Advanced Mechanics I. Phys 302

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LECTURE 1
Introduction. Vectors.

Preliminaries.

- Syllabus.
- Homework. To cheat or not to cheat?
- Homework session.
- Problems (web page).
- Office hours (closed door – no problem)
- Lecture is a conversation.

Physics.

- Introduction
- Vectors, coordinates.
- What can be done with vectors? Linearity, scalar product, vector product.
- Vector components.
  - Scalar product. Einstein notations.
  - Vector product. Determinant. Symbol Levi-Chivita.
  - Useful formulas:
$$\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}, \qquad \epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}.$$
  - Examples:
$$[\vec{a}\times\vec{b}]\cdot[\vec{c}\times\vec{d}] = \epsilon^{ijk}\epsilon^{ilm}a_j b_k c_l d_m = \left(\delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}\right)a_j b_k c_l d_m =$$
$$a_j c_j b_k d_k - a_j d_j b_k c_k = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$
$$[\vec{a}\times[\vec{b}\times\vec{c}]]^i = \epsilon^{ijk}\epsilon^{klm}a_j b_l c_m = \epsilon^{kij}\epsilon^{klm}a_j b_l c_m =$$
$$\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)a_j b_l c_m = b_i a_j c_j - c_i b_j a_j = \left[\vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})\right]^i$$
- Bilinearity.
- Differentiation of scalar and vector products.
- Differentiation of $|\vec{r}|$.

LECTURE 2

Frames of references. Principle of relativity. Newton's first and second law.

- Coordinates, Frames of reference.
- Moving frame of reference:

$$\vec{r} = \vec{R} + \vec{r}'$$
$$\dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{r}}', \qquad \vec{v} = \vec{V} + \vec{v}'$$

- Different meaning of dt and $d\vec{r}$. It is not guaranteed, that dt is the same in all frames of reference.
- If $\vec{V}$ is constant, then $\dot{\vec{v}} = \dot{\vec{v}}'$.
- The laws of physics must be the same in all inertial frames of reference.
- The laws then must be formulated in terms of acceleration.
- Initial conditions: initial position and initial velocity – we need to set up the motion.
- First Newton's law. If there is no force a body will move with constant velocity.
  - What is force? Interaction. Is there a way to exclude the interaction?
  - The existence of a special class of frames of reference – the inertial frames of reference.
- Force, as a vector measure of interaction.
- Point particle and mass.
- The requirement that the laws of physics be the same in all inertial frames of references. The second Newton's law: $\vec{F} = m\vec{a}$.

LECTURE 3
Newton's laws.

- Second Newton's law. Forces are vectors. Superposition.
- Third Newton's law.

In the following I give examples of the use of the Newton's Laws.

- Vertical motion.
- Wedge.
- Wedge with friction.
- Pulley.

LECTURE 4
Air resistance.

- $\vec{F} = m\vec{a}$ works both ways. (Wedge as an example.)
- Momentum $\vec{p} = m\vec{v}$ — usual way. $\vec{F} = \dot{\vec{p}}$.
- Water hose. Force per area
$$f = \rho v^2.$$
Force is proportional to the velocity squared.
- Force of viscous flow. Two infinite parallel plates at distance L from each other. One plate is moving with velocity v in the direction parallel to the plates. There is a viscous liquid in between the plates. What force is acting on the plates?

  The force per area of a viscous flow is proportional to the velocity difference, or derivative $f \sim \partial v_x/\partial y$. Consider a slab of liquid of thickness dy, the total force which acts on a liquid of area S of this slab is $\eta S\left(\frac{\partial v_x}{\partial y}\Big|_y - \frac{\partial v_x}{\partial y}\Big|_{y+dy}\right) = -\eta S dy \frac{\partial^2 v_x}{\partial y^2}$. This force must be equal to $a\rho S dy$. But the acceleration $a = 0$, so
$$\frac{\partial^2 v_x}{\partial y^2} = 0, \qquad v_x(y=0) = v, \qquad v_x(y=L) = 0.$$
The solution of this equation is
$$v_x(y) = v\frac{L-y}{L}.$$
The force per area then is proportional to
$$f \sim \frac{\partial v_x}{\partial y} = v/L.$$
So the force is linear in velocity.
- Air resistance.
  - Linear: $F = -\gamma v$. Finite distance.
$$m\dot{v} = -\gamma v, \qquad v(t=0) = v_0,$$
$$v(t) = v_0 e^{-\frac{\gamma}{m}t}, \qquad l(t) = \int_0^t v(t')dt' = \frac{mv_0}{\gamma}(1 - e^{-\frac{\gamma}{m}t}), \qquad l(t\to\infty) = \frac{mv_0}{\gamma}.$$
  - Quadratic: $F = -\gamma|v|v$. Infinite distance.
$$m\dot{v} = -\gamma v^2, \qquad v(t=0) = v_0,$$
$$\frac{m}{v} = \gamma t + \frac{m}{v_0}, \qquad v(t) = \frac{v_0}{1 + \frac{v_0\gamma}{m}t}, \qquad l(t) = \frac{m}{\gamma}\log\left(1 + \frac{v_0\gamma}{m}t\right).$$

LECTURE 5
Air resistance. Oscillations.

- Air resistance and gravity. Linear case.

$$m\dot{v} = -mg - \gamma v, \qquad v(t=0) = v_0,$$
$$v = v_0 e^{-\frac{\gamma}{m}t} + \frac{mg}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right).$$

  - Limit of $\gamma \to 0$
  - Time to the top. Height. At the top $v_T = 0$,

$$T = \frac{m}{\gamma}\log\left(1 + \frac{\gamma v_0}{mg}\right),$$
$$l(t) = v_0\frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right) - \frac{mg}{\gamma}\left(\frac{m}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right) + t\right)$$

  for $\frac{\gamma v_0}{mg} \ll 1$

$$l(T) \approx \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{3}\frac{\gamma v_0^3}{mg^2}$$

  - Terminal velocity.

$$t \to \infty, \qquad v_\infty = -\frac{mg}{\gamma}, \qquad mg = -v_\infty \gamma$$

Oscillators

- Equation:

$$m\ddot{x} = -kx, \qquad ml\ddot{\phi} = -mg\sin\phi \approx -mg\phi, \qquad -L\ddot{Q} = \frac{Q}{C},$$

  All of these equation have the same form

$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

LECTURE 6
Oscillations. Oscillations with friction.

Oscillations.

- Equation:

$$m\ddot{x} = -kx, \qquad ml\ddot{\phi} = -mg\sin\phi \approx -mg\phi, \qquad -L\ddot{Q} = \frac{Q}{C},$$

  All of these equation have the same form

$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- The solution

$$x(t) = A\sin(\omega t) + B\cos(\omega t) = C\sin(\omega t + \phi), \qquad B = x_0, \qquad \omega A = v_0.$$

- Oscillates forever: $C = \sqrt{A^2 + B^2}$ — amplitude; $\phi = \tan^{-1}(A/B)$ — phase.
- Energy. Conserved quantity: $E = \frac{\dot{x}^2}{2} + \frac{\omega_0^2 x^2}{2}$. It stays constant on a trajectory!

$$\frac{dE}{dt} = \dot{x}\left(\ddot{x} + \omega_0^2 x\right) = 0.$$

Oscillations with friction:

- Equation of motion.

$$m\ddot{x} = -kx - \gamma\dot{x}, \qquad -L\ddot{Q} = \frac{Q}{C} + R\dot{Q},$$

- Consider

$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- Dissipation

$$\frac{dE}{dt} = \dot{x}\left(\ddot{x} + \omega_0^2 x\right) = -2\gamma\dot{x}^2 < 0.$$

  The energy is decreasing!
- Solution: This is a linear equation with constant coefficients. We look for the solution in the form $x = \Re C e^{-i\omega t}$, where ω and C are complex constants.

$$\omega^2 + 2i\gamma\omega - \omega_0^2 = 0, \qquad \omega = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Two solutions, two independent constants.

- Two cases: $\gamma < \omega_0$ and $\gamma > \omega_0$.
- In the first case (underdamping):

$$x = e^{-\gamma t}\Re\left[C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right] = Ce^{-\gamma t}\sin\left(\Omega t + \phi\right), \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}$$

  Decaying oscillations. Shifted frequency.
- In the second case (overdamping):

$$x = Ae^{-\Gamma_- t} + Be^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} > 0$$

- For the initial conditions $x(t=0) = x_0$ and $v(t=0) = 0$ we find $A = x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}$, $B = -x_0\frac{\Gamma_-}{\Gamma_+ - \Gamma_-}$. For $t \to \infty$ the B term can be dropped as $\Gamma_+ > \Gamma_-$, then $x(t) \approx x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}e^{-\Gamma_- t}$.

LECTURE 7
Oscillations with external force. Resonance.

7.1. Different limits.

— *Overdamping:*
We found before that in the overdamped case:

$$x = Ae^{-\Gamma_- t} + Be^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} > 0$$

Consider a limit $\gamma \to \infty$. Then we have

$$\Gamma_+ \approx 2\gamma, \qquad \Gamma_- \approx \frac{\omega_0^2}{\gamma}$$

$$x_+(t) \approx Be^{-2\gamma t}, \qquad x_-(t) \approx Ae^{-\frac{\omega_0}{2\gamma}t}.$$

Let's see where these solutions came from. In the equation

$$\ddot{x} = -\omega_0^2 x - 2\gamma \dot{x}$$

in the limit $\gamma \to \infty$ the last term is huge. It must be compensated by one of the others terms. Let's see what will happen if we drop the $\omega_0^2 x$ term. Then we get the equation $\ddot{x} = -2\gamma\dot{x}$. Its solution is $\dot{x} = Be^{-2\gamma t}$. After one more integration we see, that we will get the $x_+(t)$ solution.

Now let's see what will happen if we drop the $\ddot{x}$ term. We get the equation $\dot{x} = -\frac{\omega_0^2}{2\gamma}x$. Its solution is $x = Ae^{-\frac{\omega_0^2}{2\gamma}t}$ – this is our $x_-(t)$ solution.
— *Case of* $\gamma = 0$, $\omega_0 \to 0$*:*
In this case the equation is

$$\ddot{x} = -\omega_0^2 x \to 0$$

Se we expect to have $\ddot{x} = 0$, or $x(t) = v_0 t + x_0$.

Let's see how we get it out of the exact solution:

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

If we naively take $\omega_0 \to 0$ we will get $x(t) = B$, which is incorrect. What we need to do is to first impose the initial conditions. Then we get

$$x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t) + x_0\cos(\omega_0 t).$$

Now the limit $\omega_0 \to 0$ is not so trivial, as in the first term zero is divided by zero. So we need to use the Taylor expansion $\sin(\omega_0 t) \approx \omega_0 t$. Then we get

$$x(t) = v_0 t + x_0.$$

7.2. External force.

In equilibrium everything is at the minimum of the potential energy, so we have the harmonic oscillator with dissipation. All we measure are the response functions, so we need the know how the harmonic oscillator behaves under external force.

- Let's add an external force:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- The full solution is the sum of the solution of the homogeneous equation with any solution of the inhomogeneous one. This full solution will depend on two arbitrary constants. These constants are determined by the initial conditions.
- Let's assume, that $f(t)$ is not decaying with time. Any solution of the homogeneous solution will decay in time. There is, however, a solution of the inhomogeneous equation which will not decay in time. So in a long time $t \gg 1/\gamma$ the solution of the homogeneous equation can be neglected. In particular this means that the asymptotic of the solution does not depend on the initial conditions.
- Let's now assume that the force $f(t)$ is periodic with some period. It then can be represented by a Fourier series. As the equation is linear the solution will also be a series, where each term corresponds to a force with a single frequency. So we need to solve

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$

  where f is the force's amplitude.

LECTURE 8
Resonance. Response.

8.1. Resonance.

— *Resonance:*

- In the previous lecture we found that for arbitrary $f(t)$ we need to solve:
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$
where f is the force's amplitude.
- Let's look at the solution in the form $x = f\Im C e^{-i\Omega_f t}$, and use $\sin(\Omega_f t) = \Im e^{-i\Omega_f t}$. We then get
$$C = \frac{1}{\omega_0^2 - \Omega_f^2 - 2i\gamma\Omega_f} = |C|e^{i\phi},$$
$$|C| = \frac{1}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}}, \quad \tan\phi = \frac{2\gamma\Omega_f}{\omega_0^2 - \Omega_f^2}$$
$$x(t) = f\Im|C|e^{-i\Omega_f t + i\phi} = f|C|\sin\left(\Omega_f t - \phi\right),$$
- Resonance frequency for the position measurement
$$\Omega_f^r = \sqrt{\omega_0^2 - 2\gamma^2}.$$
- Phase changes sign at $\Omega_f^\phi = \omega_0$.

