Advanced Mechanics I. Phys 302

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Contents

LECTURE 1
Introduction. Vectors.

Preliminaries.

- Contact info.
- Syllabus. Homework 40%, First exam 30%, Final 30%.
- Attendance policy.
- Zoom. Mute. Use space bar. Pin video. No video.
- Lecture, feedback. Going too fast, etc.
- Office hours (Mondays 12 pm on Zoom)
- Canvas.
- Homework submissions through Canvas. PDF SINGLE FILE.
- Homeworks due by Wednesday's lectures.
- Homework session. Wednesday 12pm on Zoom.
- Homeworks: To cheat or not to cheat? collaborations!!!!! make study groups, mistakes, etc.
- Honors problems. Indicate if you are an Honors student on top of your homework.
- Homework sessions (Wednesday 12pm on Zoom)
- Grading. Every assignment is 100 pt. The points split equally between the problems in a given assignment.
- Exams. The same point system as in homework assignments. First exam is take home. Most of the problems are taken from the problem bank. The bank is on the web.
- Book. Lecture notes. Zoom and lecture notes.
- Language.
- Course content and philosophy.
- Questions: profound vs. stupid.
- Lecture is a conversation.

LECTURE 2

Coordinates. Frames of references. Newton's first and second laws.

2.1. Coordinates, scalars, vectors.

- Coordinates. Coordinate systems. You chose a coordinate system to describe a process (positions, motion, fields, etc) The physical process does not depend on the system of coordinates you use!
- Scalars. Vectors.
- Vector components.
- Vectors and scalars are independent of the coordinate systems. Vector components do depend on the coordinate system which you use.
- What can be done with vectors? Linearity, scalar (dot) product, vector (cross) product. The idea is to keep their independence from the coordinate system.
  - Scalar (dot) product. Coordinate independent definition. Bilinear. Symmetric.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\phi) = \sum_{i=1}^{3} a^i b^i \equiv a^i b^i \text{ — Einstein notations.}$$

  - Vector (cross) product. Coordinate independent definition.

$$\vec{c} = \vec{a} \times \vec{b}, \qquad |\vec{c}| = |\vec{a}||\vec{b}| \sin(\phi), \qquad \text{Direction — right hand rule.}$$

    Bilinear. Antisymmetric – RHR, this is why it is $\sin(\phi)$ and not $\cos(\phi)$. Determinant.
- Symbol Levi-Chivita.
  - Useful formulas:

$$\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}, \qquad \epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}.$$

  Notice the use of Einstein notations.
- Examples:
  - Vector product $\vec{c} = \vec{a} \times \vec{b}$:

$$\begin{aligned} c^i &= [\vec{a} \times \vec{b}]^i = \epsilon^{ijk} a^j b^k \\ c^x &= [\vec{a} \times \vec{b}]^x = \epsilon^{xyz} a^y b^z + \epsilon^{xzy} a^z b^y = a^y b^z - a^z b^y, \quad \text{etc.} \end{aligned}$$

    Importance of the order of indexes.

- Scalar product of two vector products:

$$[\vec{a}\times\vec{b}]\cdot[\vec{c}\times\vec{d}] = [\vec{a}\times\vec{b}]^i[\vec{c}\times\vec{d}]^i = \epsilon^{ijk}\epsilon^{ilm}a_jb_kc_ld_m = \left(\delta^{jl}\delta^{km}-\delta^{jm}\delta^{kl}\right)a_jb_kc_ld_m =$$
$$a_jc_jb_kd_k - a_jd_jb_kc_k = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

- Triple vector product:

$$[\vec{a}\times[\vec{b}\times\vec{c}]]^i = \epsilon^{ijk}\epsilon^{klm}a_jb_lc_m = \epsilon^{kij}\epsilon^{klm}a_jb_lc_m =$$
$$\left(\delta^{il}\delta^{jm}-\delta^{im}\delta^{jl}\right)a_jb_lc_m = b_ia_jc_j - c_ib_ja_j = \left[\vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})\right]^i$$

so

$$[\vec{a}\times[\vec{b}\times\vec{c}]] = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

- Bilinearity.
- Differentiation of scalar and vector products.
  - Example: Consider a unit vector $\vec{n}(t)$ which depends on time t (or any other parameter). As $\vec{n}$ is a unit vector we have $\vec{n}\cdot\vec{n}=1$. Differentiating with respect to time gives $\dot{\vec{n}}\cdot\vec{n}=0$ — the derivative is orthogonal to the vector $\vec{n}$ at all times.
  - Notations:

$$\dot{f}\equiv\frac{df}{dt}.$$

- Differentiation of $|\vec{r}|$. We start with $|\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}}$, then

$$\frac{d}{dt}|\vec{r}| = \frac{d}{dt}\sqrt{\vec{r}\cdot\vec{r}} = \frac{\frac{d(\vec{r}\cdot\vec{r})}{dt}}{2\sqrt{\vec{r}\cdot\vec{r}}} = \frac{\vec{r}\cdot\dot{\vec{r}}}{|\vec{r}|}.$$

2.2. Frames of reference.

Definitions:

If $\vec{r}$ is a position vector, then

$$\dot{\vec{r}}\equiv\vec{v}\text{ — velocity, the rate of change of the position,}$$
$$\dot{\vec{v}}=\ddot{\vec{r}}\equiv\vec{a}\text{ — acceleration, the rate of change of the velocity.}$$

All three: the position $\vec{r}$, the velocity $\vec{v}$, and the acceleration $\vec{a}$ are vectors!!!

- Moving frame of reference:

$$\vec{r} = \vec{R}+\vec{r}'$$
$$\dot{\vec{r}} = \dot{\vec{R}}+\dot{\vec{r}}', \qquad \vec{v}=\vec{V}+\vec{v}'$$

- Different meaning of dt and $d\vec{r}$. It is not guaranteed, that dt is the same in all frames of reference.
- If $\vec{V}$ is constant, then $\dot{\vec{v}} = \dot{\vec{v}}'$.
- The laws of physics must be the same in all inertial frames of reference.
- The laws then must be formulated in terms of acceleration.
- Initial conditions: initial position and initial velocity – we need to set up the motion.
- First Newton's law. If there is no force a body will move with constant velocity.
  - What is force? Interaction. Is there a way to exclude the interaction?
  - The existence of a special class of frames of reference – the inertial frames of reference.

- Force, as a vector measure of interaction.
- Point particle and mass.
- The requirement that the laws of physics be the same in all inertial frames of references. The second Newton's law: $\vec{F} = m\vec{a}$.

