the electron, and ℓ and m are the usual quantum numbers corresponding to $\hat{L}^2$ and $\hat{L}_z$. It is possible to show (although we shall not do this) that the energy is given by $E = (2n_r + \ell + \frac{3}{2})\hbar\omega$.

4. Make a table showing the total energy, values of the quantum numbers n_r, ℓ and m, and the degeneracy, for the three lowest energy levels of the 3D harmonic oscillator.
   Are the degeneracies the same as you found in question (b)?
   Can you identify the state $|n_r\,\ell\,m\rangle$ that corresponds to $|\psi\rangle$ from question (c)?

Quantum dots with diameter $d = 7.5$ nm will emit orange light with wavelength $\lambda = 590$ nm. Assume that the characteristic length associated with the harmonic oscillator is related to the diameter by $\sqrt{\hbar/(m\omega)} = \frac{1}{4}d$, and that the emitted light is associated with a electron transition from an initial state with $n_r = 2$ and $\ell = 1$ to a final state with $n_r = 0$ and $\ell = 0$.

5. Estimate the effective mass of the electron in the quantum dot. Note that this is not necessarily the same as the "free" electron mass, m_e.

6. Derive the expressions for $\hat{L}_z$ and $\hat{L}^2$ given above.

Problem 123. *by Morten Eskildsen* ID:QM-U-1457

Consider element number 44 (Ruthenium).

1. What is the electron configuration for Ruthenium? Only electrons in excess of those for ^{36}Kr need to be specified.

2. Use Hund's rules to find the total spin (S), the total orbital angular momentum (L), and the combined total angular momentum (J) for the ground state of Ruthenium.

Problem 124. *by Morten Eskildsen* ID:QM-U-1460

Consider a particle in a one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2x^2$. The particle is now subjected to the perturbation

$$H' = V_0\, e^{-2m\omega x^2/\hbar}.$$

1. Use perturbation theory to find the first-order correction to the ground state energy.

Consider now a spin-0 particle with charge q in a two-dimensional harmonic oscillator potential, $V(x, y) = \frac{1}{2}m\omega^2(x^2+y^2)$. A magnetic field, B_0, is applied along the z-axis, ie. perpendicular to the harmonic oscillator plane.

2. What is the perturbation, H', due to the magnetic field?

3. Use perturbation theory to find the first- and second-order corrections to the ground state energy due to the magnetic field. Don't let the result surprise you! (Hint: This and the following question are most easily solved using Dirac notation and the raising and lowering operators $a_\pm^{(x)}$ and $a_\pm^{(y)}$).

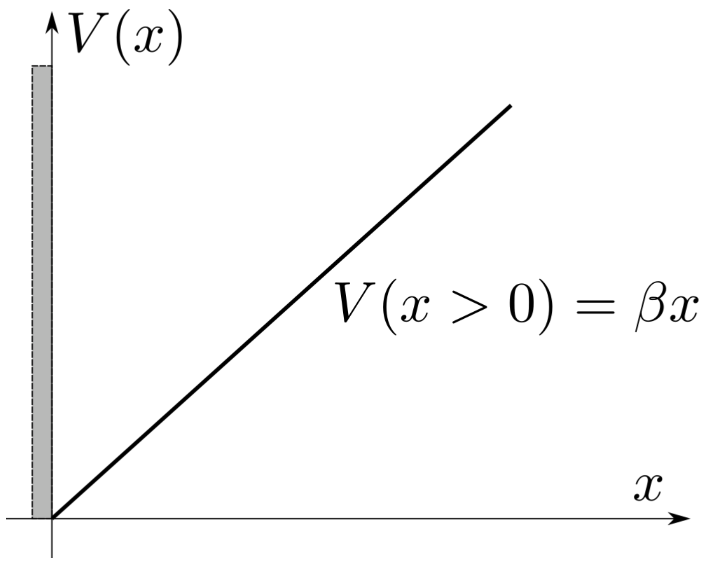
4. Calculate the first-order correction to the energy of the doubly degenerate first excited state. Use as basis vectors $\psi_a = |1, 0\rangle$ and $\psi_b = |0, 1\rangle$, in the notation $|n_x, n_y\rangle$.

5. What are the "good" linear combinations of ψ_a and ψ_b for this problem? Find a hermition operator for which these "good" linear combinations are the eigenstates.

6. If the particle was spin-$\frac{1}{2}$ in stead of spin-0, would there be a first-order correction to the harmonic oscillator energies due to spin-orbit coupling? The answer must be based on physical arguments, and no calculations are allowed for this question.

Problem 125. *by Morten Eskildsen* ID:QM-U-1463

In this problem you will consider a triangular potential well, which is a good model for a two-dimensional electron gas (2DEG) confined close to the interface between two semiconductor materials. The potential shown in the figure goes to infinity for $x < 0$, and increases linearly as $V(x) = \beta x$ for $x > 0$.



1. Find an upper bound on the ground state energy using the variational method and the trial wavefunction,

$$\psi(x) = Ax\, e^{-ax}.$$

2. Use the WKB approximation to find the allowed energies for the triangular potential well.

3. Compare the results obtained in questions (a) and (b). Which method provides the lowest bound on the ground state energy?

4. Which value for the ground state energy do you think is the most accurate? No new calculations are necessary here, just a physical argument.

Problem 126. *by Morten Eskildsen* ID:QM-U-1466

A hydrogen atom is placed in a time-dependent homogeneous electric field parallel to the z-axis

$$\mathbf{E}(t) = E_0 \frac{\tau^2}{t^2 + \tau^2} \hat{k},$$

where E_0 and τ are constants. Initially ($t = -\infty$) the hydrogen atom is in the ground state.

1. What is the perturbation Hamiltonian, $H'(t)$, due to the electric field?

2. Use selection rules to find the possible final states to which the hydrogen atom can be excited by the electric field. Specify these states by their quantum numbers n, l and m. *Note: Consider only states which can be excited directly from the ground state, i.e. not through some intermediary state.*

3. Consider the transition from the ground state, $|nlm\rangle = |100\rangle$, to the $|210\rangle$ state, and calculate $\langle 210 \,| H'(t) |\, 100\rangle$. Express your result in terms of e, E_0, t, τ and the Bohr radius, a.

4. What is the characteristic angular frequency, ω_0, associated with the transition to the $|210\rangle$ state? Please give a numerical value.

5. Use first-order perturbation theory to calculate the probability that the hydrogen atom have been excited to the $|210\rangle$ state at $t = \infty$. Express your answer in terms of ω_0 and the constants used in question (c).

We now assume that the hydrogen atom have been excited to the $|322\rangle$ state.

6. What is the probability of stimulated emission if the hydrogen atom is subjected to a sinusiodally oscillating electric field along the z-axis?

7. What is the lifetime of the $|322\rangle$ state? Give your result in seconds.

Problem 127. *by Morten Eskildsen* ID:QM-U-1469

Consider a particle which is scattered by a spherically symmetric potential at sufficiently low energy that only S- and P-wave phase shifts (i.e. δ_0 and δ_1) are nonzero.

1. Show that the differential scattering cross-section has the form

$$D(\theta) = A + B\cos\theta + C\cos^2\theta,$$

and determine A, B and C in terms of the phase shifts.

2. Determine the the total scattering cross-section in terms of A, B and C.

Problem 128. *by Morten Eskildsen* ID:QM-U-1472

Consider a particle which is scattered by the potential

$$V(r) = \frac{C}{r^2},$$

where C is a constant, corresponding to a $1/r^3$ force.

1. Use the Born approximation to find the differential scattering cross-section.

2. Find the total scattering cross section.

3. Is the differential scattering cross-section the same as we would have found classically? Not further calculations are necessary to answer this question, but you must justify your answer.

Problem 129. *by Morten Eskildsen* ID:QM-U-1475

When an atom is placed in a uniform external electric field $\mathbf{E}_{\text{ext}}$, the energy levels are shifted. This phenomenon is known as the Stark effect and is the electric analogue to the magnetic Zeeman effect. In this problem you will analyze the Stark effect for the $n = 1$ and $n = 2$ states of Hydrogen. If we choose the z direction to be along the electric field the perturbation to the Bohr (Hydrogen) Hamiltonian is

$$H'_S = eE_{\text{ext}}z = eE_{\text{ext}}r\,\cos\theta.$$

Spin is irrelevant to this problem and should be ignored. You should also neglect the fine structure.

1. Show that to first order the ground state energy is not affected by the electric field.

The first excited state is four-fold degenerate: $|1\rangle = \psi_{200}, |2\rangle = \psi_{210}, |3\rangle = \psi_{211}, |4\rangle = \psi_{21-1}$.

2. Calculate the matrix elements W_{11}, W_{12}, W_{13} and W_{23}, where $W_{ij} = \langle i|H'_S|j\rangle$. Hint: Look carefully at the integrals in this problem before you do any calculations.

The remaining elements of the matrix $\mathbf{W}$ are zero, except of course the hermitian conjugates of any non-zero elements you found above, such that

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} & W_{13} & 0 \\ W_{21} & 0 & W_{23} & 0 \\ W_{31} & W_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Determine the first-order corrections to the energy. Into how many levels does E_2 split? Sketch how the energy levels evolve with increasing E_{ext} and identify the states for which the energy remains unchanged.

Problem 130. *by Morten Eskildsen* ID:QM-U-1478

In this problem you will consider a particle with mass m in a general power-law potential

$$V(x) = \alpha |x|^{\nu},$$

where ν is a positive integer.

The first three questions of this problem deals with the variational method.

1. Calculate the expectation value $\langle H \rangle$ using the trial wave function

$$\psi(x) = A\, e^{-bx^2}.$$

Note that you are only asked to calculate $\langle H \rangle$ but *not* to minimize it!

2. Show that $\langle H \rangle$ is minimized when

$$b = \frac{1}{2} \left[\frac{2m\alpha\nu}{\sqrt{\pi}\hbar^2}\, \Gamma\left(\frac{\nu+1}{2}\right) \right]^{\frac{2}{\nu+2}}$$

3. Find the minimum value of $\langle H \rangle$ for the case $\nu = 2$. Comment on the result.

The remaining questions of this problem should be completed using the WKB approximation.

4. Find expressions for the turning points as well as the momentum $p(x)$, for a particle with energy E.

5. Show that the allowed energies are given by

$$E_n = \alpha \left[(n - 1/2)\hbar \sqrt{\frac{\pi}{2m\alpha}}\, \frac{\Gamma\left(\frac{1}{\nu} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu} + 1\right)} \right]^{\frac{2\nu}{\nu+2}}.$$

Here you may have to use the Gamma function relation $\Gamma(1/n) = n\,\Gamma((1/n)+1)$.

6. Simplify the result above for the case $\nu = 2$. Comment on the result.

Problem 131. *by Morten Eskildsen* ID:QM-U-1481

With modern technology it has become possible to fabricate materials where electrons are confined to two, one and even zero dimensions. In this problem you will consider the 2-dimensional case, with $\sigma = Nq/A$ being the number of free electrons per unit area.

1. Calculate the Fermi energy for the two-dimensional free electron gas.

2. Find the average energy per free electron and express your result as a fraction of the Fermi energy.

Problem 132. *by Morten Eskildsen* ID:QM-U-1484

Consider a neutral spin-$\frac{1}{2}$ at rest in a static magnetic field of magnitude B_0 applied along the z-axis. The magnetic dipole moment of the particle is given by $\boldsymbol{\mu} = \gamma\,\mathbf{S}$, where γ is the relevant gyromagnetic ratio.

At $t = 0$, an additional field of magnitude B' is applied along the x-axis.

1. Construct the 2×2 Hamiltonian matrix for this system for $t > 0$. Specify how the Hamiltonian can be separated into matrices $\mathbf{H}^0$ and $\mathbf{H}'$ corresponding to B_0 and B' respectively.

2. If we write the spinor (wave function) at time $t > 0$ as

$$\chi(t) = \begin{pmatrix} c_a(t)\, e^{i\omega_0 t/2} \\ c_b(t)\, e^{-i\omega_0 t/2} \end{pmatrix},$$

show that

$$\begin{aligned} \dot{c}_a &= \frac{i\omega'}{2}\, e^{-i\omega_0 t}\, c_b \\ \dot{c}_b &= \frac{i\omega'}{2}\, e^{i\omega_0 t}\, c_a. \end{aligned}$$

Give expressions for ω_0 and ω'.

3. Assume that the system initially is in the spin-up state: $c_a(t \le 0) = 1$; $c_b(t \le 0) = 0$. Find the probability to first order that the system is in the spin-down state at time $t > 0$. What is the necessary condition for this result to be valid?

4. Find exact solutions to the differential equations in (b) with $B_0 = 0$. Under which condition does the result from (c) in the limit $B_0 \to 0$ agree with the exact solution?

Problem 133. *by Morten Eskildsen* ID:QM-U-1487

In this problem you will use partial wave analysis to calculate the low-energy scattering from a spherical well potential:

$$V(r) = \begin{cases} -V_0 & r \le a \\ 0 & r > a. \end{cases}$$

1. Combine the expressions for the partial wave function for $r > a$

$$\psi^{(l)}(r,\theta) = A\, i^l (2l+1) \left[j_l(kr) + ik\, a_l\, h_l^{(1)}(kr) \right] P_l(\cos\theta)$$

and the relation between the partial wave amplitude and the phase shift

$$a_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{k}\, e^{i\delta_l} \sin(\delta_l),$$

to show

$$\psi^{(l)}(r,\theta) = B_l \left[j_l(kr) - \tan(\delta_l)\, n_l(kr) \right] P_l(\cos\theta).$$

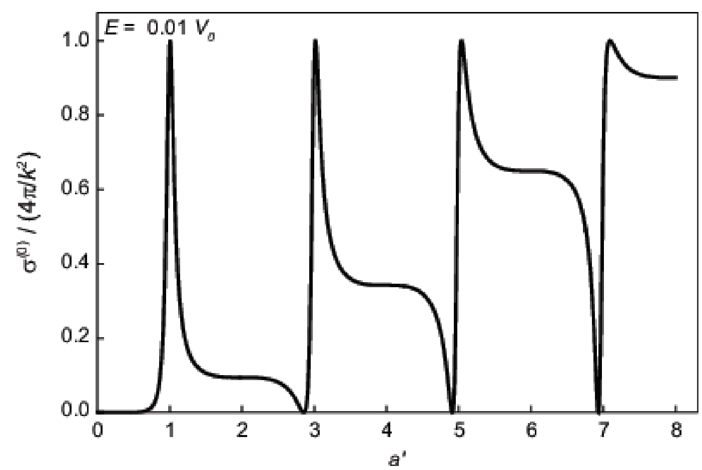
Give an expression for B_l in terms of A and δ_l.