— *Resonance in velocity measurement*

- The velocity is given by
$$v(t) = \dot{x}(t) = -f\Im i\Omega_f C e^{-i\Omega_f t}.$$
- The velocity amplitude is given by
$$f\Omega_f|C| = f\frac{\Omega_f}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}} = f\frac{1}{\left[(\Omega_f - \omega_0^2/\Omega_f)^2 + 4\gamma^2\right]^{1/2}}$$
- The maximum is when $\Omega_f - \omega_0^2/\Omega_f = 0$, so the resonance frequency for the velocity is ω_0 — without the damping shift.
- Current is velocity.

— *Analysis for small γ.*

- To analyze resonant response we analyze $|C|^2$.
- The most interesting case $\gamma \ll \omega_0$, then the response $|C|^2$ has a very sharp peak at $\Omega_f \approx \omega_0$:

$$|C|^2 = \frac{1}{(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2} \approx \frac{1}{4\omega_0^2}\frac{1}{(\Omega_f - \omega_0)^2 + \gamma^2},$$

  so that the peak is very symmetric.
- $|C|^2_{\max} \approx \frac{1}{4\gamma^2\omega_0^2}$.
- to find HWHM we need to solve $(\Omega_f - \omega_0)^2 + \gamma^2 = 2\gamma^2$, so HWHM $= \gamma$, and FWHM $= 2\gamma$.
- Q factor (quality factor). The good measure of the quality of an oscillator is $Q = \omega_0/\text{FWHM} = \omega_0/2\gamma$. (decay time) $= 1/\gamma$, period $= 2\pi/\omega_0$, so $Q = \pi\frac{\text{decay time}}{\text{period}}$.
- For a grandfather's wall clock $Q \approx 100$, for the quartz watch $Q \sim 10^4$.

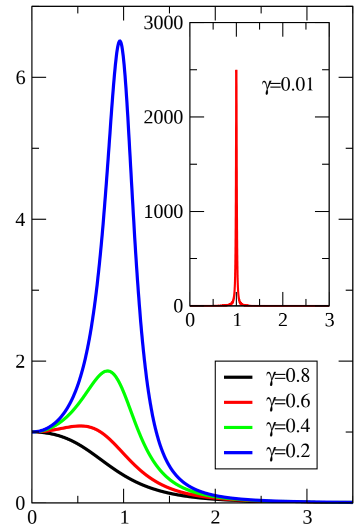


Figure: Resonant response. For insert $Q = 50$.

8.2. Useful points.

- The *complex* response function

$$C(\Omega_f) = \frac{1}{\omega_0^2 - \Omega_f^2 - 2i\gamma\Omega_f}$$

  as a function of *complex* frequency Ω_f has simple poles at $\Omega_f^p = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$. Both poles are in the lower half plane of the complex Ω_f plane. This is always so for any linear response function. It is the consequence of causality!
- The resonator with a high Q is a filter. One can tune this filter by changing the parameters of the resonator.
- By measuring the response function and its HWHM we can measure γ. By changing the parameters such as temperature, fields, etc. we can measure the dependence of γ on these parameters. γ comes from the coupling of the resonator to other degrees of freedom (which are typically not directly observable) so this way we learn something about those other degrees of freedom.

LECTURE 9
Momentum Conservation. Rocket motion. Charged particle in magnetic field.

9.1. Momentum Conservation.

It turns out that the mechanics formulated by Newton implies certain conservation laws. These laws allows us to find answers to many problems/questions without solving equations of motion. Moreover, they are very useful even when it is impossible to solve the equations of motion, as happens, for example, in Stat. Mech. But the most aspect of the conservation laws is that they are more fundamental than the Newtonian mechanics itself. In Quantum mechanics or Relativity, or quantum field theory the very same conservation laws still hold, while the Newtonian mechanics fails.

- Momentum conservation. A bunch of bodies with no external forces. Then for each we have $\dot{\vec{p}} = \vec{F} = \sum_j \vec{F}_{ij}$, where $\vec{F}_{ij}$ is the force with which a body j acts on the body i (we take $F_{ii} = 0$).
- According to the Newton's third law $\vec{F}_{ij} = -\vec{F}_{ji}$.
- Consider the total momentum of the whole bunch $\vec{P} = \sum_i \vec{p}_i$, then

$$\dot{\vec{P}} = \sum_i \dot{p}_i = \sum_{i,j} \vec{F}_{ij} = 0.$$

- So the momentum of a closed system is conserved.
- Examples of the momentum conservation law.

9.2. Rocket motion.

- A rocket burns fuel. The spent fuel is ejected with velocity V in the **frame of reference of the rocket**.
- Let's assume that at some time the velocity of the rocket is v and its mass is m, Its momentum at this moment is mv.
- During a small time interval dt the mass changes by dm and becomes $m+dm$ (where dm is negative), its velocity becomes $v + dv$.
- So it's momentum becomes $(m + dm)(v + dv) \approx mv + mdv + vdm$, and change of the rocket's momentum is $mdv + vdm$.

- The spent fuel has a mass dm_f and has a momentum $(v-V)dm_f$. As the total mass of a rocket with the fuel does not change $dm + dm_f = 0$. So the change of the fuel momentum is $-(v-V)dm$.
- As there is no external force the change of the total momentum must be zero, so

$$
mdv + vdm - (v-V)dm = 0, \qquad mdv = -Vdm,
$$
$$
dv = -V\frac{dm}{m}, \qquad v = V\log\frac{m_{\text{initial}}}{m_{\text{final}}}.
$$

- Notice, that the answer does not depend on the exact form of the function $m(t)$. It depends only on the ratio of the initial mass to the final mass.
- Consider now that there is an external force F_{ex} acting on the rocket. Then we will have

$$
mdv = -Vdm + F_{ex}dt, \qquad m\frac{dv}{dt} = F_{ex} - V\frac{dm}{dt}.
$$

- This equation looks like the second Newton law if we say that there is a new force "thrust"$= -V\frac{dm}{dt}$, which acts on the rocket. Notice, that $\frac{dm}{dt} < 0$, so this force is positive.

9.3. Charged particle in magnetic field.

- Lorentz force: $\vec{F} = q\vec{v}\times\vec{B} + q\vec{E}$.
- No electric field. Trajectories. $gvB = m\omega^2 R = \omega v$. Cyclotron frequency $\omega_c = \frac{qB}{m}$. Cyclotron radius $r_c = \frac{mv}{qB}$.
- Boundary effect.

LECTURE 10
Kinematics in cylindrical coordinates. Vector of angular velocity.

- In $2D$ we can use r and ϕ as coordinates. We can introduce e_r and e_ϕ as unit coordinate vectors. Then

$$\begin{array}{l} e_r = e_x \cos\phi + e_y \sin\phi \\ e_\phi = -e_x \sin\phi + e_y \cos\phi \end{array} ; \qquad \begin{array}{l} e_x = e_r \cos\phi - e_\phi \sin\phi \\ e_y = e_r \sin\phi + e_\phi \cos\phi \end{array}$$

$$\dot{e}_r = \dot{\phi} e_\phi, \qquad \dot{e}_\phi = -\dot{\phi} e_r$$

- The radius vector $\vec{r} = r e_r$. Let's calculate $\vec{v} = \dot{\vec{r}} = \dot{r} e_r + r \dot{e}_r = \dot{r} e_r + r\dot{\phi} e_\phi$.
- Acceleration

$$\vec{a} = \dot{\vec{v}} = \left(\ddot{r} - r\dot{\phi}^2\right) e_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) e_\phi$$

- In the case $r = \text{const}$, $\dot{\phi} = \omega$, $\vec{a} = -r\omega^2 e_r + r\dot{\omega} e_\phi$.
- Free motion: $\vec{a} = 0$,

$$\begin{array}{l} r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \\ \ddot{r} - r\dot{\phi}^2 = 0 \end{array} , \qquad \begin{array}{l} r^2\dot{\phi} = \text{const} = A \\ \ddot{r} - \frac{A^2}{r^3} = 0 \end{array}$$

- Now I will do the following trick. Instead of two functions $r(t)$ and $\phi(t)$ I will consider a function $r(\phi)$ — the trajectory — and use

$$\frac{\partial}{\partial t} = \frac{\partial\phi}{\partial t}\frac{\partial}{\partial\phi} = \dot{\phi}\frac{\partial}{\partial\phi} = \frac{A}{r^2}\partial_\phi; \qquad \dot{r} = \frac{A}{r^2}\partial_\phi r = -A\partial_\phi \frac{1}{r}; \qquad \ddot{r} = -\frac{A^2}{r^2}\partial_\phi^2 \frac{1}{r},$$

then we get

$$\frac{A^2}{r^2}\partial_\phi^2\frac{1}{r} - \frac{A^2}{r^3} = 0, \qquad \partial_\phi^2\frac{1}{r} = -\frac{1}{r}, \qquad \frac{1}{r} = B\cos(\phi - \phi_0)$$

- This is an equation of the straight line in the polar coordinates.
- Notice, if $\dot{\phi} = \omega = \text{const}$, then $a_\phi = 2\dot{r}\omega$ – this is the origin of the Coriolis force.

LECTURE 11
Angular velocity. Angular momentum.

11.1. Angular velocity. Rotation.

- Vector of angular velocity $\vec{\omega}$. For $|\vec{r}| = \text{const.}$:

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

- Sum of two vectors

$$\vec{v}_{13} = \vec{v}_{12} + \vec{v}_{23}, \qquad (\vec{\omega}_{13} - \vec{\omega}_{12} - \vec{\omega}_{23}) \times \vec{r} = 0, \qquad \vec{\omega}_{13} = \vec{\omega}_{12} + \vec{\omega}_{23}$$

- We have a frame rotating with angular velocity $\vec{\omega}$ with respect to the rest frame. A vector $\vec{l}$ constant in the rotating frame will change with time in the rest frame and

$$\dot{\vec{l}} = \vec{\omega} \times \vec{l}.$$

- $\omega = \frac{d\phi}{dt}$, if ω is a vector $\vec{\omega}$, then $d\phi$ must be a vector $\vec{d\phi}$. Notice, that ϕ is not a vector!
- If we rotate one frame with respect to another by a small angle $\vec{d\phi}$, then a vector $\vec{l}$ will change by

$$d\vec{l} = \vec{d\phi} \times \vec{l}.$$

11.2. Angular momentum.

- Consider a vector $\vec{J} = \vec{r} \times \vec{p}$ – vector of angular momentum.
- Consider a bunch of particles which interact with central forces: $\vec{F}_{ij} \parallel \vec{r}_i - \vec{r}_j$. There is also external force $\vec{F}_i^{ex}$ acting on each particle.
- Consider the time evolution of the vector of the total angular momentum $\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i$:

$$\dot{\vec{J}} = \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \left[\sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{ex}\right] = \sum_{i \neq j} \vec{r}_i \times \vec{F}_{ij} + \sum_i \vec{r}_i \times \vec{F}_i^{ex}$$

- The sum $\sum_i \vec{r}_i \times \vec{F}_i^{ex}$ is called torque. Here it is the torque of external forces $\vec{\tau}^{ex}$.

- Consider now the first sum in the RHS. Remember that $\vec{F}_{ij} = -\vec{F}_{ji}$

$$\sum_{i\neq j} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2}\sum_{i\neq j} \vec{r}_i \times \vec{F}_{ij} + \frac{1}{2}\sum_{i\neq j} \vec{r}_j \times \vec{F}_{ji} = \frac{1}{2}\sum_{i\neq j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = 0$$

- So we have

$$\dot{\vec{J}} = \vec{\tau}^{ex}$$

- If the torque of external forces is zero, then the angular momentum is conserved.

LECTURE 12
Moment of inertia. Kinetic energy.