LECTURE 3
Newton's laws.

- Second Newton's law. You must have/identify the object! Forces are vectors. Superposition.
- $\vec{F} = m\vec{a}$ — tree second order non-linear differential equations.
- Third Newton's law.

In the following I give very simple examples of the use of the Newton's Laws. $\vec{F} = m\vec{a}$ works both ways.

- Given the motion we can find the total force.
  - Going around a circle.
  - Archimedes law.
- Given the force we can find the motion.
  - Vertical motion.
  - Wedge.
  - Wedge with friction.
  - Pulley.

LECTURE 4
Air resistance.

- Momentum $\vec{p} = m\vec{v}$ — usual way. $\vec{F} = \dot{\vec{p}}$.
- Water hose. Force per area
$$f = \rho v^2.$$
Force is proportional to the velocity squared.
- Force of viscous flow. Two infinite parallel plates at distance L from each other. The plate is moving with velocity v in the direction parallel to the plates, which we will take as $\hat{x}$ direction. There is a viscous liquid in between the plates. What force is acting on the plates?

  The force per area of a viscous flow is proportional to the velocity difference, or derivative $f \sim -\partial v_x/\partial y$. Consider a slab of liquid of thickness dy, the total force which acts on a liquid of area S of this slab is $\eta S\left(-\frac{\partial v_x}{\partial y}\Big|_y + \frac{\partial v_x}{\partial y}\Big|_{y+dy}\right) = \eta S dy \frac{\partial^2 v_x}{\partial y^2}$. This force must be equal to $a\rho S dy$. But the acceleration $a = 0$, so
$$\frac{\partial^2 v_x}{\partial y^2} = 0, \qquad v_x(y=0) = 0, \qquad v_x(y=L) = v.$$
The solution of this equation is
$$v_x(y) = v\frac{y}{L}.$$
The force per area then is proportional to
$$f \sim -\frac{\partial v_x}{\partial y} = -v/L.$$
So the force is linear in velocity.

LECTURE 5
Air resistance.

- Air resistance. We consider two model cases: the air resistance is proportional to v, or to v^2 – linear, or quadratic. These forms of the air resistance should not be taken literary. These two cases are just models we will use to learn how the motion depends on the forms of the air resistance.
  - Linear: $F = -\gamma v$. Finite distance.
    Units of $[\gamma] = kg/s$.

$$m\dot{v} = -\gamma v, \qquad v(t=0) = v_0, \qquad l(t=0) = 0,$$

$$v(t) = v_0 e^{-\frac{\gamma}{m}t}, \qquad l(t) = \int_0^t v(t')dt' = \frac{mv_0}{\gamma}(1 - e^{-\frac{\gamma}{m}t}), \qquad l(t \to \infty) = \frac{mv_0}{\gamma}.$$

    If $\frac{\gamma}{m}t \ll 1$, then

$$v(t) \approx v_0 - v_0\frac{\gamma t}{m},$$
$$l(t) \approx v_0 t - \frac{1}{2}v_0 t\frac{\gamma t}{m}.$$

  - Quadratic: $F = -\gamma|v|v$. Infinite distance.
    Units of $[\gamma] = kg/m$

$$m\dot{v} = -\gamma v^2, \qquad v(t=0) = v_0, \qquad l(t=0) = 0,$$

$$\frac{m}{v} = \gamma t + \frac{m}{v_0}, \qquad v(t) = \frac{v_0}{1 + \frac{v_0\gamma}{m}t}, \qquad l(t) = \frac{m}{\gamma}\log\left(1 + \frac{v_0\gamma}{m}t\right).$$

    If $\frac{v_0\gamma}{m}t \ll 1$, then

$$v(t) \approx v_0 - v_0\frac{v_0\gamma}{m}t,$$
$$l(t) \approx v_0 t - v_0 t\frac{1}{2}\frac{v_0\gamma}{m}t.$$

- Air resistance and gravity. Linear case.

$$m\dot{v} = -mg - \gamma v, \qquad v(t=0) = v_0, \qquad l(t=0) = 0.$$

so

$$v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{mg}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right).$$
$$l(t) = v_0 \frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right) - \frac{mg}{\gamma}\left(\frac{m}{\gamma}\left(e^{-\frac{\gamma}{m}t} - 1\right) + t\right)$$

– Limit of $\gamma t/m \ll 1$:

$$v \approx v_0 - gt$$
$$l(t) \approx v_0 t - \frac{gt^2}{2}$$

In our condition $\gamma t/m \ll 1$ what t should we use? – depends on the problem.

– Time to the top. Height. At the top $v_T = 0$,

$$T = \frac{m}{\gamma}\log\left(1 + \frac{\gamma v_0}{mg}\right),$$

for $\frac{\gamma v_0}{mg} \ll 1$

$$T \approx \frac{m}{\gamma}\frac{\gamma v_0}{mg} - \frac{1}{2}\frac{m}{\gamma}\left(\frac{\gamma v_0}{mg}\right)^2 = \frac{v_0}{g} - \frac{1}{2}\frac{v_0}{g}\frac{\gamma v_0}{mg},$$
$$l(T) \approx \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{3}\frac{\gamma v_0^3}{mg^2}$$

– Terminal velocity.

$$t \to \infty, \qquad v_\infty = -\frac{mg}{\gamma}, \qquad mg = -v_\infty \gamma$$

LECTURE 6
Oscillations. Oscillations with friction.

Oscillations.

- Equation:
$$m\ddot{x} = -kx, \qquad ml\ddot{\phi} = -mg\sin\phi \approx -mg\phi, \qquad -L\ddot{Q} = \frac{Q}{C},$$
All of these equation have the same form
$$\ddot{x} = -\omega_0^2 x, \qquad \omega_0^2 = \begin{cases} k/m \\ g/l \\ 1/LC \end{cases}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$
- The general solution is
$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) = C\sin(\omega_0 t + \phi),$$
where A and B are arbitrary constants. $C = \sqrt{A^2 + B^2}$ — amplitude; $\phi = \tan^{-1}(A/B)$ — phase.
- The velocity as a function of time is
$$v(t) = \dot{x} = \omega_0 A\cos(\omega_0 t) - \omega_0 B\sin(\omega_0 t).$$
- Our initial conditions give
$$x(t=0) = B = x_0, \qquad v(t=0) = A\omega_0 = v_0,$$
so the arbitrary constants are given by
$$B = x_0, \qquad A = \frac{v_0}{\omega_0}.$$
(check units)
- Oscillates forever. The frequency of oscillations does not depend on the initial conditions and can be read straight from the equation of motion. This is the property of harmonic oscillations. It also means, that the frequency is the property of the system itself, not of the way we set up the motion.
- Energy. Conserved quantity: $E = \frac{\dot{x}^2}{2} + \frac{\omega_0^2 x^2}{2}$. It stays constant on a trajectory!
$$\frac{dE}{dt} = \dot{x}\left(\ddot{x} + \omega_0^2 x\right) = 0.$$