2. Show that the S-wave ($l = 0$) partial wave phase shift for the spherical well potential is given by

$$\tan(\delta_0) = \frac{k\,\tan(Ka) - K\,\tan(ka)}{K + k\,\tan(ka)\,\tan(Ka)},$$

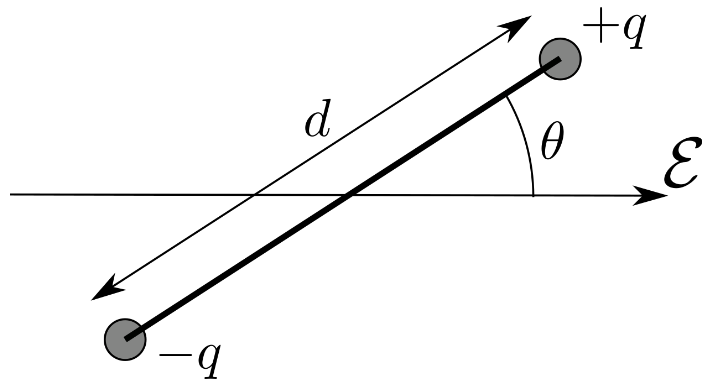
and provide expressions for k and K.

3. The figure below shows the total S-wave scattering cross-section for $E = 0.01V_0$ as a function of the dimensionless well radius $a' = a/(\pi\hbar/\sqrt{8mV_0})$. What is your best *guess* at the mechanism responsible for the resonances occurring at odd a'? Do not calculate anything unless you have already finished all other questions on the exam.



Problem 134. *by Morten Eskildsen* ID:QM-U-1490

In this problem you will use perturbation theory to calculate the polarisability (α) in 2 and 3 dimensions, of a rigid rotator with dipole moment $\boldsymbol{d}$ and subjected to an electric field $\mathcal{E}$. The rotator consists of two masses carrying opposite charges, separated by a fixed distance and free to rotate around the center of mass as shown in the figure below (you can think of the rotator as a simple example of a polarisable molecule).



The unperturbed Hamiltonian is given by $\hat{H}^0 = \hat{L}^2/(2I)$, where $\hat{L} = (\hbar/i)\,\partial/\partial\theta$ is the angular momentum operator and I is the moment of inertia for the rotator. The perturbation is given by $H' = -\boldsymbol{d}\cdot\mathcal{E} = -d\mathcal{E}\cos\theta$.
Begin by considering the two-dimensional case.

1. Show that
$$\psi_l^0(\theta) = \frac{1}{\sqrt{2\pi}}\,e^{il\theta} \qquad \text{with } l = 0, \pm1, \pm2, \ldots$$
are normalized eigenstates for the unperturbed Hamiltonian with energies
$$E_l^0 = \frac{(l\hbar)^2}{2I}.$$
What is the degeneracy of the ground state? Of the excited states?

2. Using *non-degenerate* perturbation theory, calculate the first-order correction to the energy eigenvalues (E^1).

3. Using *non-degenerate* perturbation theory, calculate the second-order correction to the energy eigenvalues (E^2). Why is it possible to use non-degenerate perturbation theory for this and the previous question?

4. Use the results from (b) and (c) and the definition of the polarisability, $E^1 + E^2 = -(1/2)\,\alpha\mathcal{E}^2$ to show that
$$\alpha_l = -\frac{2Id^2}{\hbar^2(4l^2-1)}.$$
The polarisability is a measure of the tendency for the dipole to be aligned parallel ($\alpha > 0$) or antiparallel ($\alpha < 0$) to the electric field. Give a physical explanation for why α is positive for $l = 0$ but negative for $l \geq 2$. Hint: Think of this in classical terms and compare fast and slow rotation.

Now consider the three-dimensional case. Here the normalized eigenstates for the unperturbed Hamiltonian are given by the spherical harmonics $Y_l^m(\theta,\phi)$ with eigenenergies $E_l^0 = l(l+1)\hbar^2/(2I)$.

5. Use non-degenerate perturbation theory to calculate the polarisability for the lm-state and show that

$$\alpha_{lm} = \frac{2Id^2}{\hbar^2}\left[\frac{(l+1)^2 - m^2}{(2l+3)(2l+1)(l+1)} - \frac{l^2 - m^2}{(2l+1)(2l-1)l}\right].$$

Don't worry about the $l = 0$ case. It can either be worked out separately or found by setting $m = 0$ and letting $l \to 0$. Just like in the two-dimensional case the polarisability can be either negative or positive depending on the choice of l and m (compare for instance the case with $l = 1$ and m equal to either 0 or ± 1).

In this problem the following property of the spherical harmonics might be helpful:

$$\cos\theta\, Y_l^m = \sqrt{\frac{(l+1)^2 - m^2}{(2l+3)(2l+1)}}\, Y_{l+1}^m + \sqrt{\frac{l^2 - m^2}{4l^2 - 1}}\, Y_{l-1}^m.$$

Problem 135. *by Morten Eskildsen* ID:QM-U-1493

Consider a particle in a harmonic oscillator potential $V(x) = (1/2)m\omega^2 x^2$ and use the variational method to answer the following questions.

1. Find the best bound on the ground state energy of the harmonic oscillator using a trial wave function given by

$$\psi_0(x) = \begin{cases} A(c^2 - x^2) & |x| < c \\ 0 & |x| \geq c \end{cases}$$

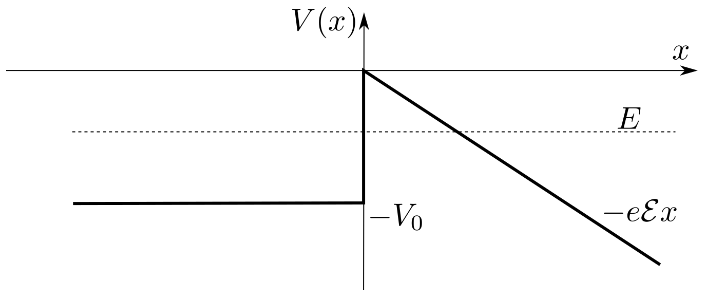
Express your results in term of the exact ground state energy. By how many percent does it exceed the exact value?

2. Suggest a trial wave function which would be suitable for finding a bound on the first excited state of the harmonic oscillator. Make sure you argue why this is valid trial function.

3. Calculate the best bound on the energy of the first excited state of the harmonic oscillator using a the trial wave function from (b). Express your results in term of the exact result.

Problem 136. *by Morten Eskildsen* ID:QM-U-1496

In this problem we consider the transmission of electrons through a metal surface in the presence of an electric field of magnitude $\mathcal{E}$. In the figure below $x < 0$ corresponds to the metal.



1. Find the effective barrier thickness x_0 as a function of the electron energy E $(-V_0 < E < 0)$.

2. Use the WKB approximation to find the transmission probability T for an electron with energy E and show that

$$T = \exp\left[-\frac{\sqrt{32m|E|^3}}{3e\mathcal{E}\hbar}\right].$$

3. Calculate the characteristic inverse length $\alpha = (2m|V'(x_0)|/\hbar^2)^{1/3}$ for this problem.

4. The WKB expression for the transmission probability is valid in the limit $T \ll 1$. Show that the this corresponds to having $x_0 \gg \alpha^{-1}$.

Problem 137. *by Morten Eskildsen* ID:QM-U-1499

Lithium is a metal with a Fermi energy measured to be $E_F = 4.72$ eV. The density of lithium is 0.53 g/cm^3 its atomic weight is 6.941.

1. Based on the electron configuration for lithium, how many valence electrons do you *think* each Li atom contributes.

2. Use the free electron model to calculate the number of valence (conduction) electrons per unit volume in lithium.

3. Calculate the number of free electrons per lithium atom and compare to your answer to (a).

Problem 138. *by Morten Eskildsen* ID:QM-U-1502

Consider a particle with mass m and charge q in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. Initially the particle is in the first excited state ($n = 1$). At time $t = 0$ an electric field $\mathcal{E}$ is applied in the positive x-direction. At time $t = T$ the electric field is switched off.

1. Consider the electric field as a perturbation and specify H'.

2. What is the probability, to first order, that the system is in the ground state ($n = 0$) after the electric field is switched off? Consider the harmonic oscillator as a two-level system, consisting only of the ground state and the first excited state. What is the shortest time T for which $P_{1\to 0}$ reaches its maximum value?

3. What is the probability, to first order, that the system is in the second excited state ($n = 2$) after the electric field is switched off? Consider the harmonic oscillator as a two-level system, consisting only of the first and second excited states.

4. Generalize the time-dependent perturbation theory to non-degenerate multilevel systems where
$$\Psi(t) = \sum_n c_n(t)\, \psi_n\, e^{-iE_n t/\hbar},$$
and show that
$$\dot{c}_m = -\frac{i}{\hbar} \sum_n c_n\, H'_{mn}\, e^{i(E_m - E_n)t/\hbar}.$$
Here $H_0\psi_n = E_n\psi_n$ with $\langle\psi_n|\psi_m\rangle = \delta_{nm}$ and $H'_{nm} \equiv \langle\psi_m|H'|\psi_n\rangle$. Does this generalization justify the approach in questions (b) and (c) where we considered adjacent states as two-level systems? Which order of time-dependent perturbation theory is required to obtain a non-zero probability of exciting the system from the $n = 1$ to the $n = 3$ state (no calculation allowed)?

Problem 139. *by Morten Eskildsen* ID:QM-U-1505

In this problem we will use the first Born approximation to determine the one-dimensional reflection coefficient R for a potential $V(x)$ that vanishes everywhere except in the vicinity of the origin.

1. Show that we can write the solution to the one-dimensional Schrödinger equation in the form
$$\psi(x) = A\, e^{ikx} + \int_{-\infty}^{\infty} G(x - x_0)\, Q(x_0)\, dx_0$$
where $k = \sqrt{2mE}/\hbar$ and $Q(x) = (2m/\hbar^2)\, V(x)\, \psi(x)$ and the Green's function satisfies
$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) G(x) = \delta(x).$$

2. Show that
$$g(s) = \frac{1}{\sqrt{2\pi}(k^2 - s^2)},$$
where $g(s)$ is the Fourier transform of the Green's function defined by
$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, g(s)\, ds.$$

Use Cauchy's integral formula to evaluate the integral and show

$$G(x) = \begin{cases} \frac{1}{2ik}\, e^{ikx}, & x > 0 \\ \frac{1}{2ik}\, e^{-ikx}, & x < 0 \end{cases}$$

3. In the first Born approximation we set

$$Q(x) = \frac{2mV(x)}{\hbar^2}\, A\, e^{ikx}.$$

Show that this leads to

$$\psi(x) \xrightarrow[x\to-\infty]{} A\, e^{ikx} + A\, \frac{m}{i\hbar^2 k}\, e^{-ikx} \int_{-\infty}^{\infty} V(x_0)\, e^{2ikx_0}\, dx_0$$

and consequently

$$R = \left| \frac{m}{\hbar^2 k} \int_{-\infty}^{\infty} V(x_0)\, e^{2ikx_0}\, dx_0 \right|^2 .$$

4. Use the result from the third question to calculate the reflection coefficient for a square barrier:

$$V(x) = \begin{cases} V_0 & 0 < x < a \\ 0 & \text{elsewhere} \end{cases} .$$

Compare this to the exact result for the transmission coefficient $T = 1 - R$

$$T = \frac{1}{1 + [V_0^2/4E(E - V_0)]\, \sin^2 \sqrt{(2m/\hbar^2)(E - V_0)}a}$$

in the limit $E \gg V_0$. Comment on the result.

Problem 140. *by Morten Eskildsen* ID:QM-U-1508

In this problem you will consider and compare two different perturbations to the simple harmonic oscillator Hamiltonian,

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2.$$

Consider first the perturbation $\hat{H}_1 = b_1\, (4m^2\omega^3/\hbar)\, \hat{x}^4$, where $b_1 \ll 1$.

1. What is the order of the lowest, non-vanishing correction to the energy? In other words, which is the smallest i for which $E_n^{(i)} \neq 0$?

2. Show that the lowest order, non-vanishing correction to the energy is given by

$$E_n^{(i)} = 3b_1(1 + 2n + 2n^2)\hbar\omega.$$

3. Will the above result be valid for for all values of n? Provide a physical justification for your answer.

Consider now the perturbation $\hat{H}_2 = b_2\sqrt{2m\hbar\omega^3}\,\hat{x}$, where $b_2 \ll 1$.

4. Find the lowest order, non-vanishing correction to the energy.
5. Will the above result be valid for for all values of n? Provide a physical justification for your answer.

Problem 141. *by Morten Eskildsen* ID:QM-U-1511

In this problem you will consider a particle with mass m in two-dimensional linear potential given by

$$V(x) = \beta\,|x|\,|y|.$$

Such a situation could for example describe an electron at the interface between two semiconductors, and with the in-plane motion confined by an electrostatic potential.

1. Use a Gaussian trial function

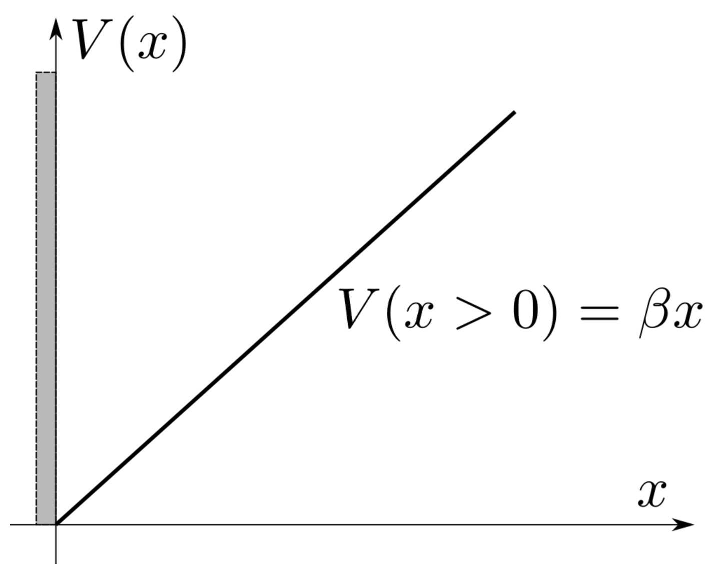
$$\psi(x,y) = Ae^{-b(x^2+y^2)}$$

   to calculate the expectation value for the total energy $\langle H\rangle$.
2. Find the value of the variational parameter b that minimizes $\langle H\rangle$.
3. Show that upper limit on the ground state energy is given by

$$\langle H\rangle_{\min} = \hbar\sqrt{\frac{2\beta}{\pi m}}.$$

Problem 142. *by Morten Eskildsen* ID:QM-U-1514

In this problem you will consider a triangular potential well, as shown above. The potential goes to infinity for $x < 0$, and increases linearly as $V(x) = \beta x$ for $x > 0$.