12.1. Moment of inertia.

- Consider a ridged set of particles of masses m_i — the distances between the particles are fixed and do not change. The whole system rotates with the angular velocity $\vec{\omega}$. Each particle has a radius vector $\vec{r}_i$. Let's calculate the angular momentum of the whole system.
$$\vec{J} = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times [\vec{\omega} \times \vec{r}_i] = \sum_i m_i \left(\vec{\omega} \vec{r}_i^{\,2} - \vec{r}_i (\vec{\omega} \cdot \vec{r}_i) \right)$$
or in components
$$J^\alpha = \sum_i \omega^\alpha \vec{r}^{\,2} - r^\alpha \omega^\beta r^\beta = \sum_i m_i \left(\delta^{\alpha\beta} \vec{r}_i^{\,2} - r_i^a r_i^\beta \right) \omega^\beta = I^{\alpha\beta} \omega^\beta,$$
$$I^{\alpha\beta} = \sum_i m_i \left(\delta^{\alpha\beta} \vec{r}_i^{\,2} - r_i^a r_i^\beta \right)$$
- The moment of inertia is a symmetric 3×3 tensor!
$$\hat{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, \qquad I^{\alpha\beta} = I^{\beta\alpha}.$$
$$\vec{J} = \hat{I} \vec{\omega}.$$
- The direction of the angular momentum $\vec{J}$ and direction of the angular velocity $\vec{\omega}$ do not in general coincide!
- It is $\vec{J}$ which is constant when there is no external forces, not $\vec{\omega}$!
- Let's calculate the projection of the angular momentum on $\vec{\omega}$. I denote $\hat{\omega}$ the unit vector along $\vec{\omega}$, so $\vec{\omega} = \omega \hat{\omega}$. Then we want to calculate $\vec{J} \cdot \hat{\omega}$:
$$\vec{J} \cdot \hat{\omega} = \omega \sum_i m_i \left(\vec{r}_i^{\,2} - (\hat{\omega} \cdot \vec{r}_i)(\hat{\omega} \cdot \vec{r}_i) \right) = \omega \sum_i m_i r_{i\perp}^2.$$
- Moment of inertia of a continuous body. Examples.
$$I^{\alpha\beta} = \int \left(\delta^{\alpha\beta} \vec{r}_i^{\,2} - r_i^a r_i^\beta \right) dm.$$
  - A thin ring: $I_{zz} = mR^2$, $I_{xx} = I_{yy} = \frac{1}{2} mR^2$, all off diagonal elements vanish.

  - A disc: $I_{zz} = \frac{1}{2}mR^2$, $I_{xx} = I_{yy} = \frac{1}{4}mR^2$, all off diagonal elements vanish.
  - A sphere: $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mR^2$, all off diagonal elements vanish.
  - A stick at the end: $I_{xx} = I_{yy} = \frac{1}{3}mL^2$.
  - A stick at the center: $I_{xx} = I_{yy} = \frac{1}{12}mL^2$.
- Role of symmetry.

12.2. Kinetic energy.

- Consider the kinetic energy of the moving body.

$$K = \frac{1}{2}\sum_i m_i \vec{v}_i^2 = \frac{1}{2}\sum_i m_i[\vec{\omega} \times \vec{r}_i]^2 = \frac{1}{2}\sum_i m_i[\vec{\omega}^2\vec{r}^2 - (\vec{\omega}\cdot\vec{r})^2] = \frac{I^{\alpha\beta}\omega^\alpha\omega^\beta}{2}$$

- In terms of angular momentum:

$$K = \frac{1}{2}\left(\hat{I}^{-1}\right)^{\alpha\beta} J^\alpha J^\beta.$$

LECTURE 13
Work energy theorem. Energy conservation. Potential energy.

13.1. Mathematical preliminaries.

- Functions of many variables.
- Differential of a function of many variables.
- Examples.

13.2. Work.

- A work done by a force: $\delta W = \vec{F} \cdot d\vec{r}$.
- Superposition. If there are many forces, the total work is the sum of the works done by each.
- Finite displacement. Line integral.

13.3. Change of kinetic energy.

- If a body of mass m moves under the force $\vec{F}$, then.

$$m\frac{d\vec{v}}{dt} = \vec{F}, \qquad md\vec{v} = \vec{F}dt, \qquad m\vec{v} \cdot d\vec{v} = \vec{F} \cdot \vec{v}dt = \vec{F} \cdot d\vec{r} = \delta W.$$

So we have

$$d\frac{mv^2}{2} = \delta W$$

- The change of kinetic energy equals the total work done by all forces.

13.4. Conservative forces. Energy conservation.

- Fundamental forces. Depend on coordinate, do not depend on time.
- Work done by the forces over a closed loop is zero.
- Work is independent of the path.
- Consider two paths: first dx, then dy; first dy then dx

$$\delta W = F_x(x, y)dx + F_y(x + dx, y)dy = F_y(x, y)dy + F_x(x, y + dy)dx, \qquad \left.\frac{\partial F_y}{\partial x}\right|_{x,y} = \left.\frac{\partial F_x}{\partial y}\right|_{x,y}.$$

- So a small work done by a conservative force:

$$\delta W = F_x dx + F_y dy, \qquad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

  is a full differential!

$$\delta W = -dU$$

- It means that there is such a function of the coordinates $U(x, y)$, that

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad \text{or} \quad \vec{F} = -\text{grad}U \equiv -\vec{\nabla}U.$$

- So on a trajectory:

$$d\left(\frac{mv^2}{2} + U\right) = 0, \qquad K + U = \text{const}.$$

- If the force $\vec{F}(\vec{r})$ is known, then there is a test for if the force is conservative.

$$\nabla \times \vec{F} = 0.$$

  In $1D$ the force that depends only on the coordinate is always conservative.
- Examples.

LECTURE 14
One-dimensional motion.

- Last lecture we found that for a conservative (zero work on a closed loop) force there exists a function U such that
$$\delta W = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz = -dU$$
- We then must have
$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad \text{or} \quad \vec{F} = -\text{grad}U \equiv -\vec{\nabla}U.$$
- So as on a trajectory: $\delta W = d\frac{m\vec{v}^2}{2}$ we have
$$d\left(\frac{mv^2}{2} + U\right) = 0, \qquad K + U = \text{const.}$$
- If the force $\vec{F}(\vec{r})$ is known, then there is a test for if the force is conservative.
$$\nabla \times \vec{F} = 0.$$
In $1D$ the force that depends only on the coordinate is always conservative.
- Examples.
- In $1D$ in the case when the force depends only on coordinates the equation of motion can be solved in quadratures.
- The number of conservation laws is enough to solve the equations.
- If the force depends on the coordinate only $F(x)$, then there exists a function — potential energy — with the following property
$$F(x) = -\frac{\partial U}{\partial x}$$
Such function is not unique as one can always add an arbitrary constant to the potential energy.
- The total energy is then conserved
$$K + U = \text{const.}, \qquad \frac{m\dot{x}^2}{2} + U(x) = E$$
- Energy E can be calculated from the initial conditions: $E = \frac{mv_0^2}{2} + U(x_0)$
- The allowed areas where the particle can be are given by $E - U(x) > 0$.
- Turning points. Prohibited regions.

- Notice, that the equation of motion depends only on the difference $E - U(x) = \frac{mv_0^2}{2} + U(x_0) - U(x)$ of the potential energies in different points, so the zero of the potential energy (the arbitrary constant that was added to the function) does not play a role.
- We thus found that

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

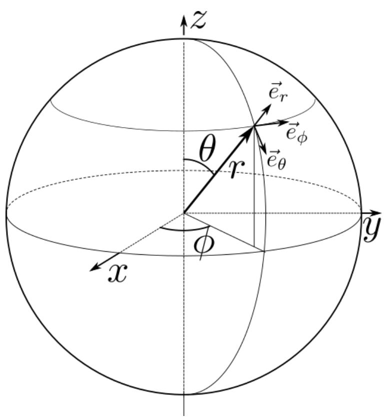
- Energy conservation law cannot tell the direction of the velocity, as the kinetic energy depends only on absolute value of the velocity. In $1D$ it cannot tell which sign to use "$+$" or "$-$". You must not forget to figure it out by other means.
- We then can solve the equation

$$\pm\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E - U(x)}} = dt, \qquad t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}}$$

- Examples:
  - Motion under a constant force.
  - Oscillator.
  - Pendulum.
- Periodic motion. Period.

LECTURE 15
Central forces. Effective potential.

15.1. Spherical coordinates.



- The spherical coordinates are given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \quad .$$

- The coordinates r, θ, and ϕ, the corresponding unit vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$.
- the vector $d\vec{r}$ is then

$$d\vec{r} = \vec{e}_r dr + \vec{e}_\theta r d\theta + \vec{e}_\phi r\sin\theta d\phi.$$
$$d\vec{r} = \vec{e}_x dx + \vec{e}_y dy + \vec{e}_z dz$$

- Imagine now a function of coordinates U. We want to find the components of a vector $\vec{\nabla}U$ in the spherical coordinates.
- Consider a function U as a function of Cartesian coordinates: $U(x, y, z)$. Then

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \vec{\nabla}U \cdot d\vec{r}.$$
$$\vec{\nabla}U = \frac{\partial U}{\partial x}\vec{e}_x + \frac{\partial U}{\partial y}\vec{e}_y + \frac{\partial U}{\partial z}\vec{e}_z$$

- On the other hand, like any vector we can write the vector $\vec{\nabla}U$ in the spherical coordinates.
$$\vec{\nabla}U = (\vec{\nabla}U)_r\vec{e}_r + (\vec{\nabla}U)_\theta\vec{e}_\theta + (\vec{\nabla}U)_\phi\vec{e}_\phi,$$
where $(\vec{\nabla}U)_r$, $(\vec{\nabla}U)_\theta$, and $(\vec{\nabla}U)_\phi$ are the components of the vector $\vec{\nabla}U$ in the spherical coordinates. It is those components that we want to find
- Then
$$dU = \vec{\nabla}U \cdot d\vec{r} = (\vec{\nabla}U)_r dr + (\vec{\nabla}U)_\theta r d\theta + (\vec{\nabla}U)_\phi r\sin\theta d\phi$$
- On yet the other hand if we now consider U as a function of the spherical coordinates $U(r,\theta,\phi)$, then
$$dU = \frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \theta}d\theta + \frac{\partial U}{\partial \phi}d\phi$$
- Comparing the two expressions for dU we find
$$\begin{array}{l} (\vec{\nabla}U)_r = \frac{\partial U}{\partial r} \\ (\vec{\nabla}U)_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta} \\ (\vec{\nabla}U)_\phi = \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} \end{array}.$$
- In particular
$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\vec{e}_r - \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\vec{e}_\phi.$$

15.2. Central force

- Consider a motion of a body under central force. Take the origin in the center of force.
- A central force is given by
$$\vec{F} = F(r)\vec{e}_r.$$
- Such force is always conservative: $\vec{\nabla}\times\vec{F} = 0$, so there is a potential energy:
$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\vec{e}_r, \quad \frac{\partial U}{\partial \theta} = 0, \quad \frac{\partial U}{\partial \phi} = 0,$$
so that potential energy depends only on the distance r, $U(r)$.
- The torque of the central force $\tau = \vec{r}\times\vec{F} = 0$, so the angular momentum is conserved: $\vec{J} = \text{const}$.
- The motion is all in one plane! The plane which contains the vector of the initial velocity and the initial radius vector.
- We take this plane as $x-y$ plane.
- The angular momentum is $\vec{J} = J\vec{e}_z$, where $J = |\vec{J}| = \text{const.}$. This constant is given by initial conditions $J = m|\vec{r}_0 \times \vec{v}_0|$.
- In the $x-y$ plane $\theta = \pi/2$ we can use only r and ϕ coordinates — the polar coordinates.
$$mr^2\dot{\phi} = J, \qquad \dot{\phi} = \frac{J}{mr^2}$$
- The velocity in these coordinates is
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi = \dot{r}\vec{e}_r + \frac{J}{mr}\vec{e}_\phi$$

- The kinetic energy then is

$$K = \frac{m\vec{v}^2}{2} = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2}$$

- The total energy then is

$$E = K + U = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + U(r).$$

- If we introduce the effective potential energy

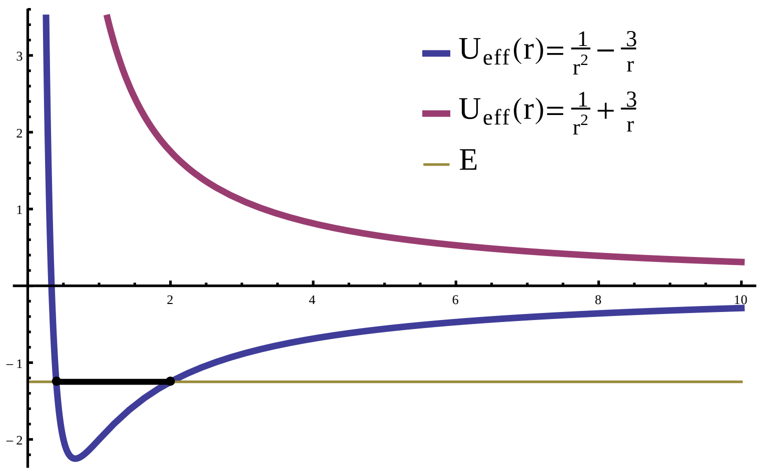
$$U_{eff}(r) = \frac{J^2}{2mr^2} + U(r),$$

then we have

$$\frac{m\dot{r}^2}{2} + U_{eff}(r) = E, \qquad m\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

- This is a one dimensional motion which was solved before.