Oscillations with friction:

- Equation of motion.

$$m\ddot{x} = -kx - \gamma\dot{x}, \qquad -L\ddot{Q} = \frac{Q}{C} + R\dot{Q},$$

- Consider

$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}, \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$

- Units of γ are s^{-1} – the same as for ω_0.
- Dissipation

$$\frac{dE}{dt} = \dot{x}\left(\ddot{x} + \omega_0^2 x\right) = -2\gamma\dot{x}^2 < 0.$$

If $\gamma > 0$, the energy is decreasing! – dissipation!
- Solution: This is a linear equation with constant coefficients. We look for the solution in the form $x = \Re C e^{-i\omega t}$, where ω and C are complex constants.

$$\omega^2 + 2i\gamma\omega - \omega_0^2 = 0, \qquad \omega = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

- Two solutions, two independent constants.
- Two cases: $\gamma < \omega_0$ and $\gamma > \omega_0$.
- In the first case $\gamma < \omega_0$ (underdamping):

$$x = e^{-\gamma t}\Re\left[C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right] = Ce^{-\gamma t}\sin\left(\Omega t + \phi\right), \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}$$

Decaying oscillations. Shifted frequency. For $\gamma \ll \omega_0$ we can use the Taylor expansion

$$\Omega \approx \omega_0 - \frac{1}{2}\frac{\gamma^2}{\omega_0}$$

- In the second case $\gamma > \omega_0$ (overdamping):

$$x = Ae^{-\Gamma_- t} + Be^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} > 0, \qquad \Gamma_+ > \Gamma_-.$$

- For the initial conditions $x(t=0) = x_0$ and $v(t=0) = 0$ we find

$$A = x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}, \qquad B = -x_0\frac{\Gamma_-}{\Gamma_+ - \Gamma_-}$$

For $t \to \infty$ the B term can be dropped as $\Gamma_+ > \Gamma_-$, then $x(t) \approx x_0\frac{\Gamma_+}{\Gamma_+ - \Gamma_-}e^{-\Gamma_- t}$.

LECTURE 7
Oscillations with external force. Resonance.

7.1. Different limits.

— *Overdamping:*
We found before that in the overdamped case:

$$x = Ae^{-\Gamma_- t} + Be^{-\Gamma_+ t}, \qquad \Gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} > 0$$

Consider a limit $\gamma \to \infty$. Then we have

$$\Gamma_+ \approx 2\gamma, \qquad \Gamma_- \approx \frac{\omega_0^2}{\gamma}$$

$$x_+(t) \approx Be^{-2\gamma t}, \qquad x_-(t) \approx Ae^{-\frac{\omega_0}{2\gamma}t}.$$

Let's see where these solutions came from. In the equation

$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}$$

in the limit $\gamma \to \infty$ the last term is huge. It must be compensated by one of the others terms. Let's see what will happen if we drop the $\omega_0^2 x$ term. Then we get the equation $\ddot{x} = -2\gamma\dot{x}$. Its solution is $\dot{x} = Be^{-2\gamma t}$. After one more integration we see, that we will get the $x_+(t)$ solution.

Now let's see what will happen if we drop the $\ddot{x}$ term. We get the equation $\dot{x} = -\frac{\omega_0^2}{2\gamma}x$. Its solution is $x = Ae^{-\frac{\omega_0^2}{2\gamma}t}$ – this is our $x_-(t)$ solution.
— *Case of $\gamma = 0$, $\omega_0 \to 0$:*
In this case the equation is

$$\ddot{x} = -\omega_0^2 x \to 0$$

Se we expect to have $\ddot{x} = 0$, or $x(t) = v_0 t + x_0$.

Let's see how we get it out of the exact solution:

$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

If we naively take $\omega_0 \to 0$ we will get $x(t) = B$, which is incorrect. What we need to do is to first impose the initial conditions: $x(t = 0) = x_0$ and $v(t = 0) = v_0$. Then we get

$$x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t) + x_0\cos(\omega_0 t).$$

Now the limit $\omega_0 \to 0$ is not so trivial, as in the first term zero is divided by zero. So we need to use the Taylor expansion $\sin(\omega_0 t) \approx \omega_0 t$. Then we get

$$x(t) = v_0 t + x_0.$$

7.2. External force.

In equilibrium everything is at the minimum of the potential energy, so we have the harmonic oscillator with dissipation. All we measure are the response functions, so we need the know how the harmonic oscillator behaves under external force.

- Let's add an external force:
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t), \qquad x(t=0) = x_0, \quad v(t=0) = v_0.$$
- The full solution is the sum of the solution of the homogeneous equation with any solution of the inhomogeneous one. This full solution will depend on two arbitrary constants. These constants are determined by the initial conditions.
- Let's assume, that $f(t)$ is not decaying with time. Any solution of the homogeneous equation will decay in time. There is, however, a solution of the inhomogeneous equation which will not decay in time. So in a long time $t \gg 1/\gamma$ the solution of the homogeneous equation can be neglected. In particular this means that the asymptotic of the solution does not depend on the initial conditions.
- Let's now assume that the force $f(t)$ is periodic with some period. It then can be represented by a Fourier series. As the equation is linear the solution will also be a series, where each term corresponds to a force with a single frequency. So we need to solve
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$
where f is the force's amplitude.

LECTURE 8
Resonance. Response.

8.1. Resonance.

— *Resonance:*

- In the previous lecture we found that for arbitrary $f(t)$ we need to solve:
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\sin(\Omega_f t),$$
where f is the force's amplitude.
- Let's look at the solution in the form $x = -f\Im C e^{-i\Omega_f t}$, and use $\sin(\Omega_f t) = -\Im e^{-i\Omega_f t}$. We then get
$$C = \frac{1}{\omega_0^2 - \Omega_f^2 - 2i\gamma\Omega_f} = |C|e^{i\phi},$$
$$|C| = \frac{1}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}}, \quad \tan\phi = \frac{2\gamma\Omega_f}{\omega_0^2 - \Omega_f^2}$$
$$x(t) = -f\Im|C|e^{-i\Omega_f t + i\phi} = f|C|\sin\left(\Omega_f t - \phi\right),$$
- Resonance frequency for the position measurement
$$\Omega_f^r = \sqrt{\omega_0^2 - 2\gamma^2}.$$
- Phase changes sign at $\Omega_f^\phi = \omega_0$.
- Role of the phase: delay in response. The force is zero at $t = 0$, the response $x(t)$ is zero at $t = \phi/\Omega_f > 0$, so if $\phi > 0$ the response is "delayed" in comparison to the force.