1. What is the turning point (x_0) for a particle with energy, E?
2. Provide an exact solution to the Schrödinger equation for this potential, expressing your answer in terms of the appropriate Airy function. Make sure you pay special attention the argument to these functions. Do not attempt to normalize $\psi(x)$.

The first zero of the Airy functions $Ai(z)$ and $Bi(z)$ occur for $z = -2.33811$ and $z = -1.17371$ respectively (see Fig. 8.8 in Griffiths).

3. Find an expression for the exact ground state energy for the triangular potential well. Express your answer in units of $(\hbar^2\beta^2/m)^{1/3}$.
4. Compare your result above to what you would find from the WKB approximation
$$\int_0^{x_0} p(x)\,dx = \left(n - \tfrac{1}{4}\right)\pi\hbar.$$
By how many percent does the two results differ?

Besides being a a good model for a two-dimensional electron gas (2DEG) confined close to the interface between two semiconductor materials, the triangular potential also describes the quantum mechanical analogue of the bouncing ball.

e. (5 pts. extra credit) Find the ground state energy for a "bouncing" neutron a gravitational field ($\beta = m_n\, g$). How high off the ground is the neutron, on average?

Problem 143. *by Morten Eskildsen* ID:QM-U-1517

Consider a particle with mass m and charge q in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. The particle is acted on by a time-dependent homogeneous electric field (also along the x-axis)
$$\mathcal{E} = \mathcal{E}_0 \exp[-t^2/\tau^2],$$
where $\mathcal{E}_0$ and τ are constants. At $t = -\infty$ the particle is in the ground state ($n = 0$).

In the first part of this problem we will consider the electric field as a perturbation.

1. Specify H' due to the electric field.
2. Use time-dependent perturbation theory to calculate the probability, to first order, that the system is in the first excited state ($n = 1$) as $t \to \infty$.
3. Show that the transition probability is maximized when $\tau = \sqrt{2}/\omega$.

In the second part of this problem we will treat the same problem using the adiabatic theorem.

4. Starting from Eq. (10.19) in Griffiths show that the probability that the system is in the first excited state as $t \to \infty$ is given by

$$P_{0\to 1} = \left| \int_{-\infty}^{\infty} \frac{\langle \psi_1(t)|\dot{H}|\psi_0(t)\rangle}{E_0(t) - E_1(t)} \; \exp\left[-\frac{i}{\hbar} \int_0^t (E_0(t') - E_1(t'))\, dt' \right] dt \right|^2,$$

where the dynamic phase has been set to zero at $t = 0$. Here $\psi_n(t)$ are *instantaneous* wave functions and $E_n(t)$ are the corresponding *instantaneous* energies.

5. Show that
$$\dot{H} = \frac{2tqx}{\tau^2}\, \mathcal{E}_0 \, \exp[-t^2/\tau^2],$$
and calculate
$$\langle \psi_1^0|\dot{H}|\psi_0^0\rangle.$$
Here ψ_n^0 are wave functions for the unperturbed system.
It is possible to show that $\langle \psi_1(t)|\dot{H}|\psi_0(t)\rangle = \langle \psi_1^0|\dot{H}|\psi_0^0\rangle$ but you can take that as a given.

6. Show that the full Hamiltonian can be written as
$$H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2[x - a(t)]^2 - \frac{1}{2}m\omega^2[a(t)]^2,$$
and provide an expression for the time dependent length $a(t)$ in terms of the constants in the problem (q, ω, m, $\mathcal{E}_0$ and τ).
Specify the *instantaneous* energies, $E_n(t)$ (no calculation required), and show that $E_1(t) - E_0(t) = \hbar\omega$.

7. Combine the results from questions (d) to (f) and calculate $P_{0\to 1}$.
Hint: Use partial integration.
Compare this with your result from question (b).

8. Prove that $\langle \psi_1(t)|\dot{H}|\psi_0(t)\rangle = \langle \psi_1^0|\dot{H}|\psi_0^0\rangle$.

Problem 144. *by Morten Eskildsen* ID:QM-U-1520

Here we consider the so-called "polarization" potential
$$V(r) = \frac{V_0}{(r^2 + d^2)^2},$$
where V_0 and d are constants.

1. Use the first Born approximation to calculate the scattering amplitude $f(\theta)$ for the "polarization" potential. Express you answer in terms of the magnitude of the wave vector transfer (κ), V_0, d, and any necessary fundamental constants.
Hint: This question is most easily solved by partial integration.

2. Prove the useful relation between the wave vector, wave vector transfer and scattering angle:
$$k^2 \sin\theta \, d\theta = \kappa \, d\kappa.$$

3. Show that the total scattering cross-section is given by
$$\sigma = \frac{A}{k^2}\left[1 - (4kd+1)\,e^{-4kd}\right],$$
and give an expression for the constant A.

4. Show that for large energies, E, the total scattering cross-section, $\sigma \propto E^{-1}$.

5. Show that for any spherically symmetric potential, the first Born approximation will always yield $\sigma \propto E^{-1}$ at high energies.

2 Graduate level

Problem 145. *1983-Fall-QM-G-4* ID:QM-G-2

A beam of neutrons (mass $m_N = 940$ MeV/c^2, charge $= 0$, spin $= \hbar/2$, magnetic moment $\mu_N = -0.90\ e\hbar/m_N c$) is produced at an accelerator. The neutrons move in the y-direction with a kinetic energy of 10 MeV. A Stern-Gerlach apparatus is used to select those particles whose spin projection along the z-axis is $\hbar/2$. A solenoid magnet produces a uniform magnetic field go along the y-axis, of magnitude 12000 Gauss = 1.2 Tesla, and stretching over 1 meter.

1. What will be the direction of the spin as it emerges from the solenoid?
2. What values of the x-projection of the spin may be measured, and with what probabilities?

Problem 146. *1983-Fall-QM-G-5* ID:QM-G-5

A particle of mass m, moving in one dimension, is trapped in the region $-a < x < a$ by an infinite square well potential. It also sees a positive delta function potential of strength q at the origin.

1. Find all energy eigenstates of this particle.
2. Determine the eigenenergies as explicitly as possible, but you should switch to a graphical method when you encounter an implicit equation which you cannot solve explicitly. Use this method to estimate the ground state energy for the case $\frac{\hbar^2}{mqa} = 1$ to one significant digit of accuracy in units of $\frac{\hbar^2\pi^2}{2mq^2}$.
3. Obtain the proper normalization for your eigenfunctions.
4. Indicate qualitatively what you must do if at $t = 0$ it is given that $\psi(x,0) = \frac{1}{\sqrt{a}}$ for $0 < x < a$, and 0 for $-a < x < 0$, and you are asked to find the probability of finding the particle in the left half of the well at any later time t.

Problem 147. *1984-Fall-QM-G-4* ID:QM-G-8

Consider a one-dimensional harmonic oscillator ($V(x) = \frac{kx^2}{2}$). Recall that

$$\int_{-\infty}^{\infty} U_n^* x U_m dx = \begin{cases} \left(\frac{\hbar}{m\omega}\frac{n+1}{2}\right)^{1/2}, & \text{if } m = n+1; \\ \left(\frac{\hbar}{m\omega}\frac{n}{2}\right)^{1/2}, & \text{if } m = n-1; \\ 0, & \text{otherwise;} \end{cases}$$

where U_n are energy eigenfunctions with eigenvalues $E_n = \hbar\omega(n + 1/2)$ and $\omega = \sqrt{k/m}$.

1. Prove that $\langle V\rangle_n = \frac{1}{2}E_n$, where $\langle V\rangle_n = \int_{-\infty}^{\infty} U_n^* V U_n dx$.

2. Given that $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ compute $\Delta x \Delta p$ for the harmonic oscillator ground state. State the precise inequality involving Δx and Δp to which this result can be compared.

Problem 148. *1984-Fall-QM-G-5* ID:QM-G-11

A spinless particle of mass m and charge q moves in a one-dimensional harmonic oscillator potential

$$V(x) = \frac{1}{2} m\omega^2 x^2,$$

The eigenenergies and normalized eigenvectors are

$$E_n = \hbar\omega(n + 1/2), \qquad |\Psi_n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |\Psi_0\rangle,$$

where

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\beta x + \frac{i}{\hbar\beta}\hat{p}\right), \qquad \beta = \sqrt{m\omega/\hbar}$$

and $|\Psi_0\rangle$ is the ground state eigenvector which satisfies $\hat{a}|\Psi_0\rangle = 0$. At time $t = 0$, the particle is in the ground state $|\Psi_0\rangle$. The particle is then put in a uniform electric field ϵ with strength given by $\epsilon = \sqrt{\frac{3}{2}} \frac{\hbar\omega\beta}{q}$. The motion of the particle is assumed to be restricted to the subspace spanned by $\{|\Psi_0\rangle, |\Psi_1\rangle\}$. Find the state vector of this particle at time t and the probability that the particle remains in the ground state.

Problem 149. *1984-Fall-QM-G-6* ID:QM-G-14

Consider a spin 1/2 particle of magnetic moment $\vec{m} = \gamma\vec{s}$. The spin state space is spanned by the basis of the $|+\rangle$ and $|-\rangle$ vectors, eigenvectors of $\hat{S}_z$ with eigenvalues $\hbar/2$ and $-\hbar/2$. In this basis the components of $\hat{\vec{s}}$ have the following representations

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At time $t = 0$, the particle is in a statistical mixture of states with probabilities in the states $|+\rangle$ and $|-\rangle$ being 1/3 and 2/3 respectively. A magnetic field B_0 is introduced along the x-direction. Determine the density matrix: $\rho(t)$ of the particle in the basis $\{|\pm\rangle\}$ at the time t. At this instant, S_y, is measured; find the mean value of the results.

Hint: Recall that $i\hbar\frac{d}{dt}\rho(t) = [\hat{H}, \rho(t)]$ and solve the resulting 4 equations to find the 4 elements of $\rho(t)$.

Problem 150. *1984-Spring-QM-G-4* ID:QM-G-17

In a certain 3-dimensional representation with basis vectors $|1\rangle$, $|2\rangle$, and $|3\rangle$, the physical quantities A and B are represented y the following matrices:

$$M_A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad M_B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

1. If the system is in the lowest eigenstate of B, what is the expectation value if A is measured?

2. If A and B are simultaneously measurable for a certain state $|\Psi\rangle$, what must be this state, and what must be the measured values for A and B?

3. Suppose that (1) the Hamiltonian of the system, $\hat{H}$, is equal to $\frac{h}{\tau}M_B$ for all $t \leq 0$, and the system is in the ground state of $\hat{H}$ up until $t = 0$, then (2) $\hat{H}$ is suddenly changed to $\frac{h}{\tau}M_A$, which lasts until $t = \pi\tau/2$, and then (3) $\hat{H}$ is changed back to $\frac{h}{\tau}M_B$ again. What is the probability that the system remains in the ground state of $\hat{H}$?

[Note that the Hamiltonian is essentially just (1) M_B, (2) M_A, and (3) M_B again. The factors of Planck's constant, H, and the characteristic time τ, are added only to make $\hat{H}$ have the correct units of energy.]

Hint: The eigenvalues for M_A and M_B are $(0, 0, 3)$ and $(0, 1, 2)$, respectively.

Problem 151. *1984-Spring-QM-G-5* ID:QM-G-20

The spin of an electron points in the $\hat{n}$ direction, where the unit vector $\hat{n}$ has standard spherical angles θ and ϕ with respect to the fixed coordinate system x, y, z. Write the spinor wavefunction for this state as a linear combination of spin up and spin down states (along $\hat{z}$ and opposite to $\hat{z}$ direction respectively). To what extent is this wavefunction unique?

Problem 152. *1984-Spring-QM-G-6* ID:QM-G-23

An electron moves in a potential which (like that in the hydrogen atom) has both a central part and a spin-orbit part (proportional to $\vec{L} \cdot \vec{S}$).

1. Name a complete set of commuting observables (C.S.C.0.), one of which is the Hamiltonian.

2. Define a basis of eigenstates of the C.S.C.0. What are the eigenvalues of each observable, for each state? Assume there is exactly one bound state. plus a continuum of all positive energies.

3. Express the orthonormality and closure relations for your answers to the previous part.

4. Name a complete set of commuting observables for the electron, one of which is L_z, the component of the orbital angular momentum along the z-axis.

5. Define a basis of eigenstates of the C.S.C.0. in previous part. What are the eigenvalues of each observable, for each state?

6. Express the orthonormality and closure relations for your basis of the previous part.

7. Describe, as completely as you can, the basis transformation between your answers to the second and the fifth. Give the matrix elements in terms of standard functions and tabulated coefficients, to the extent possible without knowing more about the Hamiltonian.

Problem 153. *1985-Fall-QM-G-4.jpg* ID:QM-G-26

For spin 1/2 particles

$$\hat{S}^2|1/2,m\rangle = \frac{3}{4}\hbar^2|1/2,m\rangle, \qquad \hat{S}_z|1/2,m\rangle = m\hbar|1/2,m\rangle$$

and

$$\hat{S}_x + i\hat{S}_y|1/2,1/2\rangle = 0, \qquad \hat{S}_x - i\hat{S}_y|1/2,1/2\rangle = \hbar|1/2,-1/2\rangle.$$

1. Find the matrix representation of σ_x, σ_y and σ_z where $\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$ .In other words find the matrix elements

$$\begin{array}{ll} \langle 1/2,1/2|\sigma_x|1/2,1/2\rangle, & \langle 1/2,1/2|\sigma_x|1/2,-1/2\rangle, \\ \langle 1/2,-1/2|\sigma_x|1/2,1/2\rangle, & \langle 1/2,-1/2|\sigma_x|1/2,-1/2\rangle, \end{array}$$

   etc.