LECTURE 16
Kepler orbits.



Historically, the Kepler problem — the problem of motion of the bodies in the Newtonian gravitational field — is one of the most important problems in physics. It is the solution of the problems and experimental verification of the results that convinced the physics community in the power of Newton's new math and in the correctness of his mechanics. For the first time people could understand the observed motion of the celestial bodies and make accurate predictions. The whole theory turned out to be much simpler than what existed before.

- In the Kepler problem we want to consider the motion of a body of mass m in the gravitational central force due to much larger mass M.
- As $M \gg m$ we ignore the motion of the larger mass M and consider its position fixed in space (we will discuss what happens when this limit is not applicable later)
- The force that acts on the mass m is given by the Newton's law of gravity:

$$\vec{F} = -\frac{GmM}{r^3}\vec{r} = -\frac{GmM}{r^2}\vec{e}_r$$

  where $\vec{e}_r$ is the direction from M to m.
- The potential energy is then given by

$$U(r) = -\frac{GMm}{r}, \qquad -\frac{\partial U}{\partial r} = -\frac{GmM}{r^2}, \qquad U(r\to\infty)\to 0$$

- The effective potential is

$$U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GMm}{r},$$

  where J is the angular momentum.

- For the Coulomb potential we will have the same r dependence, but for the like charges the sign in front of the last term is different — repulsion.
- In case of attraction for $J \neq 0$ the function $U_{eff}(r)$ always has a minimum for some distance r_0. It has no minimum for the repulsive interaction.
- Looking at the graph of $U_{eff}(r)$ we see, that
  - for the repulsive interaction there can be no bounded orbits. The total energy E of the body is always positive. The minimal distance the body may have with the center is given by the solution of the equation $U_{eff}(r_{min}) = E$.
  - for the attractive interaction if $E > 0$, then the motion is not bounded. The minimal distance the body may have with the center is given by the solution of the equation $U_{eff}(r_{min}) = E$.
  - for the attractive for $U_{eff}(r_{min}) < E < 0$, the motion is bounded between the two real solutions of the equation $U_{eff}(r) = E$. One of the solution is larger than r_0, the other is smaller.
  - for the attractive for $U_{eff}(r_{min}) = E$, the only solution is $r = r_0$. So the motion is around the circle with fixed radius r_0. For such motion we must have

    $$\frac{mv^2}{r_0} = \frac{GmM}{r_0^2}, \qquad \frac{J^2}{mr_0^3} = \frac{GmM}{r_0^2}, \qquad r_0 = \frac{J^2}{Gm^2M}$$

    and

    $$U_{eff}(r_0) = E = \frac{mv^2}{2} - \frac{GmM}{r_0} = -\frac{1}{2}\frac{GmM}{r_0}$$

    these results can also be obtained from the equation on the minimum of the effective potential energy $\frac{\partial U_{eff}}{\partial r} = 0$.
- In the motion the angular momentum is conserved and all motion happens in one plane.
- In that plane we describe the motion by two time dependent polar coordinates $r(t)$ and $\phi(t)$. The dynamics is given by the angular momentum conservation and the effective equation of motion for the r coordinate

$$\dot{\phi} = \frac{J}{mr^2}, \qquad m\ddot{r} = -\frac{\partial U_{eff}(r)}{\partial r}.$$

- For now I am not interested in the time evolution and only want to find the trajectory of the body. This trajectory is given by the function $r(\phi)$. In order to find it I will use the trick we used before

$$\dot{r} = \frac{dr}{dt} = \frac{d\phi}{dt}\frac{dr}{d\phi} = \frac{J}{mr^2}\frac{dr}{d\phi} = -\frac{J}{m}\frac{d(1/r)}{d\phi}, \qquad \frac{d^2r}{dt^2} = \frac{d\phi}{dt}\frac{d\dot{r}}{d\phi} = -\frac{J^2}{m^2r^2}\frac{d^2(1/r)}{d\phi^2}$$

- On the other hand

$$\frac{\partial U_{eff}}{\partial r} = -\frac{J^2}{m}(1/r)^3 + GMm\,(1/r)^2.$$

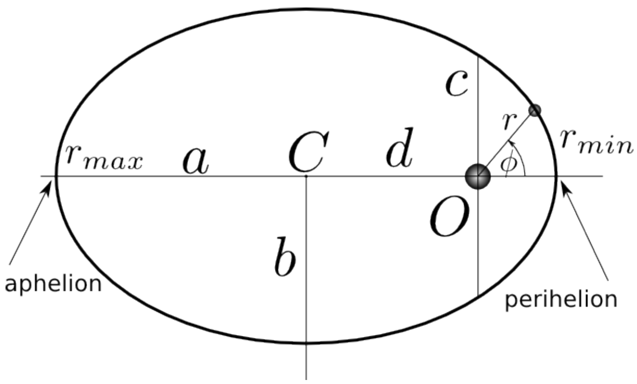
- Now I denote $u(\phi) = 1/r(\phi)$ and get

$$-\frac{J^2}{m}u^2\frac{d^2u}{d\phi^2} = \frac{J^2}{m}u^3 - GMmu^2$$

or

$$u'' = -u + \frac{GMm^2}{J^2}$$

LECTURE 17
Kepler orbits continued



- We stopped at the equation

$$u'' = -u + \frac{GMm^2}{J^2}$$

- The general solution of this equation is

$$u = \frac{GMm^2}{J^2} + A\cos(\phi - \phi_0)$$

- We can put $\phi_0 = 0$ by redefinition. So we have

$$\frac{1}{r} = \gamma + A\cos\phi, \qquad \gamma = \frac{GMm^2}{J^2}$$

  If $\gamma = 0$ this is the equation of a straight line in the polar coordinates.
- A more conventional way to write the trajectory is

$$\frac{1}{r} = \frac{1}{c}\left(1 + \epsilon\cos\phi\right), \qquad c = \frac{J^2}{GMm^2} = \frac{1}{\gamma}$$

  where $\epsilon > 0$ is dimentionless number – eccentricity of the ellipse, while c has a dimension of length
- We see that
  - If $\epsilon < 1$ the orbit is periodic.
  - If $\epsilon < 1$ the minimal and maximal distance to the center — the perihelion and aphelion are at $\phi = 0$ and $\phi = \pi$ respectively.

$$r_{min} = \frac{c}{1+\epsilon}, \qquad r_{max} = \frac{c}{1-\epsilon}$$

  – If $\epsilon > 1$, then the trajectory is unbounded.
- If we know c and ϵ we know the orbit, so we must be able to find out J and E from c and ϵ. By definition of c we find $J^2 = cGMm^2$. In order to find E, we notice, that at $r = r_{min}$, $\dot{r} = 0$, so at this moment $v = r_{min}\dot{\phi} = J/mr_{min}$, so the kinetic energy $K = mv^2/2 = J^2/2mr_{min}^2$, the potential energy is $U = -GmM/r_{min}$. So the total energy is

$$E = K + U = -\frac{1-\epsilon^2}{2}\frac{GmM}{c}, \qquad J^2 = cGMm^2,$$

Indeed we see, that if $\epsilon < 1$, $E < 0$ and the orbit is bounded.
- The ellipse can be written as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with

$$a = \frac{c}{1-\epsilon^2}, \quad b = \frac{c}{\sqrt{1-\epsilon^2}}, \quad d = a\epsilon, \quad b^2 = ac.$$

- One can check, that the position of the large mass M is one of the focuses of the ellipse — NOT ITS CENTER!
- This is the **first Kepler's law**: all planets go around the ellipses with the sun at one of the foci.

17.1. Kepler's second law

The conservation of the angular momentum reads

$$\frac{1}{2}r^2\dot{\phi} = \frac{J}{2m}.$$

We see, that in the LHS rate at which a line from the sun to a comet or planet sweeps out area:

$$\frac{dA}{dt} = \frac{J}{2m}.$$

This rate is constant! So

- **Second Kepler's law:** A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

17.2. Kepler's third law

Consider now the closed orbits only. There is a period T of the rotation of a planet around the sun. We want to find this period.

The total area of an ellipse is $A = \pi ab$, so as the rate dA/dt is constant the period is

$$T = \frac{A}{dA/dt} = \frac{2\pi abm}{J},$$

Now we square the relation and use $b^2 = ac$ and $c = \frac{J^2}{GMm^2}$ to find

$$T^2 = 4\pi^2\frac{m^2}{J^2}a^3c = \frac{4\pi^2}{GM}a^3$$

Notice, that the mass of the planet and its angular momentum canceled out! so

- **Third Kepler's law:** For all bodies orbiting the sun the ration of the square of the period to the cube of the semimajor axis is the same.

This is one way to measure the mass of the sun. For all planets one plots the cube of the semimajor axes as x and the square of the period as y. One then draws a straight line through all points. The slope of that line is $GM/4\pi^2$.

LECTURE 18
Another derivation. Change of orbits. Conserved Laplace-Runge-Lenz vector.

18.1. Another way

- Another way to solve the problem is starting from the following equations:

$$\dot{\phi} = \frac{J}{mr^2(t)}, \qquad \frac{m\dot{r}^2}{2} + U_{eff}(r) = E$$

- For now I am not interested in the time evolution and only want to find the trajectory of the body. This trajectory is given by the function $r(\phi)$. In order to find it I will express $\dot{r}$ from the second equation and divide it by $\dot{\phi}$ from the first. I then find

$$\frac{\dot{r}}{\dot{\phi}} = \frac{dr}{d\phi} = r^2\sqrt{\frac{2m}{J^2}}\sqrt{E - U_{eff}(r)}$$

or

$$\frac{J}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E - U_{eff}(r)}} = d\phi, \qquad \frac{J}{\sqrt{2m}}\int^r \frac{dr'}{r'^2\sqrt{E - U_{eff}(r')}} = \int d\phi$$

The integral becomes a standard one after substitution $x = 1/r$.

18.2. Conserved vector $\vec{A}$

The Kepler problem has an interesting additional symmetry. This symmetry leads to the conservation of the Laplace-Runge-Lenz vector $\vec{A}$. If the gravitational force is $\vec{F} = -\frac{k}{r^2}\vec{e}_r$, then we define:

$$\vec{A} = \vec{p} \times \vec{J} - mk\vec{e}_r,$$

where $\vec{J} = \vec{r} \times \vec{p}$ This vector can be defined for both gravitational and Coulomb forces: $k > 0$ for attraction and $k < 0$ for repulsion.

An important feature of the "inverse square force" is that this vector is conserved. Let's check it. First we notice, that $\dot{\vec{J}} = 0$, so we need to calculate:

$$\dot{\vec{A}} = \dot{\vec{p}} \times \vec{J} - mk\dot{\vec{e}}_r$$

Now using

$$\dot{\vec{p}} = \vec{F}, \qquad \dot{\vec{e}}_r = \vec{\omega} \times \vec{e}_r = \frac{1}{mr^2} \vec{J} \times \vec{e}_r$$

We then see

$$\dot{\vec{A}} = \vec{F} \times \vec{J} - \frac{k}{r^2} \vec{J} \times \vec{e}_r = \left(\vec{F} + \frac{k}{r^2} \vec{e}_r \right) \times \vec{J} = 0$$

So this vector is indeed conserved.

The question is: Is this conservation of vector $\vec{A}$ an independent conservation law? If it is the three components of the vector $\vec{A}$ are three new conservation laws. And the answer is that not all of it.