— *Resonance in velocity measurement*

- The velocity is given by
$$v(t) = \dot{x}(t) = f\Im i\Omega_f C e^{-i\Omega_f t}.$$
- The velocity amplitude is given by
$$f\Omega_f|C| = f\frac{\Omega_f}{\left[(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2\right]^{1/2}} = f\frac{1}{\left[(\Omega_f - \omega_0^2/\Omega_f)^2 + 4\gamma^2\right]^{1/2}}$$

- The maximum is when $\Omega_f - \omega_0^2/\Omega_f = 0$, so the resonance frequency for the velocity is ω_0 — without the damping shift.
- Current is velocity.

— *Analysis for small* γ.

- To analyze resonant response we analyze $|C|^2$.
- The most interesting case $\gamma \ll \omega_0$, then the response $|C|^2$ has a very sharp peak at $\Omega_f \approx \omega_0$:

$$|C|^2 = \frac{1}{(\Omega_f^2 - \omega_0^2)^2 + 4\gamma^2\Omega_f^2} \approx \frac{1}{4\omega_0^2}\frac{1}{(\Omega_f - \omega_0)^2 + \gamma^2},$$

  so that the peak is very symmetric.
- $|C|^2_{\text{max}} \approx \frac{1}{4\gamma^2\omega_0^2}$.
- to find HWHM we need to solve $(\Omega_f - \omega_0)^2 + \gamma^2 = 2\gamma^2$, so HWHM $= \gamma$, and FWHM $= 2\gamma$.
- Q factor (quality factor). The good measure of the quality of an oscillator is $Q = \omega_0/\text{FWHM} = \omega_0/2\gamma$. (decay time) $= 1/\gamma$, period $= 2\pi/\omega_0$, so $Q = \pi\frac{\text{decay time}}{\text{period}}$.
- Quality factor Q is the property of the resonator.
- For a grandfather's wall clock $Q \approx 100$, for the quartz watch $Q \sim 10^4$.

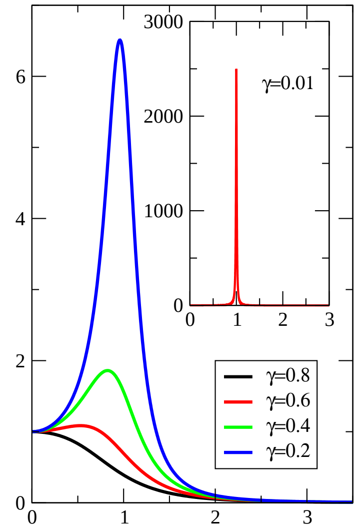


Figure: Resonant response. For insert $Q = 50$.

8.2. Useful points.

- The *complex* response function

$$C(\Omega_f) = \frac{1}{\omega_0^2 - \Omega_f^2 - 2i\gamma\Omega_f}$$

  as a function of *complex* frequency Ω_f has simple poles at $\Omega_f^p = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$. Both poles are in the lower half plane of the complex Ω_f plane. This is always so for any linear response function. It is the consequence of causality!
- The resonator with a high Q is a filter. One can tune this filter by changing the parameters of the resonator.
- By measuring the response function and its HWHM we can measure γ. By changing the parameters such as temperature, fields, etc. we can measure the dependence of γ on these parameters. γ comes from the coupling of the resonator to other degrees of freedom (which are typically not directly observable) so this way we learn something about those other degrees of freedom.

LECTURE 9
Momentum Conservation. Rocket motion. Charged particle in magnetic field.

9.1. Momentum Conservation.

It turns out that the mechanics formulated by Newton implies certain conservation laws. These laws allows us to find answers to many problems/questions without solving equations of motion. Moreover, they are very useful even when it is impossible to solve the equations of motion, as happens, for example, in Stat. Mech. But the most important aspect of the conservation laws is that they are more fundamental than the Newtonian mechanics itself. In Quantum mechanics or Relativity, or quantum field theory the very same conservation laws still hold, while the Newtonian mechanics fails.

- Momentum conservation. Consider a system of N interacting bodies
- We number the bodies with indexes $i = 1, \dots N$, etc.,
- All bodies interact with each other and with something outside of our system.
- A body j acts on a body i with a force $\vec{F}_{ij}$.
- A body i experiences an external force $\vec{F}_i^{ex}$ — this is the force with which whatever is outside of our *system* acts on the body i.
- Then for each of the bodies we have

$$\dot{\vec{p}}_i = \vec{F}_i = \vec{F}_i^{ex} + \sum_j \vec{F}_{ij}.$$

  We take $F_{ii} = 0$ — no self action.
- According to the Newton's third law $\vec{F}_{ij} = -\vec{F}_{ji}$.
- Consider the total momentum of the whole system $\vec{P} = \sum_i \vec{p}_i$, then

$$\dot{\vec{P}} = \sum_i \dot{p}_i = \sum_i \vec{F}_i^{ex} + \sum_{i,j} \vec{F}_{ij} = \sum_i \vec{F}_i^{ex}.$$

  because $\sum_{i,j} \vec{F}_{ij} = 0$ as in this sum for every term $\vec{F}_{ij}$ there is a term $\vec{F}_{ji}$.
- So internal forces in a system do not contribute to the change of the total momentum.
- The momentum of a closed system (when there is no interaction with outside $\vec{F}_i^{ex} = 0$) is conserved $\dot{\vec{P}} = 0$.
- Important points:

  - It is of paramount importance to clearly define what your system is and what the "outside" is.
  - The statement is only about the *total* momentum of the system.
  - The nature of the forces does not matter. They can be dissipative, or non-dissipative it will still work.
  - It is only the sum of all outside forces that leads to the change of the total momentum.
  - The momentum is a vector! there are three conservation laws — one for each component..
  - If only some components of the total external force are zero, then only the corresponding components of the total momentum will be conserved.
- Examples of the momentum conservation law.