2. Show that it is impossible for a spin 1/2 particle to be in a state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ such that

$$\langle\hat{S}_x\rangle = \langle\hat{S}_y\rangle = \langle\hat{S}_z\rangle = 0.$$

3. Consider a spin 1/2 particle prepared with its spin along the direction $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$ for which $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ satisfies

$$\langle\hat{\vec{S}}\cdot\hat{n}\rangle = \frac{\hbar}{2}.$$

   If a spin analyzer is placed along the z-axis what is the relative probability of finding spin up $(+z)$ compared to spin down $(-z)$?

Problem 154. *1985-Fall-QM-G-5* ID:QM-G-29

The Laplacian in cylindrical coordinates is

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

1. Consider two identical particles of mass M which are constrained to move on a surface (1.e., in two dimensions) and are attracted by a central force $V(\vec{r}_1,\vec{r}_2) = V(\vec{r}_1-\vec{r}_2) = V(\vec{\rho}) = V(\rho)$, where $\rho = (x^2+y^2)^{1/2}$ is the relative coordinate. Write the Schrödinger equation in terms of the relative coordinates and center of mass coordinates for this $2D$ problem. The wave function can be written in

terms of $\psi = \psi(X, Y, \rho, \theta)$, where X and Y are rectilinear coordinates for the center of mass and ρ and θ are the polar coordinates for the relative motion. Show that the wave function is separable, i.e.,

$$\psi(X, Y, \rho, \theta) = U_x(X)U_y(Y)R(\rho)\Phi(\theta)$$

and solve for U_x, U_y and $\Phi(\theta)$ and find the equation for $R(\rho)$.

2. Assume the particles are spin 1/2 fermions that can be in either the symmetric triplet spin state or the antisymmetric singlet spin state. Represent the spin wave functions for each of these four possibilities. If the eigenfunctions of the relative motion part of the Hamiltonian are also chosen to be eigenfunctions of $\hat{L}_z = -i\hbar\frac{\partial}{\partial\theta}$, what are the allowed values of orbital angular momentum, $\hat{L}_z$, for particles in the triplet spin state? For the singlet spin state. Why?

3. Verify that the Laplacian given above is in fact correct.

Problem 155. *1985-Spring-QM-G-4* ID:QM-G-32

In three dimensions, a spin 1/2 particle with mass m moves in the potential

$$V(x) = \begin{cases} V_0\vec{\sigma}\cdot\hat{z}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}.$$

Take the σ matrices to be

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

1. A beam of particles comes from $-\infty$ in the positive x-direction with energy $E > V_0 > 0$ and in an eigenstate of σ_x with eigenvalue 1. Write down the general wave function of such an incoming beam.

2. Write down the general solutions for the transmitted and reflected beams.

3. What are the boundary conditions at $x = 0$? Write down the equations which must be satisfied by the general amplitudes appearing in the two previous parts.

4. Solve these equations.

Problem 156. *1985-Spring-QM-G-5* ID:QM-G-35

Two distinguishable, spinless partirles. each of mass m move in one dimension in the potential $V(x_1, x_2) = U(x_1) + U(x_2)$, where

$$U(x) = \frac{1}{2}kx^2.$$

At time $t = 0$, the total energy of the two particles is measured and found to be $2h\sqrt{k/m}$. At time $t = 1$ sec, the energy E_1 of particle 1 is measured.

1. What values of E_1 may be found, and with what probabilities?
2. Suppose the result of the measurement is the lowest of your answers to the previous part. At a later time, the position of particle 2 is measured. What values may be found? What is the most likely result of this position measurement? The least likely?

Problem 157. *1985-Spring-QM-G-6.jpg* ID:QM-G-38

Consider a hydrogen atom governed by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}, \qquad r = \sqrt{x^2 + y^2 + z^2}.$$

If one includes the spin of the electron but ignores any other perturbations, then the $n = 2$ excited energy level is eight-fold degenerate: Three $l = 1$ states and one $l = 0$ state with electron spin up or down for each.

Consider now the following four perturbations $\hat{H}'$

i. $\beta \hat{L}_z/\hbar$;

ii. $\beta r^2/a_0^2$;

iii. $\beta \hat{\vec{L}} \cdot \hat{\vec{S}}/\hbar$;

iv. $\beta \hat{L}^2/\hbar$;

where $\hat{\vec{L}}$ and $\hat{\vec{S}}$ are the angular momentum and the electron spin operators, respectively, a_0 is the Bohr radius, and $\beta > 0$ is a parameter with units of energy that describes the strength of the perturbation; it is independent of spatial variables.

Calculate the change in energy of the atom to first order in β, show how the degeneracy is broken and give the remaining degeneracy of each level.

You will need the following functions:

$$Y_{l,m}: \quad Y_{0,0} = \left(\frac{1}{4\pi}\right)^{1/2}, \qquad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta, \qquad Y_{1,\pm 1} = \pm\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi},$$

and

$$R_{n,l}: \quad R_{2,0} = \left(\frac{1}{2a_0}\right)^{1/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}, \qquad R_{2,1} = \left(\frac{1}{2a_0}\right)^{2} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0}.$$

You will also need $\int_0^\infty x^n e^{-x} dx = n!$.

Problem 158. *1985-Spring-QM-G-7* ID:QM-G-41

A spinless particle of mass m moves non-relativistically in 3 dimensions in the potential

$$V(\vec{r}) = \alpha\delta(|\vec{r}| - a), \qquad \alpha < 0.$$

Find the energy E_0 and the wave function of the ground state. Under what conditions does one find a bound state?

Hint: For singular potentials, the derivative of the wave function is not necessarily continuous.

Problem 159. *1987-Fall-QM-G-4* ID:QM-G-44

The ground state hydrogenic wave function for an electron (charge q) bound around a nucleus of charge Z is

$$\psi = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}, \qquad a_0 = \frac{\hbar}{mq^2}$$

and the zrouud state energy is

$$E_0 = -\frac{Z^2 e^2}{2a_0}$$

Suppose an electron is in the ground state of a nucleus of $Z = 2$.

1. Find the expectation value for the kinetic energy T and the potential energy V of the electron for the case $Z = 2$. Check that the sum $\langle T\rangle + \langle V\rangle$ is correct.

2. Suppose the nuclear charge is instantaneously changed from $Z = 2$ to $Z = 1$, for example by radioactive decay. What is the wave function of the electron immediately after this transformation? Find the new values of $\langle T\rangle$ and $\langle V\rangle$ and their sum immediately after the transformation.

3. Calculate the probability that the electron is in the ground state of the new atom ($Z = 1$) after the transformation. (Neglect any radiative transitions.)

4. Is there a finite probability that the electron will escape from the atom? Explain.

Hint:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

$$\vec{\nabla} f = \vec{i}_r \frac{\partial f}{\partial r} + \vec{i}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{i}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}.$$

Problem 160. *1987-Fall-QM-G-5* ID:QM-G-47

A particle of mass m moves in an infinite spherically symmetric potential well

$$V(r) = \begin{cases} 0, & \text{if } r \le a; \\ \infty, & \text{if } r > a. \end{cases}$$

The Hamiltonian in spherical coordinates has the form

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}r + \frac{\hat{L}^2}{2mr^2} + V(r)$$

with

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right].$$

1. Show that $[\hat{H}, \hat{\vec{L}}] = 0$ follows from the invariance of the Hamiltonian under rotations. $\hat{\vec{L}}$ is the angular momentum operator.
2. Let $l(l+1)\hbar^2$ and $m\hbar$ be the eigenvalues of the operators $\hat{L}^2$ and $\hat{L}_z$ respectively. Denote their common eigenstates by $|l, m\rangle$, prove that
$$\hat{L}_\pm|l, m\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}|l, m \pm 1\rangle,$$
where
$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y.$$
3. Find all the allowed energy values and the corresponding normalized wave functions for the s-wave ($l = 0$). Note:
$$\int \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\cos(x)).$$

Problem 161. *1987-Fall-QM-G-6* ID:QM-G-50

A particle obeys the one-dimensional Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + \frac{2m}{\hbar^2}V(x)\right]\psi(x) = k^2\psi(x)$$

Let $V_0(x)$ be non-zero only in the interval $-a < x < a$. It is symmetric so that $V_0(-x) = V_0(x)$. Reflection and transmission amplitudes (r, t respectively) for a particle incident on V_0 from the left are defined by requiring that

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx}, & \text{if } x < -a; \\ te^{ikx}, & \text{if } x > a. \end{cases}$$

is a solution of the Schrödinger equation for $V = V_0$. (Do not concern yourself with the value of ψ in the interval $-a < x < a$.)

1. Find the solution of the Schrödinger equation with the same $V - V_0$ describing stuttering of a particle incident from the right.

2. Find the solution of the Schrödinger equation for $V = V_0(x - L)$ describing the scattering of a particle from the left. Note that the potential $V_0(x - L)$ has the same magnitude and strength as $V_0(x)$, but it is shifted by a distance L.

3. Suppose $V(x) = V_0(x) + V_0(x - L)$ and a particle is incident from the left. By considering scattering from the two potentials separately find the transmission amplitude T defined by

$$\psi = Te^{ikx}, \quad \text{for } x > L + a.$$

Problem 162. *1988-Fall-QM-G-4* ID:QM-G-53

Consider the two—dimensional harmonic oscillator. Its eigenkets and the corresponding energies are given by $|n_x, n_y\rangle$ and $E_{n_x,n_y} = (n_x + n_y + 1)\hbar\omega$, respectively with n_x, n_y non-negative integers. We now apply a perturbation

$$V = gxy.$$

1. Using the time-independent perturbation theory, find the lowest order nonvanishing contribution to the energy of the ground state and the corresponding eigenket.

2. Repeat the first part for the first excited state.

Hint: You may use $\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}\right]$.

Problem 163. *1988-Fall-QM-G-5* ID:QM-G-56

A particle of mass m moves from the left towards a one-dimensional potential

$$V(x) = g\delta(x), \qquad g > 0.$$

Calculate the transmission and reflection coefficients.

Hint: To determine the change of the derivative of the wavefunction at $x = 0$, you need to integrate the Schrödinger equation from $-\epsilon$ to ϵ, where ϵ is infinitesimally small.

Problem 164. *1989-Fall-QM-G-4* ID:QM-G-59

Consider a spin 1/2 particle in a static magnetic field oriented along the x axis $\vec{B} = B_0\hat{x}$ described by the Hamiltonian $\hat{H}(t < 0) = -\mu\hat{S}_x B_0$ with eigenstates $|\alpha\rangle$ and $|\beta\rangle$ corresponding to the S_x, eigenvalues of $+\hbar/2$ and $-\hbar/2$ respectively. The system is initially in the ground state.

1. At $t = 0$ the magnetic field rapidly rotates from the x-axis to lie along the z-axis in time Δt. Now $\hat{H}(t > \Delta t) = -\mu\hat{S}_z B_0$.

(a) Write down an approximate expression for the time dependent wavefunction for $t > 0$ in terms of the basis of the new Hamiltonian $|\uparrow\rangle$, $|\downarrow\rangle$.

(b) How rapidly must this rotation occur in order for your approximation to be valid? **Hint:** Relate the time Δt to a characteristic time of the system.

2. If instead at time $t = 0$ the magnetic field slowly rotates from the x-axis to lie along the z-axis in time Δt ($\hat{H}(t > \Delta t) = -\mu \hat{S}_z B_0$).

   (a) Write down an approximate expression for the time dependent wavefunction for $t > \Delta t$ in terms of the basis of the new Hamiltonian $|\uparrow\rangle$, $|\downarrow\rangle$.

   (b) Again how slowly must this rotation occur for your approximate solution to be valid? **Hint:** Relate the time Δt to a characteristic time of the system.

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Problem 165. *1989-Fall-QM-G-5* ID:QM-G-62

Consider a system in which the particles of mass m are confined to the surface of a sphere of radius a. The behavior of one particle can be described by the Hamiltonian

$$\hat{H}_i = \frac{\hat{L}_i^2}{2ma^2}$$

where $\hat{\vec{L}}_i$ is the angular momentum vector operator of the ith particle.

1. Write down the quantum numbers which describe the eigenstates for one particle.

2. Write down the energies and degeneracies of the ground state and first excited states.

Now assume the system contains two identical spin 1/2 fermions. Each particle is described by the single—particle Hamiltonian above and do not interact with each other.

3. Write down the two-particle wavefunctions for the ground state and first excited states. (One can use a shorthand notation using the appropriate quantum numbers. Be sure to denote both the orbital and spin components of the wavefunction.)

4. For each of the wavefunctions above write down the values for the:

   (a) total energy;

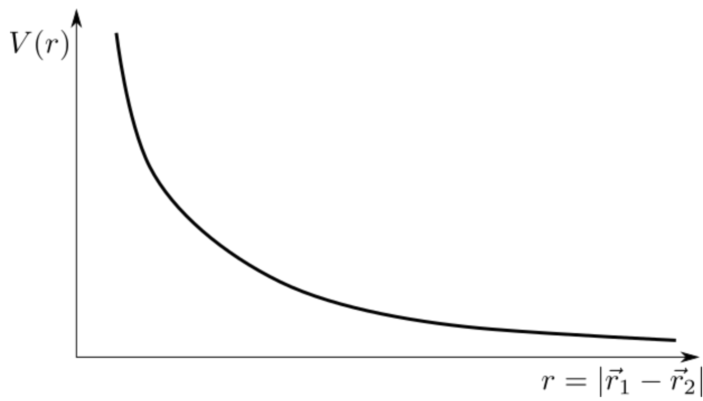
   (b) total orbital angular momentum L;

   (c) the total spin angular momentum S and

(d) the total angular momentum J. Specify which states are degenerate.

Now consider that we "turn on" a repulsive force between the two particles corresponding to the potential shown. This potential will shift the energy levels of the system and may break some (not necessarily all) of the degeneracies.

5. Qualitativelv explain how the energy of each of the above states will shift due to this potential. In particular if a degeneracy is broken, please indicate how each of degenerate states shift will respect to each other. Please include your reasoning for each. (No explicit calculations are needed.)



Problem 166. *1989-Spring-QM-G-4* ID:QM-G-65

1. State the conditions necessary for a physical quantity corresponding to an operator $\hat{Q}$ to be conserved in a system described by a Hamiltonian $\hat{H}$.