- As $\vec{J} = \vec{r} \times \vec{p}$ is orthogonal to $\vec{e}_r$, we see, that $\vec{J} \cdot \vec{A} = 0$. So the component of $\vec{A}$ perpendicular to the plane of the planet rotation is always zero.
- Now let's calculate the magnitude of this vector

$$\vec{A} \cdot \vec{A} = \vec{p}^2 \vec{J}^2 - (\vec{p} \cdot \vec{J})^2 + m^2 k^2 - 2mk\vec{e}_r \cdot [\vec{p} \times \vec{J}] = \vec{p}^2 \vec{J}^2 + m^2 k^2 - \frac{2mk}{r} \vec{J} \cdot [\vec{r} \times \vec{p}]$$

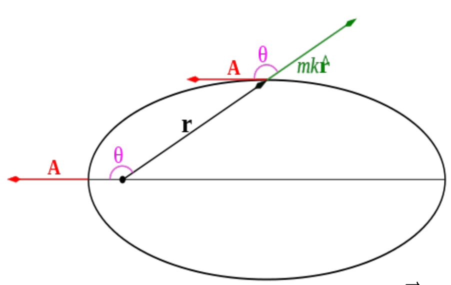
$$= 2m \left(\frac{\vec{p}^2}{2m} - \frac{k}{r} \right) \vec{J}^2 + m^2 k^2 = 2mE\vec{J}^2 + m^2 k^2 = \epsilon^2 k^2 m^2.$$

  So we see, that the magnitude of $\vec{A}$ is not an independent conservation law.
- We are left with only the direction of $\vec{A}$ within the orbit plane. Let's check this direction. As the vector is conserved we can calculate it in any point of orbit. So let's consider the perihelion. At perihelion $\vec{p}_{per} \perp \vec{r}_{per} \perp \vec{J}$, where the subscript *per* means the value at perihelion. So simple examination shows that $\vec{p}_{per} \times \vec{J} = pJ\vec{e}_{per}$. So at this point $\vec{A} = (p_{per}J - mk)\vec{e}_{per}$. However, vector $\vec{A}$ is a constant of motion, so if it has this magnitude and direction in one point it will have the same magnitude and direction at all points! On the other hand $J = p_{per} r_{min}$, so $\vec{A} = mr_{min}(2\frac{p_{per}^2}{2m} - \frac{k}{r_{min}})\vec{e}_{per} = mr_{min}(2K_{per} + U_{per})$. We know that $r_{min} = \frac{c}{1+\epsilon}$, $K_{per} = \frac{1}{2}\frac{k}{c}(1+\epsilon)^2$ and $U_{per} = -\frac{k}{c}(1+\epsilon)$. So

$$\vec{A} = mk\epsilon\vec{e}_{per}.$$

  We see, that for Kepler orbits $\vec{A}$ points to the point of the trajectory where the planet or comet is the closest to the sun.
- So we see, that $\vec{A}$ provides us with only one new independent conserved quantity.

18.2.1. Kepler orbits from $\vec{A}$



The existence of an extra conservation law simplifies many calculations. For example we can derive equation for the trajectories without solving any differential equations. Let's do just that.

Let's derive the equation for Kepler orbits (trajectories) from our new knowledge of the conservation of the vector $\vec{A}$.

$$\vec{r} \cdot \vec{A} = \vec{r} \cdot [\vec{p} \times \vec{J}] - mkr = J^2 - mkr$$

On the other hand

$$\vec{r}\cdot\vec{A} = rA\cos\theta, \quad \text{so} \quad rA\cos\theta = J^2 - mkr$$

Or

$$\frac{1}{r} = \frac{mk}{J^2}\left(1 + \frac{A}{mk}\cos\theta\right), \qquad c = \frac{J^2}{mk}, \qquad \epsilon = \frac{A}{mk}.$$

LECTURE 19
Virial theorem. Kepler orbits for comparable masses.

19.1. Change of orbits.

Consider a problem to change from an circular orbit Γ_1 of a radius R_1 to an orbit Γ_2 with a radius $R_2 > R_1$.

- For the transition we will use an elliptical orbit γ with $r_{min} = R_1$ and $r_{max} = R_2$.
- We need two boosts. One to go from Γ_1 to γ, and the second one to go from γ to Γ_2.
- The final speed on Γ_2 will be less than that on Γ_1.

19.2. Spreading of debris after a satellite explosion.

19.3. Virial theorem

Let's consider a collection of N particles interacting with each other. Let's assume that they undergo some motion with a period T — it also means that we are in the center of mass frame of reference. Then we can define an averaged quantities as follows: Let's imagine that we have a quantity $P(\vec{r}_i, \dot{\vec{r}}_i)$ which depends on the coordinates and the velocities of all particles. Then we define an average

$$\langle P \rangle = \frac{1}{T} \int_0^T P(\vec{r}_i, \dot{\vec{r}}_i) dt$$

Now let's calculate average total kinetic energy $K = \sum_i \frac{m_i \dot{\vec{r}}^2}{2}$

$$\langle K \rangle = \frac{1}{T} \int_0^T \sum_i \frac{m_i \dot{\vec{r}}_i^2}{2} dt = \sum_i \frac{m_i}{2} \frac{1}{T} \int_0^T \dot{\vec{r}}_i^2 dt = \sum_i \frac{m_i}{2} \frac{1}{T} \int_0^T \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i dt$$

Taking the last integral by parts and using the periodicity to cancel the boundary terms we get

$$\langle K \rangle = -\frac{1}{2} \sum_i \frac{1}{T} \int_0^T \vec{r}_i \cdot m_i \ddot{\vec{r}}_i dt = -\frac{1}{2} \sum_i \frac{1}{T} \int_0^T \vec{r}_i \cdot \vec{F}_i dt,$$

where $\vec{F}_i$ is the total force which acts on the particle i.

So we find

$$2\langle K \rangle = -\left\langle \sum_i \vec{r}_i \cdot \vec{F}_i \right\rangle.$$

So far it was all very general. Now lets assume that all the forces are the forces of Coulomb/Gravitation interaction between the particles.

$$\vec{F}_i = \sum_{j\neq i} \vec{F}_{ij}, \qquad \vec{F}_{ij} = -\frac{k}{r_{ij}^2}\vec{e}_{ij},$$

where $\vec{e}_{ij}$ is a unit vector pointing from j to i and $r_{ij} = |\vec{r}_i - \vec{r}_j|$. We then have for any moment of time

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \sum_{i\neq j} \vec{r}_i \cdot \vec{F}_{ij} = \sum_{i>j} (\vec{r}_i - \vec{r}_j)\cdot \vec{F}_{ij} = -\sum_{i>j} r_{ij}\frac{k}{r_{ij}^2} = U,$$

where U is the total potential energy of the system of the particles at the given moment of time. So we have

$$2\langle K\rangle = -\langle U\rangle$$

This is called the virial theorem. It also can be written as $E = -\langle K\rangle$.

It is important, that the above relation is stated for the AVERAGES only. for example in the perihelion of a Kepler orbit we know that $2K_{per}(1+\epsilon) = -U_{per}$.

On the other hand for the circular orbit kinetic and potential energies are constant in time, so the averages are just the values.

19.4. Kepler orbits for comparable masses.

If the bodies interact only with one another and no external force acts on them, then the center of mass has a constant velocity. We then can attach our frame of reference to the center of mass and work there. This way we will only be studying the relative motion of the bodies.

Let's now consider two bodies with masses m_1 and m_2 interacting by a gravitational force. We will use center of mass system of reference and place our coordinate origin at the center of mass. Then if the body m_1 has radius vector $\vec{r}_1$, and the body m_2 has a radius vector $\vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$. So the vector from 2 to 1 is $\vec{r} = \frac{M}{m_2}\vec{r}_1$, or $\vec{r}_1 = \frac{m_2}{M}\vec{r}$. The equation of motion for the mass m_1 is

$$m_1\ddot{\vec{r}}_1 = -\frac{k}{r^2}\vec{e}_r, \qquad \frac{m_1 m_2}{M}\ddot{\vec{r}} = -\frac{k}{r^2}\vec{e}_r, \qquad \mu\ddot{\vec{r}} = -\frac{k}{r^2}\vec{e}_r,$$

where μ is a "reduced mass"

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

We then see, that the problem has reduced to a motion of a single body of a "reduced mass" μ under the same force. This is our standard problem, that we have solved before.

In the case of gravitation we can go further and us $k = Gm_1m_2 = G\frac{m_1m_2}{m_1+m_2}(m_1+m_2) = G\mu M$, so the equation of motion is

$$\mu\ddot{\vec{r}} = -\frac{G\mu M}{r^2}\vec{e}_r,$$

Or just a motion of a particle of mass μ in the gravitational field of a fixed (immovable) mass M.

What one must not forget, though, is that after $\vec{r}(t)$ is found one still need to find $\vec{r}_1(t) = \frac{\mu}{m_1}\vec{r}(t)$ and $\vec{r}_2(t) = \frac{\mu}{m_2}\vec{r}(t)$ to know the positions and motions of the real bodies.

LECTURE 20
Functionals.

20.1. Difference between functions and functionals.

20.2. Examples of functionals.

- Area under the graph.
- Length of a path. Invariance under reparametrization.

It is important to specify the space of functions.

- Energy of a horizontal sting in the gravitational field.
- General form $\int_{x_1}^{x_2} L(x, y, y', y'', \dots)dx$. Important: In function L the y, y', y'' and so on are independent variables. It means that we consider a function $L(x, z_1, z_2, z_3, \dots)$ of normal variables x, z_1, z_2, z_3, …and for any function $y(x)$ at some point x we calculate $y(x)$, $y'(x)$, $y''(x)$, …and plug x and these values instead of z_1, z_2, z_3, …in $L(x, z_1, z_2, z_3, \dots)$. We do that for all points x, and then do the integration.
- Value at a point as functional. The functional which for any function returns the value of the function at a given point.
- Functions of many variables. Area of a surface. Invariance under reparametrization.

LECTURE 21
The Euler-Lagrange equations

21.1. Discretization. Fanctionals as functions.

21.2. Minimization problem

- Minimal distance between two points.
- Minimal time of travel. Ferma Principe.
- Minimal potential energy of a string.
- etc.

LECTURE 22
Euler-Lagrange equation

22.1. The Euler-Lagrange equations

- The functional $A[y(x)] = \int_{x_1}^{x_2} L(y(x), y'(x), x)dx$ with the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.
- The problem is to find a function $y(x)$ which is the stationary "point" of the functional $A[y(x)]$.
- Derivation of the Euler-Lagrange equation.
- The Euler-Lagrange equation reads

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}.$$

22.2. Example

- Shortest path $\int_{x_1}^{x_2} \sqrt{1+(y')^2}dx$, $y(x_1) = y_1$, and $y(x_2) = y_2$.

$$L(y(x), y'(x), x) = \sqrt{1+(y')^2}, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}.$$

the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = 0, \qquad \frac{y'}{\sqrt{1+(y')^2}} = \text{const.}, \qquad y'(x) = \text{const.}, \qquad y = ax + b.$$

The constants a and b should be computed from the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

LECTURE 23
Euler-Lagrange equation continued.

23.1. Example

- Shortest time to fall – Brachistochrone.
  - What path the rail should be in order for the car to take the least amount of time to go from point A to point B under gravity if it starts with zero velocity.
  - Lets take the coordinate x to go straight down and y to be horizontal, with the origin in point A.
  - The boundary conditions: for point A: $y(0) = 0$; for point B: $y(x_B) = y_B$.
  - The time of travel is

$$T = \int \frac{ds}{v} = \int_0^{x_B} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gx}} dx.$$

  - We have

$$L(y, y', x) = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gx}}, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial y'} = \frac{1}{\sqrt{2gx}} \frac{y'}{\sqrt{1 + (y')^2}}.$$

  - The Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0, \qquad \frac{1}{x}\frac{(y')^2}{1 + (y')^2} = \frac{1}{2a}, \qquad y'(x) = \sqrt{\frac{x}{2a - x}}$$

  - So the path is given by

$$y(x) = \int_0^x \sqrt{\frac{x'}{2a - x'}} dx'$$

  - The integral is taken by substitution $x = a(1 - \cos\theta)$. It then becomes $a\int(1 - \cos\theta)d\theta = a(\theta - \sin\theta)$. So the path is given by the parametric equations

$$x = a(1 - \cos\theta), \qquad y = a(\theta - \sin\theta).$$

    the constant a must be chosen such, that the point x_B, y_B is on the path.

23.2. Reparametrization

The *form* of the Euler-Lagrange equation does not change under the reparametrization.