9.2. Rocket motion.

Statement of the problem:

- A rocket burns fuel. The spent fuel is ejected with velocity V in the **frame of reference of the rocket**.
- Both the mass of the rocket $m(t)$ and its velocity $v(t)$ are functions of time t. The function $m(t)$ is in our hands – this is how we burn the fuel – how hard we press on the gas pedal.
- We want to find the function $v(t)$ — the rocket velocity as a function of time.
- The initial mass of the rocket is m_{initial}. The initial velocity of the rocket is v_{initial}.

Solution:

- At some time t the velocity of the rocket is v and its mass is m.
- Its momentum at this moment is mv.
- The engine fires constantly. At time $t+dt$ the mass of the rocket changes and becomes $m+dm$ (where dm is negative), its velocity becomes $v+dv$. The momentum of the rocket is $(m+dm)(v+dv) \approx mv + mdv + vdm$
- The spent fuel has a mass dm_f and has velocity $v-V$, so its momentum is $(v-V)dm_f$.
- As the total mass of a rocket with the fuel does not change $dm + dm_f = 0$. So the momentum of the burned fuel is $-(v-V)dm$.
- As there is no external forces acting on the system rocket+fuel the total momentum of this system must be conserved, or the total momentum mv at time t, must be equal to the total momentum $mv + mdv + vdm - (v-V)dm$ at time $t+dt$.

$$
\begin{aligned}
&mv = mv + mdv + vdm - (v-V)dm, \\
&mdv = -Vdm, \\
&dv = -V\frac{dm}{m}, \\
&v_{\text{final}} = v_{\text{initial}} + V\log\frac{m_{\text{initial}}}{m_{\text{final}}}.
\end{aligned}
$$

- Notice, that the answer does not depend on the exact form of the function $m(t)$. It depends only on the ratio of the initial mass to the final mass.

- As final moment is arbitrary we can write

$$v(t) = v_{\text{initial}} + V \log \frac{m_{\text{initial}}}{m(t)}$$

- Consider now that there is an external force F_{ex} acting on the rocket. Then we will have

$$mdv = -V dm + F_{ex} dt, \qquad m\frac{dv}{dt} = F_{ex} - V\frac{dm}{dt}.$$

- This equation looks like the second Newton law if we say that there is a new force "thrust"$= -V\frac{dm}{dt}$, which acts on the rocket. Notice, that $\frac{dm}{dt} < 0$, so this force is positive.

9.3. Charged particle in magnetic field.

- Lorentz force: $\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$.
- No electric field — $\vec{F} \perp \vec{v}$, so there is no component of the force $\vec{F}$ along the vector of velocity $\vec{v}$, so $|\vec{v}| =$ const.. Trajectories. $gvB = m\omega^2 R = m\omega v$, I used $\omega R = v$. Cyclotron frequency $\omega_c = \frac{qB}{m}$. Cyclotron radius $R_c = \frac{mv}{qB}$.
- Boundary effect.

LECTURE 10
Kinematics in cylindrical/polar coordinates.

In this lecture we will consider different coordinate systems in flat $2D$ space.

- What are coordinates?
  - The Cartesian coordinates are given by the origin and two unit vectors e_x and e_y.
  - These vectors have the following properties.

$$e_x^2 = e_y^2 = 1, \qquad e_x \cdot e_y = 0.$$

  - These two vectors e_x and e_y are the same in any point of space. (It is possible to define such vectors only because the space is flat.)
  - Any vector can be represented as

$$\vec{r} = xe_x + ye_y.$$

  - Any point can be described by the components x and y.
  - For a moving particle differentiating $\vec{r}$ we find its velocity

$$\vec{v} = \dot{x}e_x + \dot{y}e_y, \qquad v_x = \dot{x} \quad v_y = \dot{y}$$

  - Differentiating the *vector* of the velocity we find the vector of acceleration

$$\vec{a} = \dot{\vec{v}} = \ddot{x}e_x + \ddot{y}e_y, \qquad a_x = \ddot{x} \quad a_y = \ddot{y}$$

  - A trajectory is given by $x(t)$ and $y(t)$, where t is a parameter – usually time. If we are not interested on the time dependence, then we can give the trajectory as a function $y(x)$.
  - The polar coordinates are given by the origin and two vectors e_r and e_ϕ.
  - Both e_r and e_ϕ are different in different points of space. These vectors are not defined at the origin.
  - These vectors have the following properties *at every point of space*

$$e_r^2 = e_\phi^2 = 1, \qquad e_r \cdot e_\phi = 0. \tag{10.1}$$

  - In the polar coordinates we use r and ϕ to describe the position. However, the position vector $\vec{r}$ is not given by simple components as in Cartesian coordinates. Instead it is given by

$$\vec{r} = re_r(r, \phi)$$

- In $2D$ we can use r and ϕ as coordinates. Our unit vectors e_r and e_ϕ can be represented through the Cartesian vectors e_x and e_y at every point.

$$\begin{array}{l} e_r = e_x\cos\phi + e_y\sin\phi \\ e_\phi = -e_x\sin\phi + e_y\cos\phi \end{array}; \qquad \begin{array}{l} e_x = e_r\cos\phi - e_\phi\sin\phi \\ e_y = e_r\sin\phi + e_\phi\cos\phi \end{array}$$

- Differentiating these relationships with respect to a parameter (time) t we get

$$\dot{e}_r = \dot{\phi}e_\phi, \qquad \dot{e}_\phi = -\dot{\phi}e_r$$

- Notice, that if we differentiate the relationships (10.1), then we get

$$e_r\cdot\dot{e}_r = e_\phi\cdot\dot{e}_\phi = 0, \qquad e_r\cdot\dot{e}_\phi = -e_\phi\cdot\dot{e}_r$$

  The first two relations show that a derivative of a unit vector must be orthogonal to that vector (its length must not change) The second relation shows how the orthogonal unit vector must change in order to keep their orthogonality.
- The radius vector $\vec{r} = re_r$. Let's calculate the vector of velocity

$$\vec{v} = \dot{\vec{r}} = \dot{r}e_r + r\dot{e}_r = \dot{r}e_r + r\dot{\phi}e_\phi.$$

  We see that the components of the velocity are given by

$$v_r = \dot{r}, \qquad v_\phi = r\dot{\phi}$$

- Acceleration – we *must* differentiate the *vector* of the velocity!

$$\vec{a} = \dot{\vec{v}} = \left(\ddot{r} - r\dot{\phi}^2\right)e_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)e_\phi, \qquad a_r = \ddot{r} - r\dot{\phi}^2, \quad a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}$$

- In the case $r = \text{const}$, $\dot{\phi} = \omega$, $\vec{a} = -r\omega^2 e_r + r\dot{\omega}e_\phi$.
- Notice, if $\dot{\phi} = \omega = \text{const}$, then $a_\phi = 2\dot{r}\omega$ – this is the origin of the Coriolis force.