2. For each Hamiltonian

   (a) $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}kr^2$;

   (b) $\hat{H} = \frac{\hat{L}^2}{2ma^2} + \alpha\hat{\vec{L}} \cdot \hat{\vec{S}}$;

   (c) $\hat{H} = \alpha E_0 x \cos(\omega t) + \mu \hat{L}_x B_0$;

   (d) $\hat{H} = \alpha\hat{\vec{S}} \cdot \hat{\vec{p}}$;

   where $E_0 \cos(\omega t)$ represents an external time-dependent electric field, B_0 represents an external magnetic field, and $\hat{\vec{S}}$ represents the spin angular momentum, state whether the following physical quantities are conserved.

   (a) Energy: E

   (b) Momentum: $\hat{\vec{p}} = \hat{p}_x\hat{i} + \hat{p}_y\hat{j} + \hat{p}_z\hat{k}$

   (c) Orbital Angular Momentum: $\hat{\vec{L}}$

   (d) Parity: $\hat{P}$

   (e) Total Angular Momentum: $\hat{\vec{J}}$

Problem 167. *1989-Spring-QM-G-5* ID:QM-G-68

An example of a hydrogenic system is an electron attracted to a plane surface of a dielectric located in the x-y plane at z = 0. The potential is given by

$$V(x,y,z) = \begin{cases} \infty, & \text{if } z < 0; \\ -\frac{\alpha e^2}{z}, & \text{if } z > 0 \end{cases}$$

α is a positive constant.

1. Write down the appropriate Schrödinger equation for this system.
2. Describe the dependence of the wavefunction on x and y.
3. Calculate the ground state energy of the electron.
4. Calculate the most probable distance of the electron from the surface.
5. Calculate the average distance of the electron from the surface.

Problem 168. *1990-Fall-QM-G-4* ID:QM-G-71

Consider a hydrogen atom governed by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{4\pi\epsilon_0 r},$$

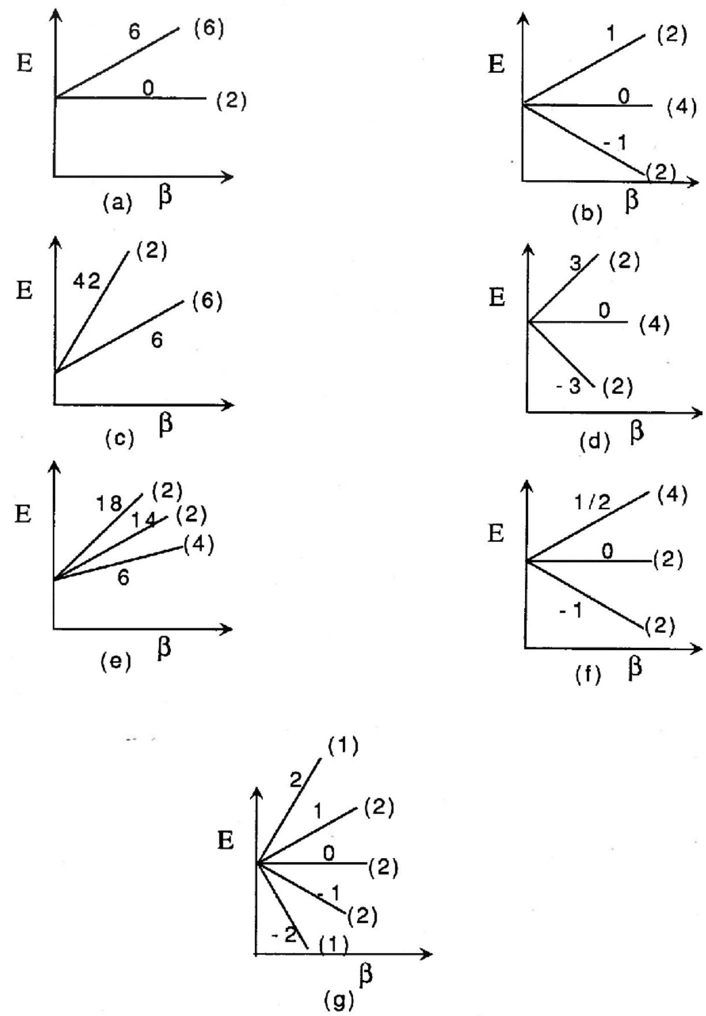
where μ is the reduced mass of the electron and $r = (x^2+y^2+z^2)^{1/2}$. If one introduces the spin of the electron but ignores any other perturbations, then the $n = 2$ first excited energy level is eight-fold degenerate.

Consider the following perturbations $\hat{H}_I$:

1. $\beta(\hat{L}_z + 2\hat{S}_z)/\hbar$;
2. $\beta\hat{L}^2/\hbar^2$;
3. $\beta z^2/a_0^2$;
4. $\beta r^2/a_0^2$;

where $\hat{\vec{L}}$ and $\hat{\vec{S}}$ are the angular momentum and electron spin operators, respectively, a_0 is the Bohr radius and $\beta \geq 0$ is a parameter with units of energy that indicates the strength of the perturbation; it is independent of spin and spatial variables and $\beta = 0$ indicates no perturbation. Figures (a)-(g) are energy-level diagrams showing how the degeneracy is broken by various perturbations. Plotted are the energies as functions of β with the slopes of the curves labeled and with the degeneracies of each level indicated in parentheses. (The figures are not drawn to scale.)

Match the four figures which correspond to the four perturbations indicated above. Discuss briefly your reasoning for each.



Problem 169. *1990-Fall-QM-G-5* ID:QM-G-74

A beam of neutrons travels along the positive z-axis with velocity 2000 meters/second. Initially it travels in a region (Region 1) of uniform magnetic field $B_0\hat{z}$, where $\hat{z}$ is the unit vector in the z direction. The spins of the neutrons are initially polarized parallel to the z axis. The beam then enters another region (Region 2) where the magnetic field is $\vec{B} = B_0\hat{z} + B_1\hat{x}$, $\hat{x}$ being the unit vector along the x axis. After traveling for 20 meters in this region without any deflection the beam enters a third region where the magnetic field is once again $B_0\hat{z}$.

1. Calculate the probability of spin-flip of the neutrons caused by the travel through Region 2.

2. You are given that $B_1 = 0.01B_0$, $\vec{\mu} = -\mu_n\vec{S}/\hbar$, and that $\frac{B_0\mu_n}{\hbar} = 60\ \text{sec}^{-1}$ where μ_n is the magnitude of the neutron magnetic moment.

Problem 170. *1990-Spring-QM-G-4* ID:QM-G-77

Two particles move in a relative s-state. Particle 1 has spin $S_1 = 1$ and magnetic moment $\vec{\mu} = -\mu\vec{S}_1$ and the second particle has spin $S_2 = 1/2$ and no magnetic moment $\vec{\mu} = 0$. The Hamiltonian of the system is $\hat{H} = A\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$.

1. List all of the constants of the motion.
2. Find the eigenstates and energies of the system. Express your eigenstates in the product basis $|1, m_1\rangle|1/2, m_2\rangle$, where $m_1 = -1, 0, 1$ and $m_2 = \pm 1/2$. Clearly state the degeneracy of each level.
3. A very strong magnetic field $\vec{B}$ is now imposed on the system. Find the approximate eigenstates and energies of the system in the limit that the $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ interaction can be ignored.

Hint: If $|J, m\rangle$ is an arbitrary angular momentum eigenstate and we define the operators $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ and $\hat{J}_- = \hat{J}_x - i\hat{J}_y$ then:

$$\hat{J}_+|J, m\rangle = [(J - m)(J + m + 1)]^{1/2} |J, m + 1\rangle,$$
$$\hat{J}_-|J, m\rangle = [(J + m)(J - m + 1)]^{1/2} |J, m - 1\rangle.$$

Problem 171. *1990-Spring-QM-G-5* ID:QM-G-80

The nucleus ^{3}H (Tritium), the mass three isotope of Hydrogen, consisting; of one proton and two neutrons is unstable. It decays by beta emission to 3He. the mass three isotope of helium which consists of two protons and one neutron. Assume that when this process takes place, it does so instantaneously. Thus there occurs a sudden doubling of the coulomb interaction between the atomic electron and the nucleus.

Note that this process produces singly ionized He, itself a hydrogenic atom.

1. If the tritium atom is in its ground state when the beta decay takes place, what is the probability that the He$^+$ ion will be in its ground state immediately after the decay?
2. After the decay has taken place, the He$^+$ ion may be found in anyone of an infinite number of possible states. Categorize all the states of the He$^+$ ion that are populated with non-vanishing probability and explain.

Hint: The normalized ground state wavefunction of a hydrogenic atom is

$$\psi = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} Y_0^0.$$

Problem 172. *1991-Fall-QM-G-4* ID:QM-G-83

A particle has a spin of $\hbar/2$. A single measurement is made of the sum of its x and z components of spin angular momentum, $(S_x + S_z)$.

1. What are the possible results of this measurement?
2. After the above measurement is made, calculate the probability that a result of $+\hbar/2$ is obtained for a measurement of S_y, only, on the same particle. (Your results for this question should include all possible outcomes of the first question.)

Pauli Matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Problem 173. *1991-Spring-QM-G-4* ID:QM-G-86

A spin-1/2 particle has wavefunction (state vector)

$$|\Psi\rangle = \frac{1}{\sqrt{5}}\left(2|+\rangle - |-\rangle\right),$$

where $|\pm\rangle$ are normalized eigenkets of $\hat{S}_z$:

$$\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle.$$

Recall that

$$\hat{S}_x|\pm\rangle = \frac{\hbar}{2}|\mp\rangle$$

1. What are the possible results of a measurement of S_z, and what is the probability of each being obtained in a given measurement?
2. What is $\langle\hat{S}_z\rangle$, the expectation value of S_z?
3. What are the possible results of a measurement of S_x and what is the probability of each being obtained in a given measurement?
4. What is $\langle\hat{S}_x\rangle$, the expectation value of S_x?
5. A measurement of S_z, is made and the result $+\hbar/2$ is obtained. Immediately afterward S_x, is measured. What are the possible results of the measurement of S_x, and what is the probability of each?

Problem 174. *1991-Spring-QM-G-5* ID:QM-G-89

A nonrelativistic particle is in a spherically symmetric potential well such as the one sketched in the figure.

1. Which property of the particle, and which two parameters characterizing the well are the most important in determining the existence of a bound state in this quantum system?

2. What qualitative conditions on each of these three parameters favor the existence of a bound state?

3. Express all three conditions you stated in the previous part in terms of a single relation involving the three parameters and any relevant fundamental constants. (Can you construct a single dimensionless quantity using the three parameters and any relevant fundamental constants?)

4. If the bound state exists, what is the symmetry of the ground state wavefunction?

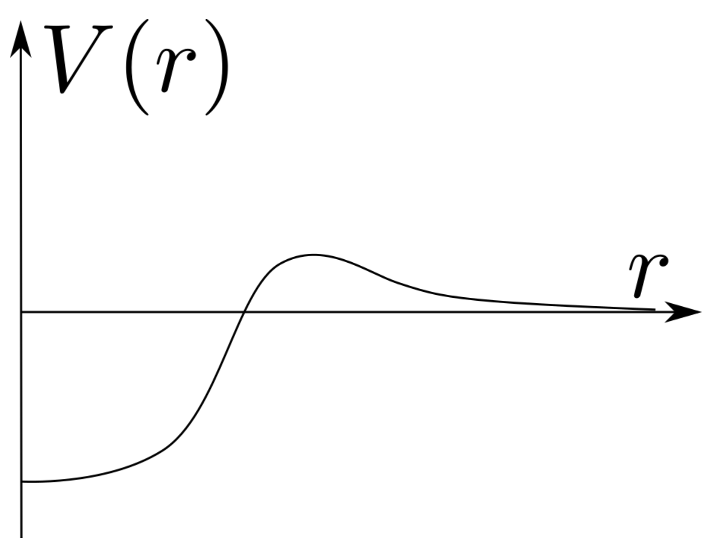
5. Sketch the radial dependence of the magnitude of this ground state wavefunction.

6. Starting from the symmetry of the problem, which quantum numbers determine degeneracy of the bound states?

7. Starting from these quantum numbers, write down the degeneracies.

8. Write the form of the energy eigenfunctions dictated by the symmetry of the problem.

9. Which matrix elements of an axially symmetric operator must vanish in the energy eigenfunction basis?



Problem 175. *1992-Fall-QM-G-4* ID:QM-G-92

Consider a system of two non-interacting spin half particles $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$. Their eigenvectors are denoted by $|\pm,\pm\rangle$. The eigenenergies are $\hbar\omega$ for $|+,+\rangle$, 0 for $|+,-\rangle$ and $|-,+\rangle$, and $-\hbar\omega$ for $|-,-\rangle$. We now take coupling of the spins into account by adding the Hamiltonian

$$\hat{W} = \alpha\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2,$$

where α is a real, positive constant. Assume that $\alpha\hbar^2 \ll \hbar\omega$, so that $\hat{W}$ can be treated like a perturbation. Find the eigenvalues to first order in α and eigenstates to zeroth order in α. Draw the energy diagram.

You may need

$$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j,m\pm\rangle.$$

Problem 176. *1992-Fall-QM-G-5* ID:QM-G-95

A particle in a one-dimensional infinite potential well with width L is in the state

$$\Psi(x) = \frac{1}{\sqrt{2}}\left[\Psi_0(x) + \Psi_1(x)\right],$$

where Ψ_0 is the normalized ground state and Ψ_1 the normalized first excited state. The wall of the box at $x = L$ is suddenly removed outward to $x = 2L$. Calculate the probability that the particle is in the ground state.

Problem 177. *1992-Spring-QM-G-4* ID:QM-G-98

Consider a linear harmonic oscillator perturbed by a uniform field $\Delta V = \epsilon kx$ so that $V = \frac{1}{2}kx^2 + \epsilon kx$.

1. Obtain the perturbed energy eigenvalues E' *exactly* in terms of the unperturbed eigenvalues E and the constants k and ϵ.

2. Use time independent perturbation theory carried to second order to find the perturbed eigenvalues, and compare them with those obtained in the previous part.

You will need the following formula, which is given here without derivation:

$$E'_n = E_n + \langle n|\Delta V|n\rangle + \sum_{i\neq n} \frac{|\langle n|\Delta V|i\rangle|^2}{E_n - E_i}$$

to second order.