Consider a functional and corresponding E-L equation

$$A = \int_{x_1}^{x_2} L(y(x), y'_x(x), x)dx, \qquad \frac{d}{dx}\frac{\partial L}{\partial y'_x} = \frac{\partial L}{\partial y(x)}$$

Let's consider a new parameter ξ and the function $x(\xi)$ converts one old parameter x to another ξ. The functional

$$A = \int_{x_1}^{x_2} L(y(x), y'_x(x), x)dx = \int_{\xi_1}^{\xi_2} L\left(y(\xi), y'_\xi \frac{d\xi}{dx}, x\right)\frac{dx}{d\xi}d\xi,$$

where $y(\xi) \equiv y(x(\xi))$. So that

$$L_\xi = L\left(y(\xi), y'_\xi \frac{d\xi}{dx}, x\right)\frac{dx}{d\xi}$$

The E-L equation then is

$$\frac{d}{d\xi}\frac{\partial L_\xi}{\partial y'_\xi} = \frac{\partial L_\xi}{\partial y(\xi)}$$

Using

$$\frac{\partial L_\xi}{\partial y'_\xi} = \frac{dx}{d\xi}\frac{\partial L}{\partial y'_x}\frac{d\xi}{dx} = \frac{\partial L}{\partial y'_x}, \qquad \frac{\partial L_\xi}{\partial y(\xi)} = \frac{dx}{d\xi}\frac{\partial L}{\partial y(x)}$$

we see that E-L equation reads

$$\frac{d}{d\xi}\frac{\partial L}{\partial y'_x} = \frac{dx}{d\xi}\frac{\partial L}{\partial y(x)}, \qquad \frac{d}{dx}\frac{\partial L}{\partial y'_x} = \frac{\partial L}{\partial y(x)}.$$

So we return back to the original form of the E-L equation.

What we found is that E-L equations are invariant under the parameter change.

23.3. The Euler-Lagrange equations, for many variables.

23.4. Problems of Newton laws.

- Not invariant when we change the coordinate system:

$$\text{Cartesian:}\begin{cases} m\ddot{x} = F_x \\ m\ddot{y} = F_y \end{cases}, \qquad \text{Cylindrical:}\begin{cases} m\left(\ddot{r} - r\dot{\phi}^2\right) = F_r \\ m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = F_\phi \end{cases}.$$

- Too complicated, too tedious. Consider two pendulums.
- Difficult to find conservation laws.
- Symmetries are not obvious.

LECTURE 24
Discussion of the Exam.

LECTURE 25
Lagrangian mechanics.

25.1. Newton second law as Euler-Lagrange equations

Second order differential equation.

25.2. Hamilton's Principle. Action.

For each conservative mechanical system there exists a functional, called action, which is minimal on the solution of the equation of motion

25.3. Lagrangian.

- Lagrangian is not energy. We do not minimize energy. We minimize action.
- Lagrangian is a function of generalized coordinates $\{q_i\}$ and generalized velocities $\{\dot{q}_i\}$. There must be no momenta in Lagrangian.

25.4. Examples.

- Free fall.
- A mass on a stationary wedge. No friction.
- A mass on a moving wedge. No friction.
- A pendulum.
- A bead on a vertical rotating hoop.

LECTURE 26
Lagrangian mechanics.

26.1. Examples.

- A pendulum on a cart.
- A bead on a vertical rotating hoop.
  - Lagrangian.
$$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mgR(1-\cos\theta).$$
  - Equation of motion.
$$R\ddot{\theta} = (\Omega^2 R\cos\theta - g)\sin\theta.$$
    There are four equilibrium points
$$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$
  - Critical Ω_c. The second two equilibriums are possible only if
$$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$

LECTURE 27
Lagrangian mechanics.

27.1. Examples.

- A bead on a vertical rotating hoop. Continued.
  - In last lecture:
    - Lagrangian.
    $$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mgR(1-\cos\theta).$$
    - Equation of motion.
    $$R\ddot{\theta} = (\Omega^2 R\cos\theta - g)\sin\theta.$$
    There are four equilibrium points
    $$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$
    - Critical Ω_c. The second two equilibriums are possible only if
    $$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$
  - The most interesting regime is $\Omega \sim \Omega_c$ and θ small.
  - Effective potential energy for $\Omega \sim \Omega_c$. From the Lagrangian we can read the effective potential energy:
  $$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta).$$
  Assuming $\Omega \sim \Omega_c$ we are interested only in small θ. So
  $$U_{eff}(\theta) \approx \frac{1}{2}mR^2(\Omega_c^2 - \Omega^2)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$
  $$U_{eff}(\theta) \approx mR^2\Omega_c(\Omega_c - \Omega)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$
  - Spontaneous symmetry breaking. Plot the function $U_{eff}(\theta)$ for $\Omega < \Omega_c$, $\Omega = \Omega_c$, and $\Omega > \Omega_c$. Discuss universality.
  - Small oscillations around $\theta = 0$, $\Omega < \Omega_c$
  $$mR^2\ddot{\theta} = -mR^2(\Omega_c^2 - \Omega^2)\theta, \qquad \omega = \sqrt{\Omega_c^2 - \Omega^2}.$$

– Small oscillations around θ_0, $\Omega > \Omega_c$.

$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta),$$

$$\frac{\partial U_{eff}}{\partial\theta} = -mR(\Omega^2 R\cos\theta - g)\sin\theta, \qquad \frac{\partial^2 U_{eff}}{\partial\theta^2} = mR^2\Omega^2\sin^2\theta - mR\cos\theta(\Omega^2 R\cos\theta - g)$$

$$\left.\frac{\partial U_{eff}}{\partial\theta}\right|_{\theta=\theta_0} = 0, \qquad \left.\frac{\partial^2 U_{eff}}{\partial\theta^2}\right|_{\theta=\theta_0} = mR^2(\Omega^2 - \Omega_c^2)$$

So the Tylor expansion gives

$$U_{eff}(\theta \sim \theta_0) \approx \text{const} + \frac{1}{2}mR^2(\Omega^2 - \Omega_c^2)(\theta-\theta_0)^2$$

The frequency of small oscillations then is

$$\omega = \sqrt{\Omega^2 - \Omega_c^2}.$$

– The effective potential energy for small θ and $|\Omega - \Omega_c|$

$$U_{eff}(\theta) = \frac{1}{2}a(\Omega_c - \Omega)\theta^2 + \frac{1}{4}b\theta^4.$$

– θ_0 for the stable equilibrium is given by $\partial U_{eff}/\partial\theta = 0$

$$\theta_0 = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \sqrt{\frac{a}{b}(\Omega - \Omega_c)} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

Plot $\theta_0(\Omega)$. Non-analytic behavior at Ω_c.

– Response: how θ_0 responses to a small change in Ω.

$$\frac{\partial\theta_0}{\partial\Omega} = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \frac{1}{2}\sqrt{\frac{a}{b}}\frac{1}{\sqrt{(\Omega-\Omega_c)}} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

Plot $\frac{\partial\theta_0}{\partial\Omega}$ vs Ω. The response *diverges* at Ω_c.

LECTURE 28
Lagrangian mechanics.

28.1. Example.

- A double pendulum.
  - Choosing the coordinates.
  - Potential energy.
  - Kinetic energy. Normally, most trouble for students.

28.2. Small Oscillations.

28.3. Generalized momentum.

LECTURE 29
Lagrangian mechanics.

29.1. Generalized momentum.

- For a coordinate q the generalized momentum is defined as

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

- For a particle in a potential field $L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r})$ we have

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

- For a rotation around a fixed axis $L = \frac{I\dot{\phi}^2}{2} - U(\phi)$, then

$$p = \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi} = J.$$

  The generalized momentum is just an angular momentum.

29.2. Ignorable coordinates. Conservation laws.

If one chooses the coordinates in such a way, that the Lagrangian does not depend on say one of the coordinates q_1 (but it still depends on $\dot{q}_1$, then the corresponding generalized momentum $p_1 = \frac{\partial L}{\partial \dot{q}_1}$ is conserved as

$$\frac{d}{dt}p_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} = 0$$

- Problem of a freely horizontally moving cart of mass M with hanged pendulum of mass m and length l.

29.3. Momentum conservation. Translation invariance

Let's consider a translationally invariant problem. For example all interactions depend only on the distance between the particles. The Lagrangian for this problem is $L(\vec{r}_1, \dots \vec{r}_i, \dot{\vec{r}}_1, \dots \dot{\vec{r}}_i)$. Then we add a constant vector ϵ to all coordinate vectors and define

$$L_\epsilon(\vec{r}_1, \dots \vec{r}_i, \dot{\vec{r}}_1, \dots \dot{\vec{r}}_i, \vec{\epsilon}) \equiv L(\vec{r}_1 + \vec{\epsilon}, \dots \vec{r}_i + \vec{\epsilon}, \dot{\vec{r}}_1, \dots \dot{\vec{r}}_i)$$

It is clear, that in the translationally invariant system the Lagrangian will not change under such a transformation. So we find

$$\frac{\partial L_\epsilon}{\partial \vec{\epsilon}} = 0.$$

But according to the definition

$$\frac{\partial L_\epsilon}{\partial \vec{\epsilon}} = \sum_i \frac{\partial L}{\partial \vec{r}_i}.$$

On the other hand the Lagrange equations tell us that

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = \frac{d}{dt}\sum_i \vec{p}_i,$$

so

$$\frac{d}{dt}\sum_i \vec{p}_i = 0, \qquad \sum_i \vec{p}_i = \text{const}.$$

We see, that the total momentum of the system is conserved!

29.4. Non uniqueness of the Lagrangian.

- Let's take a Lagrangian $L(\dot{q}, q, t)$.
- Let's take an arbitrary function $G(q, t)$.
- Let's construct a new Lagrangian $\tilde{L}(\dot{q}, q, t) = L + \dot{q}\frac{\partial G}{\partial t} + \frac{\partial G}{\partial t}$.
- The Lagrange equation

  $$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}} = \frac{\partial \tilde{L}}{\partial q}$$

  is the same as

  $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

  .
- The reason for this is that $\tilde{L} = L + \frac{dG}{dt}$ – I added a full derivative to the Lagrangian. But then the Action changes by

$$\tilde{\mathcal{A}} = \int_{t_i}^{t_f} \tilde{L} dt = \int_{t_i}^{t_f} L dt + \int_{t_i}^{t_f} \frac{dG}{dt} dt = \int_{t_i}^{t_f} L dt + G(q(t_f), t_f) - G(q(t_i), t_i) = \mathcal{A} + \text{const}.$$

  So the variation of the Action does not change, and thus the condition for the extremum — the Euler-Lagrange equation — also does not change.

So one can always add a full time derivative to a Lagrangian.

The last statement is correct only in the classical mechanics.

LECTURE 30
Lagrangian's equations for magnetic forces.

The equation of motion is

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

The question is what Lagrangian gives such equation of motion?

Consider the magnetic field. As there is no magnetic charges one of the Maxwell equations reads

$$\nabla \cdot \vec{B} = 0$$

This equation is satisfied by the following solution

$$\vec{B} = \nabla \times \vec{A},$$

for any vector field $\vec{A}(\vec{r}, t)$.

For the electric field another Maxwell equation reads

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we see that then

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t},$$

where ϕ is the electric potential.

The vector potential $\vec{A}$ and the potential ϕ are not uniquely defined. One can always choose another potential

$$\vec{A}' = \vec{A} + \nabla F, \qquad \phi' = \phi - \frac{\partial F}{\partial t}$$

Such fields are called gauge fields, and the transformation above is called gauge transformation. Such fields cannot be measured.

Notice, that if $\vec{B}$ and $\vec{E}$ are zero, the gauge fields do not have to be zero. For example if $\vec{A}$ and ϕ are constants, $\vec{B} = 0$, $\vec{E} = 0$.

Now we can write the Lagrangian:

$$L = \frac{m\dot{\vec{r}}}{2} - q(\phi - \dot{\vec{r}} \cdot \vec{A})$$

- It is impossible to write the Lagrangian in terms of the physical fields $\vec{B}$ and $\vec{E}$!

- The expression
$$\phi dt - d\vec{r}\cdot\vec{A}$$
is a full differential if and only if
$$-\nabla\phi - \frac{\partial\vec{A}}{\partial t} = 0, \qquad \nabla\times\vec{A} = 0,$$
which means that the it is full differential, and hence can be thrown out, only if the physical fields are zero!