Free motion. There is no forces, so $\vec{a} = 0$.

- In Cartesian coordinates it gives

$$\ddot{x} = 0, \quad \ddot{y} = 0, \qquad x(t) = v_{x,0}t + x_0, \quad y(t) = v_{y,0}t + y_0.$$

- Or the trajectory

$$y = y_0 + \frac{v_{y,0}}{v_{x,0}}(x - x_0).$$

  This is the equation for a straight line in the Cartesian coordinates.
- In the polar coordinates. $\vec{a} = 0$, so both components of $\vec{a}$ must be zero

$$\begin{array}{l} r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \\ \ddot{r} - r\dot{\phi}^2 = 0 \end{array}, \qquad \begin{array}{l} r^2\dot{\phi} = \text{const} = A \\ \ddot{r} - \frac{A^2}{r^3} = 0 \end{array}$$

- Notation

$$\frac{\partial}{\partial x} \equiv \partial_x$$

- Now I will do the following trick. Instead of two functions $r(t)$ and $\phi(t)$ I will consider a function $r(\phi)$ — the trajectory — and use

$$\frac{\partial}{\partial t} = \frac{\partial\phi}{\partial t}\frac{\partial}{\partial\phi} = \dot{\phi}\frac{\partial}{\partial\phi} = \frac{A}{r^2}\partial_\phi; \qquad \dot{r} = \frac{A}{r^2}\partial_\phi r = -A\partial_\phi\frac{1}{r}; \qquad \ddot{r} = -\frac{A^2}{r^2}\partial_\phi^2\frac{1}{r},$$

  then we get

$$\frac{A^2}{r^2}\partial_\phi^2\frac{1}{r} - \frac{A^2}{r^3} = 0, \qquad \partial_\phi^2\frac{1}{r} = -\frac{1}{r}, \qquad \frac{1}{r} = B\cos(\phi - \phi_0)$$

- This is the equation of the straight line in the polar coordinates.

LECTURE 11
Angular velocity. Angular momentum.

Consider a rigid body which can rotate around an axis which goes through its center of mass. We apply a force $\vec{F}$ to some point of the body.

- Depending on the direction of the force the body may or may not rotate with increasing frequency.
- In any case the body as a whole will not move.
- It means that the axis must apply a force $-\vec{F}$ to the body.
- So the sum of all forces applied to the body is zero.
- What then causes the angular velocity to change?
- Consider a small piece of the body.
- Its velocity is changing! So there must be a net force acting on it.
- This is the force of interaction of our small piece with the rest of the body.
- Such forces are very difficult to compute, but
- If the body is rigid, then me know that the relative position of the points of the body does not change.
- It turns out that this observation is enough to construct the theory of the motion of a rigid body without the reference to the internal forces.

11.1. Angular velocity. Rotation.

- Vector of angular velocity $\vec{\omega}$. For $|\vec{r}| = \text{const.}$:
$$\vec{v} = \vec{\omega} \times \vec{r}.$$
- Sum of two vectors
$$\vec{v}_{13} = \vec{v}_{12} + \vec{v}_{23}, \qquad (\vec{\omega}_{13} - \vec{\omega}_{12} - \vec{\omega}_{23}) \times \vec{r} = 0, \qquad \vec{\omega}_{13} = \vec{\omega}_{12} + \vec{\omega}_{23}$$
- We have a frame rotating with angular velocity $\vec{\omega}$ with respect to the rest frame. A vector $\vec{l}$ constant in the rotating frame will change with time in the rest frame and
$$\dot{\vec{l}} = \vec{\omega} \times \vec{l}.$$
- $\omega = \frac{d\phi}{dt}$, if ω is a vector $\vec{\omega}$, then $d\phi$ must be a vector $\vec{d\phi}$. Notice, that ϕ is not a vector!
- If we rotate one frame with respect to another by a small angle $\vec{d\phi}$, then a vector $\vec{l}$ will change by
$$d\vec{l} = \vec{d\phi} \times \vec{l}.$$

11.2. Angular momentum.

- Consider a vector $\vec{J} = \vec{r} \times \vec{p}$ – vector of angular momentum.
- Consider a bunch of particles which interact with central forces: $\vec{F}_{ij} \parallel \vec{r}_i - \vec{r}_j$. There is also external force $\vec{F}_i^{ex}$ acting on each particle.
- Consider the time evolution of the vector of the total angular momentum $\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i$:

$$\dot{\vec{J}} = \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \left[\sum_{j\neq i} \vec{F}_{ij} + \vec{F}_i^{ex}\right] = \sum_{i\neq j} \vec{r}_i \times \vec{F}_{ij} + \sum_i \vec{r}_i \times \vec{F}_i^{ex}$$

- The sum $\sum_i \vec{r}_i \times \vec{F}_i^{ex}$ is called torque. Here it is the torque of external forces $\vec{\tau}^{ex}$.
- if a force $\vec{F}$ is applied to a point with the position $\vec{r}$ with respect to the origin, then the torque of this force with respect to the same origin is given by

$$\vec{\tau} \equiv \vec{r} \times \vec{F}.$$

- Consider now the first sum in the RHS. Remember that $\vec{F}_{ij} = -\vec{F}_{ji}$

$$\sum_{i\neq j} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2}\sum_{i\neq j} \vec{r}_i \times \vec{F}_{ij} + \frac{1}{2}\sum_{i\neq j} \vec{r}_j \times \vec{F}_{ji} = \frac{1}{2}\sum_{i\neq j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = 0$$

- So we have

$$\dot{\vec{J}} = \vec{\tau}^{ex}$$

- If the torque of external forces is zero, then the angular momentum is conserved.

LECTURE 12
Moment of inertia. Kinetic energy.

In the previous lecture we considered a set of particles and showed, that if they interact through the central forces the rate of change of angular momentum equals to the total torque of external forces only. In proving this statement the condition of rigidity was not used at all. The statement $\dot{\vec{J}} = \vec{\tau}^{ex}$ is very general.

In this lecture we show how to compute the angular momentum and the kinetic energy for a rigid body. Remember, that the condition of rigidity is very strong. The equation

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

allows us to compute the velocity of *every* point of the body by knowing only one vector $\vec{\omega}$. So both the angular momentum and the kinetic energy will depend only on the vector $\vec{\omega}$ and some property of the body itself.