Facts you might find useful:

$$\psi_n = A_n H_n(s) e^{-s^2/2}, \quad s = x/a, \quad A_n = \left(2^n n! \pi^{1/2} a\right)^{-1/2}, \quad a^2 = \sqrt{\frac{\hbar}{km}},$$

$$\langle n|n'\rangle = \int_{-\infty}^{\infty} \psi_n^*(x)\psi_{n'}(x)dx = \delta_{n,n'}$$

$$s|n\rangle = \frac{nA_n}{A_{n-1}}|n-1\rangle + \frac{A_n}{2A_{n+1}}|n+1\rangle.$$

Problem 178. *1992-Spring-QM-G-5* ID:QM-G-101

A two-dimensional space has basis vectors $\{|1\rangle, |2\rangle\}$. On this basis the Hamiltonian $\hat{H}$ and another observable A have matrices

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \hat{A} = a \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$$

At time $t = 0$, the wavefunction for a system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

1. At $t = 0$, what are the possible results of a measurement of A, and what is the probability of each being obtained?
2. At $t = 0$, what is $\langle \hat{A} \rangle$?
3. What is the wavefunction at a later time t?
4. What is $\langle \hat{A} \rangle$ at time t? Is A a constant of the motion?
5. At time $t = 0$ the energy is measured. What are the possible results, and what is the probability of each result being obtained?
6. Repeat the previous part, for time t. Is the energy a constant of the motion?

Problem 179. *1993-Fall-QM-G-4* ID:QM-G-104

A particle of mass m moves in three dimensions in a potential $V(x, y, z)$. Consider the following cases:

$$V_a(x, y, z) = \begin{cases} 0, & \text{if } x^2 + y^2 + z^2 < (L/2)^2; \\ \infty, & \text{otherwise.} \end{cases}$$
$$V_b(x, y, z) = \begin{cases} 0, & \text{if } |x|, |y|, |z| < L/2; \\ \infty, & \text{otherwise.} \end{cases}$$

Let E_a be the ground state energy in potential V_a, and E_b be the ground state energy in potential V_b. Determine whether $E_a > E_b$, $E_a < E_b$, or $E_a = E_b$.

Note: You can solve this problem without doing any calculation. Of course you can also solve it "the hard way".

Problem 180. *1993-Fall-QM-G-5* ID:QM-G-107

Let $\hat{\vec{S}}$ be the spin operator for a spin one-half particle, with components $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. The spin space of the particle has basis $\{|\alpha\rangle, |\beta\rangle\}$, where the orthonormal kets $|\alpha\rangle$ and $|\beta\rangle$ satisfy the equations

$$\hat{S}_z|\alpha\rangle = +\frac{\hbar}{2}|\alpha\rangle, \qquad \hat{S}^+|\alpha\rangle = 0, \qquad \hat{S}^-|\alpha\rangle = \hbar|\beta\rangle$$
$$\hat{S}_z|\beta\rangle = -\frac{\hbar}{2}|\alpha\rangle, \qquad \hat{S}^+|\beta\rangle = \hbar|\alpha\rangle, \qquad \hat{S}^-|\beta\rangle = 0$$

with $\hat{S}^+ = \hat{S}_x + i\hat{S}_y$, and $\hat{S}^- = \hat{S}_x - i\hat{S}_y$. The particle is in an external magnetic field in the z-direction, so the Hamiltonian operator is $\hat{H} = A\hat{S}_z$, where A is a constant. At $t = 0$ the spin wavefunction for the particle is $|\psi(0)\rangle = \frac{1}{\sqrt{5}}(|\alpha\rangle - 2|\beta\rangle)$.

1. At $t = 0$ a measurement is made of the x-component of spin, $\hat{S}_x$. What are the possible results of this measurement and what is the probability of each? What is the expectation value $\langle \hat{S}_x \rangle$?

2. The measurement of $\hat{S}_x$ is not made at $t = 0$. Instead, we wait until a later time $t = \pi/A$ to measure $\hat{S}_x$. What are the possible results of this measurement and what is the probability of each? What is the expectation value $\langle \hat{S}_x \rangle$?

Problem 181. *1993-Spring-QM-G-4* ID:QM-G-110

The wavefunction of an electron is

$$\psi(\vec{r}) = R(r)\left[\sqrt{\frac{5}{7}}Y_1^0(\theta,\phi)\chi_+ + \sqrt{\frac{2}{7}}Y_1^1(\theta,\phi)\chi_-\right]$$

where the Y_l^m are spherical harmonics and the $\chi_\pm$ two-component spinors.

1. Calculate the z-component of the electron's total angular momentum, $J_z = L_z + S_z$.

2. What is the value of the orbital angular momentum?

3. What is the probability density for finding the electron with spin up at r, θ, ϕ. ? With spin down? (Leave your answer in terms of $R(r)$ and the Y_l^m.)

4. Derive the total probability density for finding the electron at r, θ, and ϕ, and show that it is independent of ϕ.

Hint: You may need these properties of angular momentum operators:

$$\begin{aligned}
Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta, \\
\hat{L}_+ &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right), \\
\hat{J}_\pm|j,m\rangle &= \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle.
\end{aligned}$$

Problem 182. *1993-Spring-QM-G-5* ID:QM-G-113

Consider a particle of mass M moving in one dimension inside an infinite square well of length L. That is, the particle sees the potential

$$V(x) = \begin{cases} 0, & \text{if } 0 < x < L; \\ \infty, & \text{otherwise} \end{cases}.$$

Now suppose that the motion of the particle is perturbed by a potential of the form

$$V'(x) = \lambda\delta(x - L/2),$$

where λ is a positive constant and $\lambda \ll \hbar^2/ML$.

1. Calculate the correction to the energy levels to first order in λ. Show that the correction is small compared to the energy of any state.

2. If we designate the unperturbed states by $|n\rangle$ and the first-order perturbed states by $|\psi_n^{(1)}\rangle$, calculate the "admixture" of state $|m\rangle$ into $|\psi_n^{(1)}\rangle$. That is, calculate $\langle m|\psi_n^{(1)}\rangle$.

3. Calculate the second-order correction to the energy levels. You may express your answer in terms of a series, but show that the correction is independent of the length of the well.

4. Calculate the exact value of the second order shift for the ground state. You may use the fact that
$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1.$$

Hint: You may need these formulas from perturbation theory:

$$\Delta E_n^{(1)} = \langle n|V'|n\rangle,$$
$$\Delta E_n^{(2)} = \sum_{n' \neq n} \frac{|\langle n'|V'|n\rangle|^2}{E_n - E_{n'}},$$
$$|\psi_n^{(1)}\rangle = |n\rangle + \sum_{n' \neq n} |n'\rangle \frac{\langle n'|V'|n\rangle}{E_n - E_{n'}}$$

Problem 183. *1994-Fall-QM-G-4* ID:QM-G-116

Consider a non-relativistic hydrogen atom placed in a uniform static electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$. Calculate the energies of each sublevel of the $n = 2$ excited state to first order in the field $\mathcal{E}$. You may ignore all spin effects.

Also calculate the perturbed eigenstates for each level in terms of the unperturbed basis $\{|n, i, m\rangle = |2, 0, 0\rangle, |2, 1, 0\rangle, |2, 1, 1\rangle, |2, 1, -1\rangle, \}$.

Take the perturbation to be $W = e\vec{\mathcal{E}}\cdot\vec{r}$ (e is the magnitude of the electron charge).

Useful information:

$$\psi_{n,l,m} = a_0^{-3/2} R_{n,l}(r) Y_{l,m}(\theta, \phi),$$

where a_0 is the Bohr radius and:

$$R_{2,0} = \frac{1}{\sqrt{2}} e^{-r/2a_0}\left(1 - \frac{r}{2a_0}\right),$$
$$R_{2,1} = \frac{1}{2\sqrt{6}} \frac{r}{a_0} e^{-r/2a_0},$$

and

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos(\theta), \qquad Y_{1,\pm 1} = \sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi}.$$

You might also need

$$\int_0^\infty x^n e^{-x} dx = n!.$$

Problem 184. *1994-Fall-QM-G-5* ID:QM-G-119

For a spin 1/2 system, the rotation matrix for an arbitrary two-component spinor is

$$U_R = \mathbf{1}\cos(\theta/2) - i\hat{n}\cdot\vec{\sigma}\sin(\theta/2),$$

where $\mathbf{1}$ is the 2×2 unit matrix, $\hat{n}$ is a unit vector in the direction of the rotation, θ is the angle of rotation, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli spin matrices defined by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider an arbitrary spinor χ in z-component basis states,

$$\chi = \begin{pmatrix} e^{i\alpha}\cos\delta \\ e^{i\beta}\sin\delta \end{pmatrix}$$

where α, β, and δ are real parameters.

1. Construct the matrix U_R that rotates χ into $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   Hint: Choose $\hat{n}$ to lie in the $x - y$ plane.

2. Verify that your result for U_R is unitary by explicit evaluation.

3. Determine the expectation value of σ_z, for the state χ.

4. Determine the expectation value of σ_y for the state χ.

5. What is the probability of observing spin-up ($+z$-direction)?

6. What is the probability of observing the spin in the $+y$-direction?

Problem 185. *1994-Spring-QM-G-4* ID:QM-G-122

Consider a hydrogen atom in its ground electronic state. There is zero orbital angular momentum. Within this state, the interaction of the nuclear magnetic moment and the electron magnetic moment may be approximated by the perturbation

$$\hat{H}' = \frac{\mathcal{A}}{\hbar^2}\hat{\vec{I}}\cdot\hat{\vec{S}}$$

where $\hat{\vec{I}}$ is the nuclear spin ($I = 1/2$), $\hat{\vec{S}}$ is the electron spin ($S = 1/2$), and where $\mathcal{A}$ is a constant determined mostly by the ground state electron wavefunction.

This perturbation leads to a splitting of the ground state, referred to as hyperfine structure. The wavelength of this transition is 21cm, and is easily detectable by radio astronomy, giving a measure of interstellar hydrogen

1. Calculate the magnitude of this splitting in terms of the constant $\mathcal{A}$.

   Hint: Recall that:

$$F^2 = \left(\hat{\vec{I}} + \hat{\vec{S}}\right)^2 = \hat{\vec{I}}^2 + 2\hat{\vec{I}} \cdot \hat{\vec{S}} + \hat{\vec{S}}^2.$$

2. Determine the degeneracy of each component.

3. Now assume this atom is placed in a magnetic field pointing in the $+z$ direction. Assume the magnetic field is weak enough that we can consider its effect on each of the above components separately. If we also neglect the interaction of the magnetic field with the nuclear spin compared to its interaction with the electron spin, we can write this perturbation as

$$\hat{H}'' = -2\omega_L \hat{S}_z$$

   where ω_L is the Lamor precession frequency of the electron in the magnetic field.

   Show that the degeneracy of each of the components found above in the previous part is completely lifted by $\hat{H}''$.

Problem 186. *1994-Spring-QM-G-5* ID:QM-G-125

In three dimensions, a spin-1/2 particle with mass m moves in the potential

$$V(x) = \begin{cases} V_0\sigma_z, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

1. A monochromatic polarized incoming plane wave comes from $-\infty$ in the positive x-direction with energy $E > V_0 > 0$ and in an eigenstate of σ_z with eigenvalue 1. Write down the general wave function of such an incoming beam.

2. Write down the general solutions for the transmitted and reflected waves.

3. What are the boundary conditions at $x = 0$? Write down the equations which must be satisfied by the general amplitudes appearing in the two previous parts.

4. Under what condition is the reflected wave an eigenstate of σ_x? Under what condition is the transmitted wave an eigenstate of σ_x?

Hint: The Pauli spin-matrices are defined by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 187. *1995-Fall-QM-G-1* ID:QM-G-128

An electron (spin 1/2) is coming from $x = -\infty$ in three dimensions. Its momentum perpendicular to the x-direction is zero and it is in an eigenstate of σ_x. At $x = 0$ it enters a magnetic field $\vec{B} = (0, 0, B_z)$. In other words it scatters off a potential step of the form:

$$V(x) = \begin{cases} 0, & \text{if } x < 0; \\ -\frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}, & \text{if } x > 0. \end{cases}$$

Calculate the transmitted and reflected waves as a function of the energy of the incident particle. Is the reflected particle still in an eigenstate of σ_x?

Ignore the Lorentz $\vec{v} \times \vec{B}$ force on the electron.

Problem 188. *1995-Fall-QM-G-2* ID:QM-G-131

Let $\vec{L}$ be the total orbital angular momentum operator of an atom, and let $\vec{S}$ be the total spin angular momentum operator of its electrons. Ignoring the spin of the nucleus write the total angular momentum operator of the atom as $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ In the absence of a magnetic field, we call $\hat{H}_0$ the Hamiltonian of the atom with eigenvalue E_0; $\hat{H}_0$ commutes with $\hat{\vec{J}}$. Assume that $\hat{H}_0$, $\hat{L}^2$, $\hat{S}^2$, $\hat{J}^2$, and $\hat{J}_z$ form a complete set of commuting operators with eigenvalues E_0, $L(L+1)\hbar$, $S(S+1)\hbar$, $J(J+1)\hbar$, and $M_J\hbar$ respectively.

Now, let there be an applied magnetic field in the $\hat{z}$ direction. The Hamiltonian becomes

$$\hat{H} = \hat{H}_0 - \frac{\mu_B}{\hbar}\left(\hat{L}_z + g_s\hat{S}_z\right)B.$$

Let $g_s = 2$, and define ω_L as the Larmor frequency $\omega_L = -\mu_B B/\hbar$, so that

$$\hat{H} = \hat{H}_0 + \omega_L\left(\hat{L}_z + 2\hat{S}_z\right)B.$$

Inside the subspace $\{E_0, L, S, J\}$, we can write

$$\hat{\vec{L}} = \frac{\langle\hat{\vec{L}}\cdot\hat{\vec{J}}\rangle}{J(J+1)\hbar}\hat{\vec{J}} \qquad \text{and} \qquad \hat{\vec{S}} = \frac{\langle\hat{\vec{S}}\cdot\hat{\vec{J}}\rangle}{J(J+1)\hbar}\hat{\vec{J}}.$$

1. Show that
$$\hat{H} = \hat{H}_0 + g_J\omega_L\hat{J}_z$$
where g_J is a real scalar quantity (called the Landé g-factor) that can be expressed in terms of L, S, and J, and derive an expression for g_J.