The generalized momenta are
$$\vec{p} = \frac{\partial L}{\partial\dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$
The Lagrange equations are :
$$\frac{d}{dt}\vec{p} = \frac{\partial L}{\partial\vec{r}}$$
Let's consider the x component
$$\frac{d}{dt}p_x = \frac{\partial L}{\partial x},$$
$$m\ddot{x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_x}{\partial y} + q\dot{z}\frac{\partial A_x}{\partial z} + q\frac{\partial A_x}{\partial t} = -q\frac{\partial\phi}{\partial x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x}$$
$$m\ddot{x} = q\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \dot{z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\right)$$
$$m\ddot{x} = q\left(E_x + \dot{y}B_z - \dot{z}B_y\right)$$

LECTURE 31
Kepler's problem in Lagrangian formalism.

31.1. Kepler's problem in Lagrangian formalism.

Coordinates r, θ, ϕ. Potential energy $U(r) = -\frac{GmM}{r}$.

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_\phi = r\sin(\theta)\dot{\phi}.$$

The Lagrangian

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - U(r)$$

ϕ is ignorable.

- Importance of Lagrangian dependence on q and $\dot{q}$, not on q and p.

LECTURE 32
Energy conservation.

32.1. Energy conservation.

We also have the time translation invariance in many systems. It means that the Lagrangian does not explicitly depends on time. So we have $L(q, \dot{q})$, and not $L(q, \dot{q}, \mathbf{t})$. However, the coordinate $q(t)$ does depend on the time. So let's see how the Lagrangian on a trajectory depends on time.

$$\frac{d}{dt}L(q,\dot{q}) = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\partial L}{\partial q}\dot{q} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) - \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) + \dot{q}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)$$

But as we are looking at the real trajectory, then according to the Lagrange equation the last term is zero, so we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L(q,\dot{q})\right) = 0$$

or

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L(q,\dot{q}) = \text{const} = E$$

Using generalized momentum we can write

$$p\dot{q} - L = E, \qquad \text{Constant on trajectory.}$$

If we have many variables q_i, then

$$E = \sum_i p_i \dot{q}_i - L$$

This is another conserved quantity.

Examples:

- A particle in a potential field.
- A particle on a circle.
- A pendulum.
- A cart (mass M) with a pendulum (mass m, length l).

$$L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

- A string with tension and gravity.

LECTURE 33
Hamiltonian.

33.1. Hamiltonian.

Given a Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ the energy

$$E = \sum_i p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is a number defined on a trajectory! One can say that it is a function of initial conditions.

We can construct a function **function** in the following way: we first solve the set of equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

with respect to $\dot{q}_i$, we then have these functions

$$\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$$

and define a function $H(\{q_i\}, \{p_i\})$

$$H(\{q_i\}, \{p_i\}) = \sum_i p_i \dot{q}_i(\{q_j\}, \{p_j\}) - L(\{q_i\}, \{\dot{q}_i(\{q_j\}, \{p_j\})\}),$$

This function is called a Hamiltonian!

The importance of variables:

- A Lagrangian is a function of generalized coordinates and velocities: q and $\dot{q}$.
- A Hamiltonian is a function of the generalized coordinates and **momenta**: q and p.

Here are the steps to get a Hamiltonian from a Lagrangian

(a) Write down a Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ – it is a function of generalized coordinates and velocities q_i, $\dot{q}_i$

(b) Find generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

(c) Treat the above definitions as equations and solve them for all $\dot{q}_i$, so for each velocity $\dot{q}_i$ you have an expression $\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$.

(d) Substitute these function $\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$ into the expression

$$\sum_i p_i \dot{q}_i - L(\{q_i\}, \{\dot{q}_i\}).$$

The resulting function $H(\{q_i\}, \{p_i\})$ of generalized coordinates and momenta is called a Hamiltonian.

33.2. Examples.

- A particle in a potential field.
- Kepler problem.
- Rotation around a fixed axis.
- A pendulum.
- A cart (mass M) with a pendulum (mass m, length l).

$$L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

LECTURE 34
Hamiltonian equations.

- If Lagrangian explicitly depends on time...
- New notation for the partial derivatives. What do we keep fixed?
- Derivation of the Hamiltonian equations.

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

- Energy conservation.
- Velocity.
- Legendre transformation construction.

LECTURE 35
Hamiltonian equations. Examples

Lagrangian→Hamiltonian, Hamiltonian→Lagrangian.

35.1. Examples.

- A particle in a potential field.
- Kepler problem.
- Rotation around a fixed axis.
- A pendulum.
- $\epsilon(p) = c\sqrt{p^2 + m^2c^2}$.
  - Equations of motion.
  - Velocity $\dot{x}$ and c.
  - Lagrangian.
  - Action. Geometrical meaning of Action.

LECTURE 36
Hamiltonian equations. Examples

36.1. Examples.

- $L = \frac{1}{2}\dot{q}_i M_{ij}(\{q_k\})\dot{q}_j - U(\{q_k\})$, where M_{ij} – a symmetric positive definite matrix.
- A cart (mass M) with a pendulum (mass m, length l).

$$L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

36.2. Phase space. Hamiltonian field. Phase trajectories.

- Motion in the phase space.
- Trajectories do not intersect. (Singular points)
- Harmonic oscillator.
- Pendulum.

LECTURE 37
Liouville's theorem. Poincaré recurrence theorem.

37.1. Liouville's theorem.

The phase space volume is conserved under the Hamiltonian flow.

Consider the Hamiltonian flow as a map of the phase space on itself: any initial point (q_0, p_0) is mapped to a point $(q(t), p(t))$ after time t, where $q(t)$ and $p(t)$ are the solutions of the Hamiltonian equations with (q_0, p_0) as initial conditions.

For a small time interval dt the map is given by

$$q_1 = q_0 + \frac{\partial H}{\partial p_0} dt, \qquad p_1 = p_0 - \frac{\partial H}{\partial q_0} dt$$

We can consider these equations as the equations for the change of variables from (q_0, p_0) to (q_1, p_1).

Consider a piece of volume at time $t = 0$: $\mathcal{A}_0 = \int dq_0 dp_0$. After time dt, this volume becomes $\mathcal{A}_1 = \int dq_1 dp_1$. We want to compute the change of this volume $d\mathcal{A} = \mathcal{A}_1 - \mathcal{A}_0$.

$$d\mathcal{A} = \int dq_1 dp_1 - \int dq_0 dp_0 = \int \left(\frac{\partial q_1}{\partial q_0} \frac{\partial p_1}{\partial p_0} - \frac{\partial q_1}{\partial p_0} \frac{\partial p_1}{\partial q_0} \right) dq_0 dp_0 - \int dq_0 dp_0.$$

Using the formulas for our change of variables we find that $d\mathcal{A} \sim (dt)^2$. So that

$$\frac{d\mathcal{A}}{dt} = 0, \qquad \mathcal{A} = \text{const.}$$

- Notice the importance of the minus sign in the Hamiltonian equations.

37.2. Poincaré recurrence theorem.

If the available phase space for the system is finite. Let's starts the motion at some point of the phase space. Let's consider an evolution of some finite but small neighborhood of this point. The volume of the neighborhood is constant, so eventually it will cover all of the available volume. Then the tube of the trajectories must intersect itself. But it cannot, as trajectories do not intersect. It means that it must return to the starting neighborhood (or intersect it at least partially.)

It means that under Hamiltonian dynamics the system will always return arbitrary close to the initial starting point.

The time it will take for the system to return is another matter.

37.3. Area

For a periodic motion the area is given by $\mathcal{A} = \int dpdq$. If we change the energy by dE the area will change the change is the sum of the vector product of the vectors (dq, dp) and $(\frac{\partial q}{\partial E}dE, \frac{\partial p}{\partial E}dE)$, so

$$d\mathcal{A} = -dE \oint \left(\frac{\partial q}{\partial E} dp - \frac{\partial p}{\partial E} dq \right) = -dE \oint \left(\frac{\partial q}{\partial E} \dot{p} - \frac{\partial p}{\partial E} \dot{q} \right) dt = dE \oint \left(\frac{\partial q}{\partial E} \frac{\partial H}{\partial q} + \frac{\partial p}{\partial E} \frac{\partial H}{\partial p} \right) dt$$
$$= dE \oint \left(\frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp \right) \frac{1}{dE} dt = dE \oint \frac{dH}{dE} dt = dE \oint dt = TdE$$

or

$$\frac{d\mathcal{A}}{dE} = T.$$

- Connection to Bohr's quantum mechanics.

LECTURE 38
Poisson brackets.Change of Variables. Canonical variables.

38.1. Poisson brackets.

Consider a function of time, coordinates and momenta: $f(t, q, p)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = \frac{\partial f}{\partial t} + \{H, f\}$$

where we defined the Poisson brackets for any two functions g and f

$$\{g, f\} = \sum_i \left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right)$$

In particular we see, that

$$\{p_i, q_k\} = \delta_{i,k}.$$

Poisson brackets are

- Antisymmetric.
- Bilinear.
- For a constant c, $\{f, c\} = 0$.
- $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.

38.2. Change of Variables.

We want to answer the following question. What change of variables will keep the Hamiltonian equations intact? Namely We have our original variables $\{p\}$ and $\{q\}$ and the original Hamiltonian $H(\{p\}, \{q\})$. The Hamiltonian equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

We want to find the new variables $\{P\}$ and $\{Q\}$, such that the *form* of the Hamiltonian equations for the new variables is the same

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial Q_i}.$$

Let's consider an arbitrary transformation of variables: $P_i = P_i(\{p\},\{q\})$, and $Q_i = Q_i(\{p\},\{q\})$. We then have

$$\dot{P}_i = \{H, P_i\}, \qquad \dot{Q}_i = \{H, Q_i\}.$$

or

$$\dot{P}_i = \sum_k \left[\frac{\partial H}{\partial p_k}\frac{\partial P_i}{\partial q_k} - \frac{\partial H}{\partial q_k}\frac{\partial P_i}{\partial p_k}\right].$$

At this point I want to make the change of variables in the Hamiltonian. For that I invert/solve the equations for the change of variables to get $p_i = p_i(\{P\},\{Q\})$ and $q_i = q_i(\{P\},\{Q\})$ and substitute these functions into the original Hamiltonian $H(\{p\},\{q\})$

$$H(\{p(\{P\},\{Q\})\},\{q(\{P\},\{Q\})\}) \equiv H(\{P\},\{Q\})$$

we then have by the chain rule

$$\frac{\partial H}{\partial p_k} = \sum_\alpha \left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial p_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial p_k}\right), \qquad \frac{\partial H}{\partial q_k} = \sum_\alpha \left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial q_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial q_k}\right)$$

Substituting this into our equation for $\dot{P}_i$ we get

$$\dot{P}_i = \sum_{k,\alpha}\left[\left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial p_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial p_k}\right)\frac{\partial P_i}{\partial q_k} - \left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial q_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial q_k}\right)\frac{\partial P_i}{\partial p_k}\right]$$
$$= -\sum_\alpha \left[\frac{\partial H}{\partial P_\alpha}\{P_i, P_\alpha\} + \frac{\partial H}{\partial Q_\alpha}\{P_i, Q_\alpha\}\right]$$

Analogously,

$$\dot{Q}_i = -\sum_\alpha \left[\frac{\partial H}{\partial Q_\alpha}\{Q_i, Q_\alpha\} + \frac{\partial H}{\partial P_\alpha}\{Q_i, P_\alpha\}\right]$$

We see, that the Hamiltonian equations keep their form if

$$\{P_i, Q_\alpha\} = \delta_{i,\alpha}, \qquad \{P_i, P_\alpha\} = \{Q_i, Q_\alpha\} = 0$$

38.3. Canonical variables.

Such Poisson brackets are called *canonical Poisson brackets*.

The variables that have such Poisson brackets are called the *canonical variables*, they are *canonically conjugated*. Transformations that keep the canonical Poisson brackets are called *canonical transformations*.

- Non-uniqueness of the Hamiltonian.
- Coordinates and momenta obtained from Lagrangian are always canonically conjugated.
- $L = p\dot{q} - H$ only if p and q are canonical variables.
- Canonical Poisson brackets are encoded in the $p\dot{q}$ term.

LECTURE 39
Hamiltonian equations. Jacobi's identity. Integrals of motion.