12.1. Angular momentum. Moment of inertia.

- Consider a ridged set of particles of masses m_i — the distances between the particles are fixed and do not change. The whole system rotates with the angular velocity $\vec{\omega}$. Each particle has a radius vector $\vec{r}_i$. Let's calculate the angular momentum of the whole system.

$$\vec{J} = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times [\vec{\omega} \times \vec{r}_i] = \sum_i m_i \left(\vec{\omega}\vec{r}_i^2 - \vec{r}_i(\vec{\omega} \cdot \vec{r}_i) \right)$$

  or in components (Einstein notations are assumed over Greek indexes)

$$J^\alpha = \sum_i m_i \left(\omega^\alpha \vec{r}_i^2 - r_i^\alpha \omega^\beta r_i^\beta \right) = \sum_i m_i \left(\delta^{\alpha\beta} \vec{r}_i^2 - r_i^a r_i^\beta \right) \omega^\beta = I^{\alpha\beta}\omega^\beta,$$

$$I^{\alpha\beta} = \sum_i m_i \left(\delta^{\alpha\beta} \vec{r}_i^2 - r_i^a r_i^\beta \right)$$

- The moment of inertia is a positive definite symmetric 3×3 tensor!

$$\hat{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, \qquad I^{\alpha\beta} = I^{\beta\alpha}.$$

  It transforms one vector into another:

$$\vec{J} = \hat{I}\vec{\omega}.$$

As for any symmetric tensor:
  - There are special coordinate axes in which the tensor has a diagonal form – only diagonal elements are nonzero, while all the off diagonal elements are zero.
  - These diagonal elements are called principle moments of inertia. The corresponding axes are called principal axes of inertia.
  - If all the principal moments are different, then the principle axes are orthogonal to each other.
  - In a degenerate case these cases can be chosen to be orthogonal.
  - These principle axes are "attached" to the body, so if the body is rotating, then these axes are also rotating with the body.
- The direction of the angular momentum $\vec{J}$ and direction of the angular velocity $\vec{\omega}$ do not in general coincide!
- It is $\vec{J}$ which is constant when there are no external torques, not $\vec{\omega}$! Let me repeat it: If there are no external torques the vector $\vec{\omega}$ may change with time — both its direction and magnitude. But the angular momentum vector $\vec{J}$ will remain constant.
  Contrast this to the usual momentum-velocity relation
$$\vec{p} = m\vec{v}$$
  where the conservation of momentum means that the velocity is also constant. This is because the mass m is a scalar, not tensor.
- This last statement makes even the kinematics (motion with no external forces) of a rigid body very complicated and highly non-trivial.
- Moment of inertia of a continuous body.
$$I^{\alpha\beta} = \int \left(\delta^{\alpha\beta}\vec{r}^2 - r^\alpha r^\beta\right) dm = \int \left(\delta^{\alpha\beta}\vec{r}^2 - r^\alpha r^\beta\right) \frac{dm}{dV} dV = \int \left(\delta^{\alpha\beta}\vec{r}^2 - r^\alpha r^\beta\right) \rho(\vec{r}) dV,$$
  where $\rho(\vec{r})$ is the mass density of the material at point $\vec{r}$ – it must be know as this is a characteristic of the body.
- How to compute the moment of inertia of an arbitrary body.
  - First you chose a system of coordinates registered with the body.
  - You chose which component of the tensor of inertia you want to compute. You have to compute all of them, but you need to start with something. Let's say it is I^{xy}.
  - Then in the expression $\int \left(\delta^{\alpha\beta}\vec{r}^2 - r^\alpha r^\beta\right) \rho(\vec{r}) dV$ we have $\alpha = x$ and $\beta = y$.
  - The first term under the integral is then zero, as $\delta^{xy} = 0$.
  - In the second term $r^\alpha = x$, and $r^\beta = y$, so we have
$$I^{xy} = -\iiint xy\rho(x, y, z) dxdydz.$$
  - Let's say we want to compute I^{xx}. Then $\alpha = x$, and $\beta = x$, so the first term $\delta^{\alpha\beta}\vec{r}^2 = x^2 + y^2 + z^2$, as $\delta^{xx} = 1$, and $\vec{r}^2 = x^2 + y^2 + z^2$. The second term is just x^2. So we need to compute
$$I^{xx} = \iiint \left(y^2 + z^2\right) \rho(x, y, z) dxdydz.$$
- Examples.
  - A thin ring: $I_{zz} = mR^2$, $I_{xx} = I_{yy} = \frac{1}{2}mR^2$, all off diagonal elements vanish.
  - A disc: $I_{zz} = \frac{1}{2}mR^2$, $I_{xx} = I_{yy} = \frac{1}{4}mR^2$, all off diagonal elements vanish.

- A sphere: $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mR^2$, all off diagonal elements vanish.
- A stick at the end: $I_{xx} = I_{yy} = \frac{1}{3}mL^2$.
- A stick at the center: $I_{xx} = I_{yy} = \frac{1}{12}mL^2$.

- Role of symmetry.

12.2. Kinetic energy.

- Consider the kinetic energy of the moving body.

$$K = \frac{1}{2}\sum_i m_i\vec{v}_i^2 = \frac{1}{2}\sum_i m_i[\vec{\omega}\times\vec{r}_i]^2 = \frac{1}{2}\sum_i m_i[\vec{\omega}^2\vec{r}^2 - (\vec{\omega}\cdot\vec{r})^2] = \frac{1}{2}\sum_i m_i[\delta^{\alpha\beta}\vec{r}^2 - r^\alpha r^\beta]\omega^\alpha\omega^\beta.$$

so we get

$$K = \frac{I^{\alpha\beta}\omega^\alpha\omega^\beta}{2}$$

(this also shows that $\hat{I}$ is positive definite)

- In terms of angular momentum:

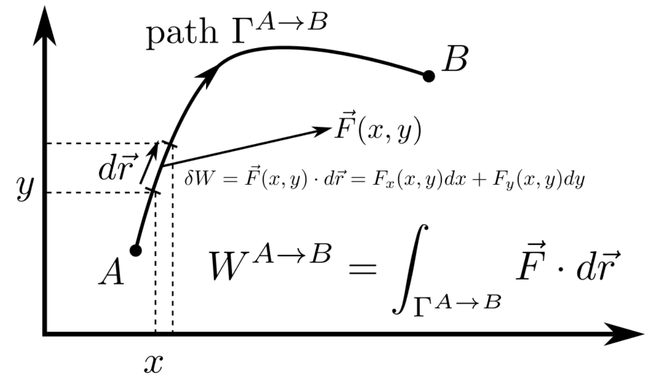
$$K = \frac{1}{2}\left(\hat{I}^{-1}\right)^{\alpha\beta} J^\alpha J^\beta.$$

LECTURE 13
Work. Potential energy.