2. What are the allowed energies of the full Hamiltonian $\hat{H}$?

3. Now we will take the nuclear spin $\hat{\vec{I}}$ into account. Let $\hat{\vec{F}} = \hat{\vec{J}} + \hat{\vec{I}}$ be the total spin. Consider the $\{E_0, J, I, F\}$ subspace, and show that the Hamiltonian becomes
$$\hat{H} = \hat{H}_0 + g_{JF}\omega_L\hat{F}_z$$
and calculate g_{JF}. (Neglect nuclear magnetic moment.)

   Hint: Use the fact that you are restricted to the $\{E_0, J, I, F\}$ subspace.

Problem 189. *1995-Fall-QM-G-3* ID:QM-G-134

You are given the following relations for the one-dimensional quantum harmonic oscillator with mass m and frequency ω:

$$\begin{aligned}
&\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right), \\
&[\hat{a}, \hat{a}^\dagger] = 1, \qquad \hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right), \\
&\hat{H}|n\rangle = (n + 1/2)\hbar\omega|n\rangle, \qquad \langle n'|n\rangle = \delta_{n',n}, \quad n, n' = 0, 1, \ldots, \\
&a|0\rangle = 0, \qquad a|n\rangle = \sqrt{n}|n-1\rangle, \quad n > 0
\end{aligned}$$

where $\hat{x}$ is the position operator, $\hat{p}$ is the momentum operator, $\hat{H}$ is the Hamiltonian operator, $\hat{a}^\dagger$ is the creation operator, and $\hat{a}$ is the annihilation operator.

Consider the eigenkets $|\alpha\rangle$ of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

(These kets $|\alpha\rangle$ correspond to the coherent or "quasi-classical" states of the oscillator.)

1. Derive an expression for $|\alpha\rangle$ in terms of the energy eigenkets $|n\rangle$. Be sure that your expression is properly normalized such that $\langle\alpha|\alpha\rangle = 1$.

2. Supposing the oscillator to be in the state $|\alpha\rangle$, evaluate the root-mean-square deviations ΔH, Δx, and Δp of energy, position, and momentum from their average values. I.e., calculate

$$\Delta O = \left[\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2\right]^{1/2} = \left[\langle\alpha|\hat{O}^2|\alpha\rangle - \langle\alpha|\hat{O}|\alpha\rangle^2\right]^{1/2},$$

   where $\hat{O}$ is $\hat{H}$, $\hat{x}$, and $\hat{p}$.

3. Calculate and comment on the value obtained for the uncertainty product $\Delta x \Delta p$. Also calculate the ratio $\Delta H/\langle\hat{H}\rangle$, for large $|\alpha|$, and comment on the result.

Problem 190. *1995-Spring-QM-G-1* ID:QM-G-137

Working in the momentum representation, calculate the ground state energy and wavefunction of a particle in the one-dimensional attractive potential with coordinate space representation $V(x) = V_0\delta(x)$, ($V_0 < 0$) You do not need to normalize the wavefunction.

Problem 191. *1995-Spring-QM-G-2* ID:QM-G-140

At $t = 0$, the measurement of S_z, for a spin-1/2 particle gives the result $\hbar/2$. A uniform magnetic field, of strength B, is then applied in the xy-plane. The magnetic field makes an angle θ with the x-axis.

1. Calculate the state vector $|\psi(t)\rangle$ for the spin of the particle at time t. Express your results in terms of eigenkets of $\hat{S}_z$.

A useful notation is $\omega = -\gamma B$. Recall that for a spin $\hat{\vec{S}}$ in a magnetic field $\vec{B}$ the Hamiltonian is $\hat{H} = -\gamma \vec{B} \cdot \hat{\vec{S}}$.

Recall that $\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$; where $\vec{\sigma}$ are the Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. At time t what are the possible results of a measurement of S_z, and what is the probability of each being obtained?

Problem 192. *1995-Spring-QM-G-3* ID:QM-G-143

Consider a hydrogen atom governed by the Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{r},$$

where μ is the electron reduced mess and $r = (x^2 + y^2 + z^2)^{1/2}$. If one includes the spin of the electron, then the $n = 2$ first excited energy level is eight-fold degenerate: Three $l = 1$ states, and one $l = 0$ state, with electron spin up or down for each.

Consider the following four perturbations $\hat{W}$:

1. $\hat{W} = \frac{\lambda}{\hbar}\hat{L}_z$;
2. $\hat{W} = \frac{\lambda}{\hbar^2}\hat{\vec{L}} \cdot \hat{\vec{S}}$;
3. $\hat{W} = \lambda\frac{z}{a_0}$;
4. $\hat{W} = \frac{\lambda}{\hbar^2}\hat{L}^2$,

where $\hat{\vec{L}}$ and $\hat{\vec{S}}$ are the angular momentum and electron spin operator, respectively, a_0 is the Bohr radius, and $\lambda > 0$ is a parameter with units of energy that describes the strength of the perturbation; it is independent of spatial and spin variables.

Calculate the splitting of the $n = 2$ (first excited) level due to each of these perturbations to first order in λ, and sketch the λ dependence of the perturbed energy levels and indicate the degeneracy of each curve.

Hint:

$$Y_{0,0}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}, \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta,$$

$$Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta, \qquad Y_{l,-m}(\theta,\phi) = (-1)^m Y_{l,m}^*(\theta,\phi),$$

$$R_{2,0} = \left(\frac{1}{2a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}, \qquad R_{2,1} = \left(\frac{1}{2a_0}\right)^{3/2}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0},$$

$$\int_0^\infty x^n e^{-x}dx = n!, \qquad \hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{1}{2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right),$$

where $\hat{\vec{J}}$ is the total angular momentum.

Problem 193. *1996-Fall-QM-G-1* ID:QM-G-146

A particle of mass m moves in one dimension under the influence of the potential:

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \alpha\frac{m^2\omega^3}{\hbar}x^4,$$

where $0 < \alpha \ll 1$. Find the ground state, expressed in terms of harmonic oscillator eigenstates, correct to $O(\alpha)$. Also find the energy of the ground state, correct to $O(\alpha^2)$.

Reminder: Let $|n\rangle$ be an eigenfunction of the harmonic oscillator number operator $\hat{N}$, with $\hat{N}|n\rangle = n|n\rangle$, $n = 0, 1, 2, 3, \ldots$ Then:

$$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right).$$

Problem 194. *1996-Fall-QM-G-2* ID:QM-G-149

1. Find the phase shifts δ_l for the potential

$$V(r) = \frac{A}{r^2}; \quad A > 0.$$

2. Consider the limit of large l i.e. $(l + 1/2)^2 \gg 2mA/\hbar^2$ and show that $\delta_l \to 0$ as $l \to \infty$.

Problem 195. *1996-Spring-QM-G-1* ID:QM-G-152

Positronium is a hydrogen—like bound state of an electron and a positron. The effective Hamiltonian for the system in the $1S$ state in a magnetic field B can be written as

$$\hat{H} = \hat{H}_0 + \frac{\mathcal{A}}{\hbar^2}\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 + 2\omega_L\left(\hat{S}_1^z - \hat{S}_2^z\right),$$

where $\hat{\vec{S}}_1$ is the spin operator of the electron, $\hat{\vec{S}}_2$ is the spin operator of the positron, $\mathcal{A}$ is a positive constant, ω_L, is the Larmor precession frequency $\mu_B B/\hbar$, and $\hat{H}_0$ contains the kinetic energies and central force potential.

Calculate the energy levels of this system relative to $E_0 = \langle\hat{H}_0\rangle$, and sketch them as a function of B, labeling the degeneracy of each level.

Problem 196. *1996-Spring-QM-G-2* ID:QM-G-155

Under what conditions does the WKB approximation:

$$\psi(x) = \frac{\bar{A}}{(E - V(x))^{1/4}}\exp\left[i\frac{\sqrt{2m}}{\hbar}\int\sqrt{E - V(x)}dx\right]$$

apply for the wave function of a particle in the potential $V(x)$? By inserting the above ansatz into the Schrödinger equation, show that the WKB approximation requires:

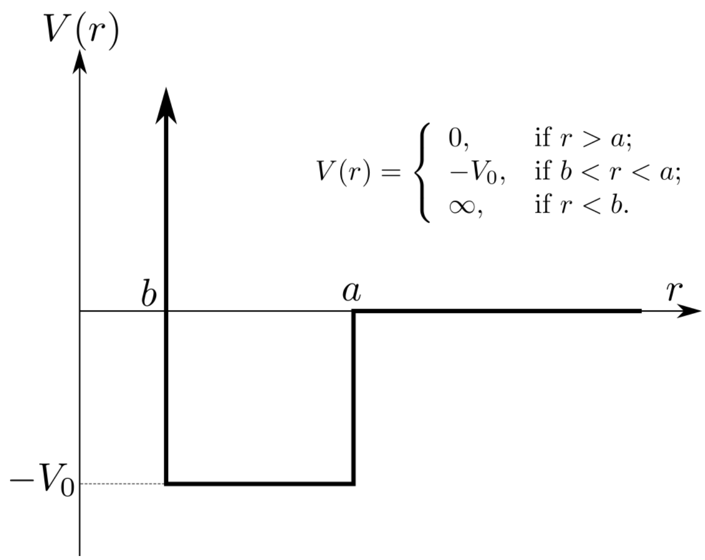
$$\left|\frac{d\lambda(x)}{dx}\right| \ll 1 \qquad \text{and} \qquad \left|\lambda(x)\frac{d^2\lambda(x)}{dx^2}\right| \ll 1$$

where

$$\lambda(x) = \frac{\hbar}{\sqrt{E - V(x)}}.$$

Problem 197. *1996-Spring-QM-G-3* ID:QM-G-158

The n-p (neutron-proton) potential has an attractive tail and a hard core interior. Approximate the central symmetric potential by the function on the figure and find the phase shift for n-p scattering for this potential.



Problem 198. *1997-Fall-QM-G-4* ID:QM-G-161

An incident beam of particles of momenta $p = \hbar k$ scatter off a spherical shell potential:

$$V(r) = -V_0\delta(r - a)$$

1. Write the radial equation of motion and continuity conditions. Define δ_l according to
$$R_l \xrightarrow[r\to\infty]{} B_l \frac{\sin(kr - \pi l/2 + \delta_l)}{kr}.$$
2. Calculate the phase shifts δ_l. Note that the differential cross-section relates to δ_l via $\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_l (2l+1)e^{i\delta_l}\sin(\delta_l)P_l(\cos\theta)\right|^2$.

Useful formulas: With $(2l+1)!! = (2l+1)(2l-1)...(5)(3)(1)$

$$j_l \xrightarrow[r\ll 1]{} \frac{x^l}{(2l+1)!!}; \qquad n_l \xrightarrow[r\ll 1]{} -\frac{(2l+1)!!}{x^{l+1}};$$

$$j_l \xrightarrow[r\gg 1]{} \frac{\sin(x - \pi l/2)}{x}; \qquad n_l \xrightarrow[r\gg 1]{} -\frac{\cos(x - \pi l/2)}{x}.$$

Problem 199. *1997-Fall-QM-G-5* ID:QM-G-164

A spin 1 particle placed in a magnetic field has a Hamiltonian $\hat{H} = A\hat{S}_z$, where A is some constant. At $t = 0$, the state vector of the particle is given by:

$$|\Phi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|11\rangle + |10\rangle\right).$$

1. Write down an expression for the state vector at future times.
2. Find the expectation values of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$, all as functions of time.

Problem 200. *1997-Spring-QM-G-4* ID:QM-G-167

Consider a two-level system with Hamiltonian $\hat{H}_0$, eigenvectors $|\phi_i\rangle$ and eigenvalues E_i, with $i = 1, 2$. That is

$$\hat{H}_0|\phi_i\rangle = E_i|\phi_i\rangle, \quad i = 1, 2.$$

Now suppose that a coupling $\hat{W}(t)$ is added such that $\hat{H} = \hat{H}_0 + \hat{W}$, with

$$\langle\phi_1|\hat{W}|\phi_1\rangle = \langle\phi_2|\hat{W}|\phi_2\rangle = 0, \qquad \text{and} \qquad \langle\phi_1|\hat{W}|\phi_2\rangle = \hbar\frac{\omega_R}{2}e^{i\omega t}$$

Take both ω_R and ω to be real. Assume that at time $t = 0$ the system is in state $|\phi_1\rangle$.

Calculate the probability as a function of time of finding the system in the state $|\phi_2\rangle$. The solution is referred to as Rabi's formula.

Hint: Use the notation $\omega_0 = (E_1 - E_2)/\hbar$ and $\Delta = \omega - \omega_0$. This cleans up the equations a bit. (ω_0 is the resonance frequency of the system and Δ is the detuning of the coupling from resonance.)

Problem 201. *1997-Spring-QM-G-5* ID:QM-G-170

Energy levels of high Z He-like atoms are determined to zero'th order by the nuclear Coulomb potential seen by each electron. In addition there are perturbations due to the electrons Coulomb repulsion, spin-orbit and spin-spin interactions. Write down the form of the zero'th order wave functions for the following states using as angular basis the total spin S, total orbital angular momentum L and total angular momentum J:

1. $n_1 = 1 = n_2$
2. $n_1 = 1$, $n_2 = 2$; $n_1 = 2$, $n_2 = 1$
3. $n_1 = 2 = n_2$

where n_1, n_2 are the radial quantum numbers of the two electrons. Express your answer in the form

$$\psi = Y_{S,L,J}R_{n_1,n_2}$$

where Y is a spin and angular function and R involves products of Coulomb radial wave functions. (Do not try to work out the analytic form of Y or R!)

Why is this basis a useful one to use?

Problem 202. *1998-Fall-QM-G-4* ID:QM-G-173

1. A quantum-mechanical point particle of mass m is in the nth eigenstate of a one dimensional harmonic potential $V(x) = \frac{1}{2}kx^2$. Apply the Rayleigh-Schrödinger perturbation theory to calculate the corrected nth eigenenergy of this particle **up to second order in** k', when a perturbation potential $V(x) = \frac{1}{2}k'x^2$ is applied to the particle. You should use the raising and lowering operators for this calculation.