39.1. Hamiltonian mechanics

- The Poisson brackets are property of the phase space and have nothing to do with the Hamiltonian.
- The Hamiltonian is just a function on the phase space.
- Given the phase space p_i, q_i, the Poisson brackets and the Hamiltonian. We can construct the equations of the Hamiltonian mechanics:
$$\dot{p}_i = \{H, p_i\}, \qquad \dot{q}_i = \{H, q_i\}.$$
- Time evolution of any function $f(p, q, t)$ is given by the equation
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}.$$
difference between the full and the partial derivatives!
- In this formulation there is no need to distinguish between the coordinates and momenta. So we can use $\xi_1 \dots \xi_{2N}$ instead of $q_1 \dots q_N$ and $p_1 \dots p_N$, with given Poisson brackets $\{\xi_i, \xi_j\}$.
- We need a way to compute the Poisson bracket between any two functions f and g if we know all $\{\xi_i, \xi_j\}$. The answer is
$$\{f, g\} = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}.$$
(summation over the repeated indexes is implied.) Let's prove it.

We start with the Poisson bracket of $\{\xi_j, g\}$. In order to compute it we consider ξ_j as our Hamiltonian. This Hamiltonian then gives a flow $\frac{dg}{dt} = \{\xi_j, g\}$. On the other hand, by the chain rule
$$\frac{dg}{dt} = \frac{\partial g}{\partial \xi_i}\frac{d\xi_i}{dt} = \frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\},$$
so we see, that
$$\{\xi_j, g\} = \frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$

To compute the Poisson bracket $\{g, f\}$ we consider the function g as the Hamiltonian, then $\frac{df}{dt} = \{g, f\}$. On the other hand, by the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial \xi_j}\frac{d\xi_j}{dt} = \frac{\partial f}{\partial \xi_j}\{g, \xi_j\} = -\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$

so that

$$\{f, g\} = \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$

- The Poisson brackets must satisfy:
  - Antisymmetric.
  - Bilinear.
  - For a constant c, $\{f, c\} = 0$.
  - $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.

  There is one more identity the Poisson brackets must satisfy – the Jacobi's identity.

39.2. Jacobi's identity

Using the definition of the Poisson brackets in the canonical coordinates it is easy, but lengthy to prove, that for any three functions f, g, and h:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

As it holds for any functions this is the property of the phase space and the Poisson brackets.

Let's take $\xi_1 \ldots \xi_{2N}$, to be *canonical coordinates*, so that $\{\xi_i, \xi_j\} = \text{const}$. Then we can write

$$\{h, \{f, g\}\} = \frac{\partial h}{\partial \xi_p}\frac{\partial}{\partial \xi_l}\left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\right)\{\xi_p, \xi_l\}.$$

Taking the derivative, remembering that $\{\xi_i, \xi_j\} = \text{const}$ and cycle permuting the functions we get

$$\{h, \{f, g\}\} = \frac{\partial h}{\partial \xi_p}\frac{\partial^2 f}{\partial \xi_i \partial \xi_l}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial h}{\partial \xi_p}\frac{\partial f}{\partial \xi_i}\frac{\partial^2 g}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}$$

$$\{g, \{h, f\}\} = \frac{\partial g}{\partial \xi_p}\frac{\partial^2 h}{\partial \xi_i \partial \xi_l}\frac{\partial f}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial g}{\partial \xi_p}\frac{\partial h}{\partial \xi_i}\frac{\partial^2 f}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}, \quad p \to j, j \to l, l \to i, i \to p$$

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial \xi_p}\frac{\partial^2 g}{\partial \xi_i \partial \xi_l}\frac{\partial h}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial^2 h}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}$$

Combining the terms with the same second derivatives, relabeling the indexes, and using antisymmetry of the Poisson brackets we see, that the Jacobi identity is satisfied.

39.3. Commutation of Hamiltonian flows.

For a Hamiltonian H we can introduce the operator $\hat{\zeta}_H$ of the Hamiltonian flow by the following definition: for any function g

$$\hat{\zeta}_H g \equiv \{H, g\}$$

Let's now consider two Hamiltonians H_1 and H_2 and compute the commutator of their flows. Namely, for any function g we have (using Jacobi's identity)

$$\hat{\zeta}_{H_1}\hat{\zeta}_{H_2} g - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1} g = \{H_1, \{H_2, g\}\} - \{H_2, \{H_1, g\}\} = \{\{H_1, H_2\}, g\} = \hat{\zeta}_{\{H_1, H_2\}} g.$$

As this is true for *any* function g we have

$$\hat{\zeta}_{H_1}\hat{\zeta}_{H_2} - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1} = \hat{\zeta}_{\{H_1,H_2\}}.$$

So the commutator of the Hamiltonian flows is also a Hamiltonian flow.

LECTURE 40
Integrals of motion. Angular momentum.

40.1. Time evolution of Poisson brackets.

Consider two arbitrary functions $f(q,p,t)$ and $g(q,p,t)$. Let's compute the time evolution of their Poisson bracket under the Hamiltonian H.

$$\frac{d}{dt}\{f,g\} = \frac{\partial}{\partial t}\{f,g\} + \{H,\{f,g\}\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} + \{\{H,f\},g\} + \{f,\{H,g\}\}$$
$$= \left\{\frac{\partial f}{\partial t} + \{H,f\}, g\right\} + \left\{f, \frac{\partial g}{\partial t} + \{H,g\}\right\}$$

So we get

$$\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}.$$

- Notice, that these are the full derivatives, not partial!!

40.2. Integrals of motion.

A conserved quantity is such a function $f(q,p,t)$, that $\frac{df}{dt} = 0$ under the evolution of a Hamiltonian H. So we have if we have to conserved quantities $f(q,p,t)$ and $g(q,p,t)$, then

$$\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\} = 0$$

So if we have two conserved quantities we can construct a new conserved quantity! Sometimes it will turn out to be an independent conservation law!

40.3. Angular momentum.

Let's calculate the Poisson brackets for the angular momentum: $\vec{M} = \vec{r} \times \vec{p}$.

Coordinate $\vec{r}$ and momentum $\vec{p}$ are canonically conjugated so

$$\{p^i, r^j\} = \delta_{ij}, \qquad \{p^i, p^j\} = \{r^i, r^j\} = 0.$$

So

$$\{M^i, M^j\} = \epsilon^{ilk}\epsilon^{jmn}\{r^l p^k, r^m p^n\} = \epsilon^{ilk}\epsilon^{jmn}\left(r^l\{p^k, r^m p^n\} + p^k\{r^l, r^m p^n\}\right) =$$
$$\epsilon^{ilk}\epsilon^{jmn}\left(r^l p^n\{p^k, r^m\} + r^l r^m\{p^k, p^n\} + p^k p^n\{r^l, r^m\} + p^k r^m\{r^l, p^n\}\right) =$$
$$\epsilon^{ilk}\epsilon^{jmn}\left(r^l p^n\delta_{km} - p^k r^m\delta_{ln}\right) = \left(\epsilon^{ilk}\epsilon^{jkn} - \epsilon^{ikn}\epsilon^{jlk}\right)p^n r^l = p^i r^j - r^i p^j = -\epsilon^{ijk}M^k$$

In short

$$\{M^i, M^j\} = -\epsilon^{ijk}M^k$$

We can now consider a Hamiltonian mechanics, say for the Hamiltonian

$$H = \vec{h}\cdot\vec{M}.$$

In this case

$$\dot{M}^i = \{H, M^i\} = h_j\{M^j, M^i\} = -h_j\epsilon^{jik}M^k,$$

or

$$\dot{\vec{M}} = \vec{h}\times\vec{M}.$$

40.4. Euler equations

Consider a free rigid body with tensor of inertia $\hat{I}$. The Hamiltonian is just the kinetic energy.

$$H = \frac{1}{2}M^i(\hat{I}^{-1})_{ij}M^j.$$

The equations of motion then is

$$\dot{M}^k = \{H, M^k\} = \frac{1}{2}\{M^i, M^k\}(\hat{I}^{-1})_{ij}M^j + \frac{1}{2}M^i(\hat{I}^{-1})_{ij}\{M^j, M^k\} = \epsilon^{kil}M^l(\hat{I}^{-1})_{ij}M^j.$$

Let's write this equation in the system of coordinates of the principal axes of the body. Then the tensor of inertia is diagonal, and for x component we get

$$\dot{M}^x = M^z I_{yy}^{-1}M^y - M^y I_{zz}^{-1}M^z.$$

or, using that $M^x = I_{xx}\Omega^x$, etc we get

$$I_{xx}\dot{\Omega}^x = (I_{zz} - I_{yy})\Omega^z\Omega^y,$$

and two more equations under the cyclic permutations.

- Three degrees of freedom. We must have three second order differential equations for complete description. We have only three first order equations. Three more equations are missing.
- The equations are written for the components of $\vec{\Omega}$ in the internal system of coordinates which is rotating with $\vec{\Omega}$.
- In order to find the orientation of the rigid body as a function of time we need to write and solve three more first order differential equations.

LECTURE 41
Hamilton-Jacobi equation

41.1. Momentum.

Consider an action

$$\mathcal{S} = \int_{t_0}^{t_1} L(q, \dot{q}, t)dt, \qquad q(t_0) = q_0, \quad q(t_1) = q_1.$$

Consider the value of the action on the trajectory as a function of q_1. We want to see how this value changes when we change the q_1.

If we change the upper limit from q_1 to $q_1 + dq_1$ the trajectory will also change from $q(t)$ to $q(t) + \delta q(t)$, where $\delta q(t_0) = 0$, and $\delta q(t_1) = dq_1$. The change of the action then is

$$d\mathcal{S} = \int_{t_0}^{t_1} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_0}^{t_1} L(q, \dot{q}, t)dt = \frac{\partial L}{\partial \dot{q}} \delta q(t)\bigg|_{t_0}^{t_1} = p dq_1.$$

So we have

$$\frac{\partial \mathcal{S}}{\partial q} = p.$$

41.2. Energy.

Consider an action

$$\mathcal{S} = \int_{t_0}^{t_1} L(q, \dot{q}, t)dt, \qquad q(t_0) = q_0, \quad q(t_1) = q_1.$$

Consider the value of the action on the trajectory as a function of t_1.

Notice, that t_1 is there in two places: as the upper limit of integration and in the boundary condition. We do not change the value of q at the upper limit! but the trajectory changes!. So we have

$$\mathcal{S}(t_1 + dt_1) = \int_{t_0}^{t_1 + dt_1} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt = L dt_1 + \int_{t_0}^{t_1} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt.$$

Using the usual trick we will get

$$d\mathcal{S} = L dt_1 + p\delta q|_{t_0}^{t_1} = L dt_1 - p\dot{q} dt_1.$$

So we have

$$\frac{\partial \mathcal{S}}{\partial t} = -H.$$

41.3. Hamilton-Jacobi equation

We have

$$-\frac{\partial \mathcal{S}}{\partial t} = H(p, q, t),$$

but $p = \frac{\partial \mathcal{S}}{\partial q}$, so we can write

$$-\frac{\partial \mathcal{S}}{\partial t} = H\left(\frac{\partial \mathcal{S}}{\partial q}, q, t\right).$$

This is a partial differential equation for the function $\mathcal{S}(q, t)$. This equation is called Hamilton-Jacobi equation.

Let's imagine, that we solved this equation and found the function $\mathcal{S}(q, t, \alpha_1 \dots \alpha_N)$, where N is the number of the coordinates. Let's see how $\frac{\partial \mathcal{S}}{\partial \alpha_i}$ depends on time. We have from the Hamilton-Jacobi equation.

$$-\frac{\partial}{\partial t}\frac{\partial \mathcal{S}}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} H\left(\frac{\partial \mathcal{S}}{\partial q}, q, t\right) = \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\frac{\partial \mathcal{S}}{\partial \alpha_i} = \dot{q}\frac{\partial}{\partial q}\frac{\partial \mathcal{S}}{\partial \alpha_i}.$$

Or we see, that

$$\frac{d}{dt}\frac{\partial \mathcal{S}}{\partial \alpha_i} = \left(\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q}\right)\frac{\partial \mathcal{S}}{\partial \alpha_i} = 0.$$

So all $\frac{\partial \mathcal{S}}{\partial \alpha_i}$ do not change with time and are constants. Then the N equations

$$\frac{\partial \mathcal{S}}{\partial \alpha_i} = \beta_i$$

are implicit definitions of the solutions of the equations of motions $q(t, \alpha_i, \beta_i)$ that depend on $2N$ arbitrary constants, which are given by initial conditions.

The quasiclassical approximation of quantum mechanics $\hbar \to 0$ transfers the Schrödinger equation into the Hamilton-Jacobi equation.