   Hint: The lowering operator is defined by $\hat{a} = \hat{Q} + i\hat{P}$, where $\hat{Q} \equiv \sqrt{m\omega/\hbar}\hat{x}$, and $\hat{P} \equiv \sqrt{1/m\omega\hbar}\hat{p}$ (with $\hat{p}$ being the momentum operator and $\hat{x} = x$). $\omega = \sqrt{k/m}$. The raising operator $\hat{a}^\dagger$ is the hermitian conjugate of $\hat{a}$. Also, it is given that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

2. This perturbed Hamiltonian can, of course, be diagonalized exactly. Obtain the exact eigenenergy of the nth eigenstate, and then expand it to second order in k'. Show that the two approaches give the same answer

Problem 203. *1998-Fall-QM-G-5* ID:QM-G-176

A Helium atom has two electrons in orbit around the nucleus. Assume that the electrons do not interact and both electrons can be in the ground state or one can be in the ground state and the other in an excited state. Allow for excited states up through principal quantum number $n = 3$. (You may want to sketch the one-electron energy levels through $n = 3$ showing the various states being considered)

1. In terms of spin-angular momentum states $|1/2, m\rangle$, what are the state vectors for the allowed total spin states for He?

2. Give the values allowed for the total angular momentum J seen up to $n = 3$. What is the angular momentum state $|J, m_J\rangle$ for the ground state? For the largest value of L, find the angular momentum states for $|J = L, m_J = 1, S = 1\rangle$ and $|J = L, m_J = 1, S = 0\rangle$ in terms of the products of $|L, m_L\rangle$ and $|S, m_s\rangle$ states. For your reference, the (raising, lowering) operators of angular momentum have the following properties: $\hat{J}_\pm|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$, but you do not need to use this relation since the Clebsch-Gordan coefficients are available in the Handbook provided.

3. Show what an energy level diagram would look like for the states up through $n = 3$. (Ignore J splitting.) Be sure to show all the levels separately even if they are (nearly) degenerate. Also show the allowed $E1$ transitions between states on your energy level diagram. Do you expect any long-lived states? (You may want to draw two separate diagrams to clarify the role of spin.)

4. The $\hat{\vec{L}} \cdot \hat{\vec{S}}$ interaction removes some of the degeneracy in the states for the second part. What is the value of $\langle \hat{\vec{L}} \cdot \hat{\vec{S}} \rangle$ for the $n = 3$ states of maximum L?

Problem 204. *1998-Spring-QM-G-4* ID:QM-G-179

Consider an atom with one electron in a p-state (orbital angular momentum equal to $1\hbar$). Suppose the spin-orbit interaction can be written as:

$$\hat{H}_{so} = A\hat{\vec{L}} \cdot \hat{\vec{S}}$$

where A is a constant. Further suppose that there is a large magnetic field B and the spin-orbit interaction is weak compared to the energy shifts due to this field,

$$\mu_0 B \gg \hbar^2 A$$

where μ_0 is the Bohr magneton. This is called the Paschen-Back regime. [Recall that $\hat{L}_\pm = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l, m \pm 1$ defines the normalization condition for the ladder operator.]

1. Assuming that the gyromagnetic ratio for the electron is exactly 2, what is the Hamiltonian associated with the magnetic field and what are the eigenfunctions and energies allowed for the p-state?
2. If the spin-orbit interaction is now included, calculate the energy levels of the system to first order in $\hat{H}_{so}$ and identify the degeneracy of each level. You can ignore all effects due to the atomic nucleus and atomic structure; just consider the spin and orbital angular momenta.

Problem 205. *1998-Spring-QM-G-5* ID:QM-G-182

1. **Define** the "phase shift" δ_l, for a quantum mechanical particle scattered by a short-range spherically symmetric potential. That is, the potential $V(r)$ depends only on the radial coordinate r in spherical coordinates (r, θ, ϕ), and it is vanishingly small for r larger than some finite radius a. Explain also why it is called a phase shift. You may need the information given below:

   (a)
   $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right];$$
   $$\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}(\theta,\phi).$$

   (b) The radial part of the free particle Schrödinger equation has special solutions $j_l(kr)$ and $n_l(kr)$, which have the following asymptotic behavior:
   $$\lim_{x\to 0} j_l(x) = \frac{2^l l!}{(2l+1)!}x^l; \qquad \lim_{x\to 0} n_l(x) = -\frac{(2l)!}{2^l l!}\frac{1}{x^{l+1}};$$
   $$\lim_{x\to\infty} j_l(x) = \frac{\sin(x - \pi l/2)}{x}; \qquad \lim_{x\to\infty} n_l(x) = -\frac{\cos(x-\pi l/2)}{x}.$$

2. Specialize now to the case of a hard-sphere potential, i.e., $V(r) = 0$ for $r > a$ and $V(r) = \infty$ for $r < a$, find the explicit solution for δ_l.

3. If the scattering potential is not spherically symmetric, can you still define the phase shift this way? Why or why not?

Problem 206. *1999-Fall-QM-G-4* ID:QM-G-185

Consider a particle described by the time-independent Schrödinger equation in one space-like dimension. If the particle is traveling from left to right and encounters a potential step at $z = 0$, i.e.,

$$V(x) = \begin{cases} 0, & \text{if } -\infty < x \leq 0; \\ V_0, & \text{if } 0 < x < \infty. \end{cases}$$

and if $E > V_0$, then

1. what fraction of its probability current is transmitted to the right of the step.

2. what fraction of its probability is reflected to the left of the step?

Problem 207. *1999-Fall-QM-G-5* ID:QM-G-188

A tritium atom undergoes β decay. The atom is initially in the ground state. The β decay emits an energetic electron and neutrino, which both leave the atom without disturbing the orbital electron or causing recoil of the nucleus. Identify which of the following states of the resulting He^+ ion are possible, and calculate the probabilities for each state:

n	l	m
1	0	0
2	0	0
2	1	0
2	1	1

Note that the normalized wave function of a hydrogen-like atom of charge Z can be written as $\Psi_{n,l,m}(r) = R_{n,l}(r)Y_l^m(\Omega)$ with

$$R_{1,0}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2e^{-Zr/a_0};$$

$$R_{2,0}(r) = \left(\frac{Z}{2a_0}\right)^{3/2} (2 - Zr/a_0)e^{-Zr/2a_0};$$

$$R_{2,1}(r) = \left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{\sqrt{3}a_0} e^{-Zr/2a_0},$$

where a_0 is the Bohr radius. You may find the following integral useful

$$\int_0^\infty dx x^n e^{-x} = n!.$$

Problem 208. *1999-Spring-QM-G-4* ID:QM-G-191

Consider a particle of mass m moving in a spherically symmetric Coulomb potential

$$V(r) = \begin{cases} -\frac{\alpha}{r}, & \text{if } r > R; \\ -\frac{\alpha}{R}, & \text{if } r < R. \end{cases}$$

1. The ground state wave function of the particle moving in the pure Coulomb potential
$$V_0(r) = -\frac{\alpha}{r}$$
is
$$\Psi_0 = A\exp\left(-r/a\right)$$
Express A, a, and the energy of the state in terms of m, α, and $\hbar$.

2. Use first order perturbation theory to find the energy shift of the ground state with respect to that of the pure Coulomb potential.

3. Find the conditions under which the above perturbative result is valid. Compare the energy shift with the ground state energy of the pure Coulomb potential.

Hint:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right];$$
$$\int dx x e^{-x} = -(x+1)e^{-x}, \qquad \int dx x^2 e^{-x} = -(x^2+2x+2)e^{-x}.$$

Problem 209. *1999-Spring-QM-G-5* ID:QM-G-194

Two identical non-interacting spin-1/2 particles are bound in a one dimensional harmonic oscillator potential. The energy eigenvalues for a single particle are

$$E_n = \hbar\omega(n+1/2)$$

(Other useful information is provided below.)

1. What are the energies of the two particle system when it is (i) in the ground state, (ii) in the first excited state, (iii) in the second excited state?

2. What are the degeneracies of these states?

3. What is the total spin S of the ground state? How many of the first and second excited states have total spin $S = 0$ and how many have $S = 1$?

4. Write down the spatial put of the wave function corresponding to a first excited state with $S = 1$. In this state evaluate the ratio of the probabilities of finding the particles in locations (ii) and (iii) with respect to location (i), where the locations are: i) one at $x = 0$ and the other at $x = a$, ii) both at $x = a$, iii) one at $x = a$ and the other at $x = -a$.

5. Evaluate the probability that in this state both particles are simultaneously in the region $x = 0$.

Useful information: The first two normalized energy eigenfunctions of a single particle in a harmonic oscillator potential are:

$$u_0 = \left(a\sqrt{\pi}\right)^{-1/2} \exp\left(-y^2/2\right), \qquad u_1 = \sqrt{2}\left(a\sqrt{\pi}\right)^{-1/2} y \exp\left(-y^2/2\right),$$

where $y = x/a$ and $a^2 = \frac{\hbar}{m\omega}$.

$$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}, \qquad \int_0^\infty y e^{-y^2} dy = \frac{1}{2}, \qquad \int_0^\infty y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{4}.$$

Problem 210. *2000-Fall-QM-G-4* ID:QM-G-197

Two electrons move in a one-dimensional infinite potential well $V = \infty$ for $x < 0$, $x > L$, and $V = 0$ for $0 \le x \le L$. Ignore the Coulomb potential.

1. Write down the energy and the spatial part of the ground-state wave function when they are constrained to a triplet spin state.
2. Repeat the previous part when they are in a singlet spin state.
3. Find the probability that both electrons are found in the region $0 \le x \le L/2$ for the case of the first part only.

You may use

$$\int dx \sin^2(x) = \frac{1}{2}(x - \sin(x)\cos(x)).$$

Problem 211. *2000-Fall-QM-G-5* ID:QM-G-200

A particle moves in a two-dimensional harmonic oscillator potential

$$V(x, y) = \frac{m\omega^2}{2}(x^2 + y^2).$$

A weak perturbation

$$\delta V = \lambda xy + \frac{\lambda^2}{\hbar\omega} x^2 y^2$$

is applied. Use perturbation theory to calculate the energy in the ground state to order $O(\lambda^2)$.

Note that for eigenkets of a one-dimensional harmonic oscillator

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$$

with

$$x = \frac{x_0}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}.$$

Problem 212. *2000-Spring-QM-G-4* ID:QM-G-203

Consider a spin-1/2 particle with magnetic moment μ in a rotating magnetic field

$$\vec{B} = B_0\hat{z} + B_0\left(\hat{x}\cos\omega t + \hat{y}\sin\omega t\right).$$

Assume that $\omega = 2\mu B_0/\hbar$ and that at $t = 0$ the particle is in the state $|1/2\rangle$. Find the exact probabilities of finding the particle in states $|1/2\rangle$ and $|-1/2\rangle$ at an arbitrary time t.

Hint: The Hamiltonian describing the system is $\hat{H} = -\mu\vec{B}\cdot\vec{\sigma}$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 213. *2000-Spring-QM-G-5* ID:QM-G-206

Consider the $2P$ excited state of a hydrogen atom. The letter P here means that the orbital angular momentum of the electron in this state is $L = 1$.

1. Consider the total angular momentum $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ where $\hat{\vec{S}}$ is the spin of the electron. Find all possible values of J and J_z and the corresponding eigenfunctions in terms of $|L_z, S_z\rangle$ where $L_z = 0, \pm 1$, and $S_z = \pm 1/2$ are the projections of the orbital angular momentum and the spin on the arbitrarily chosen axis. We set $\hbar = 1$.

   Hint:
   $$\hat{L}_{\pm}|L, m\rangle = \sqrt{(L \mp m)(L \pm m + 1)}|L, m \pm 1\rangle.$$

2. Take into account spin-orbit interaction described by the Hamiltonian $H_{so} = \lambda\hat{\vec{L}}\cdot\hat{\vec{S}}$. Find the eigenstates of this Hamiltonian and the corresponding eigenvalues.

Problem 214. *2001-Fall-QM-G-4* ID:QM-G-209

Consider a spinless particle of mass m in a one-dimensional harmonic oscillator potential of constant frequency ω:

$$V(x) = \frac{1}{2}m\omega^2 x^2.$$

In the non-relativistic limit, its eigenstates and eigenenergies are denoted by $|n\rangle$ and $\hbar\omega(n + 1/2)$, respectively. Treating the *relativistic* corrections, $-p^4/(8m^3c^2)$, to the kinetic energy as a perturbation, find the leading order correction to

1. the ground state energy,

2. the ground state eigenket.

Hint: In terms of the creation and annihilation operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i\hat{p}}{m\omega}\right), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i\hat{p}}{m\omega}\right),$$

one has

$$[\hat{a}, \hat{a}^\dagger] = 1, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

Problem 215. *2001-Fall-QM-G-5* ID:QM-G-212
A particle of mass m is scattered by a soft-sphere potential

$$V(r) = \begin{cases} V_0, & \text{if } r \le a; \\ 0, & \text{if } r > a, \end{cases}$$

where V_0 and a are constants.

1. Using the first-order Born approximation, find the transcendental equation satisfied by the momentum transfer at which the cross section vanishes.
2. Find the approximate values for the momentum transfer at which the cross section vanishes. Express your answers in terms of the range of the potential a.

Problem 216. *2001-Spring-QM-G-4* ID:QM-G-215
Find the transcendental equation for the energies at which a particle is not reflected from a potential barrier:

$$V(x) = \alpha\left[\delta(x) + \delta(x-a)\right].$$

Hint: Start with a plane wave coming from the left with no reflected part.

Problem 217. *2001-Spring-QM-G-5* ID:QM-G-218
The interaction Hamiltonian of a two-level atom interacting with a radiation field of frequency ω is given by

$$\hat{H} = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| + dE\left(|a\rangle\langle b|e^{-i\omega t} + |b\rangle\langle a|e^{i\omega t}\right),$$

where $|a\rangle$ and $|b\rangle$ are upper and lower atomic levels with energies $\hbar\omega_a$ and $\hbar\omega_b$, respectively; d is the electric dipole moment; and E is the electric field amplitude. Consider the resonant case when $\omega = \omega_a - \omega_b$. Initially the atom is in the lower level $|b\rangle$. At any later time the state vector can be written as

$$|\Psi(t)\rangle = c_a(t)|a\rangle + c_b(t)|b\rangle.$$

1. Find $c_a(t)$ and $c_b(t)$ in terms of the coupling constant $g = dE/\hbar$.
2. What is the probability to find the atom in the upper level?