13.1. Mathematical preliminaries.

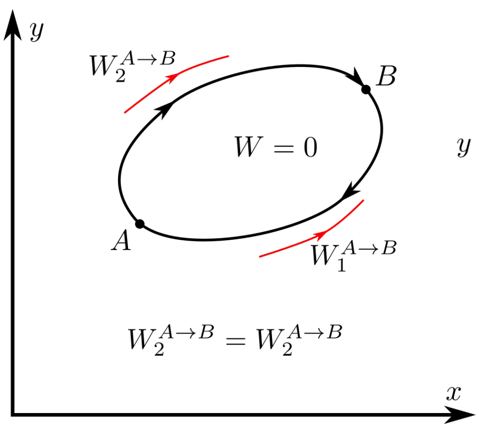
- Functions of many variables, say $U(x, y)$
- Differential of a function of many variables.
$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy.$$
- Consider an expression
$$\delta G = A(x, y)dx + B(x, y)dy.$$
where A and B are some arbitrary functions. The question is: is this a differential of some function? The answer is: not necessarily. The proof:
  - Let's assume that δG is a differential of some function U, then we must have
$$A = \frac{\partial U}{\partial x}, \qquad B = \frac{\partial U}{\partial y}.$$
  - But then
$$\frac{\partial A}{\partial y} = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial B}{\partial x}.$$
  - So δG is a differential of some function if (and only if)
$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$
  - In other words, if the condition above is satisfied, then there exists a function $U(x, y)$ such that
$$A(x, y) = \frac{\partial U(x, y)}{\partial x}, \qquad B(x, y) = \frac{\partial U(x, y)}{\partial y}.$$
  - Then the statement that the form δG is a differential is a very strong statement, as it tells you that in order to know two functions $A(x, y)$ and $B(x, y)$ you need to know only one function $U(x, y)$.
- Examples.
  - $\delta G = xdy + ydx$ is a differential $U = xy$.
  - $\delta G = xdy - ydx$ is not a differential. The function U does not exist.

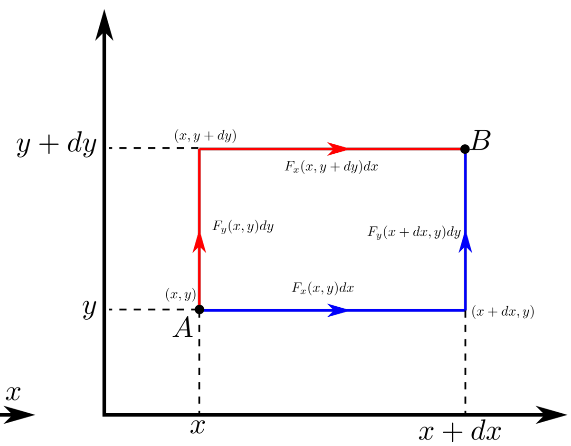
13.2. Work.



- A work done by a force: $\delta W = \vec{F}\cdot d\vec{r}$.
- Notice, that although $\delta W = F_x dx + F_y dy + F_z dz$ this is not necessarily a full differential.
- Superposition. If there are many forces, the total work is the sum of the works done by each.
- Finite displacement. Line integral.
- In general case work depends on path!!!!!

13.3. Conservative forces. Energy conservation.





- Fundamental forces. Depend on coordinate, do not depend on time.
- Work done by the forces over a closed loop is zero.
- It means that work is independent of the path.
- Consider two paths: first dx, then dy; first dy then dx

$$\delta W_1 = F_x(x,y)dx + F_y(x+dx,y)dy$$
$$\delta W_2 = F_y(x,y)dy + F_x(x,y+dy)dx.$$

- The works must be equal to each other, so

$$F_x(x,y)dx + F_y(x,y)dy + \frac{\partial F_y}{\partial x}dydx = F_y(x,y)dy + F_x(x,y)dx + \frac{\partial F_x}{\partial y}dydx$$

where we used $F_y(x+dx,y) \approx F_y(x,y) + \frac{\partial F_y}{\partial x}dx$, and $F_x(x,y+dy) \approx F_x(x,y)dx + \frac{\partial F_x}{\partial y}dydx$. So in order for the works to be equal to each other we must have

$$\left.\frac{\partial F_y}{\partial x}\right|_{x,y} = \left.\frac{\partial F_x}{\partial y}\right|_{x,y}$$

- So a small work done by a conservative force:

$$\delta W = F_x dx + F_y dy, \qquad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

is a full differential!
- So there exist a function U such that

$$\delta W = -dU$$

(the minus sign is for further convenience)
- It means that there is such a function of the coordinates $U(x,y)$, that

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad \text{or} \quad \vec{F} = -\text{grad}U \equiv -\vec{\nabla}U.$$

LECTURE 14
Energy Conservation. One-dimensional motion.

- Last lecture we found, that there exists a special class of forces (which depend only on coordinates) which are called "conservative forces".
  - Not all forces are conservative! Friction!
  - All fundamental forces are conservative.
- A conservative force is such a force that its work around any closed loop is zero.
- Last lecture we found that for a conservative (zero work on a closed loop) force there exists a function U — called "potential energy" such that

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad \text{or} \quad \vec{F} = -\text{grad}U \equiv -\vec{\nabla}U.$$

  Such function is not unique as one can always add an arbitrary constant to the potential energy.
- Under a small displacement $d\vec{r}$ a work done by such a force is

$$\delta W = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz = -dU.$$

- If the force $\vec{F}(\vec{r})$ is known, then there is a test for if the force is conservative.

$$\nabla \times \vec{F} = 0.$$

14.1. Change of kinetic energy.

- If a body of mass m moves under the force $\vec{F}$, then.

$$m\frac{d\vec{v}}{dt} = \vec{F}, \qquad md\vec{v} = \vec{F}dt, \qquad m\vec{v} \cdot d\vec{v} = \vec{F} \cdot \vec{v}dt = \vec{F} \cdot d\vec{r} = \delta W.$$

  So we have

$$d\frac{mv^2}{2} = \delta W$$

- The change of kinetic energy $K = \frac{mv^2}{2}$ equals the total work done by all forces.
- In general case this is not very useful, as we need to know the path in order to compute work.

$$W = \int_{\Gamma^{A\to B}} \vec{F} \cdot d\vec{r}.$$

  In order to know the path we need to solve the equations of motion.

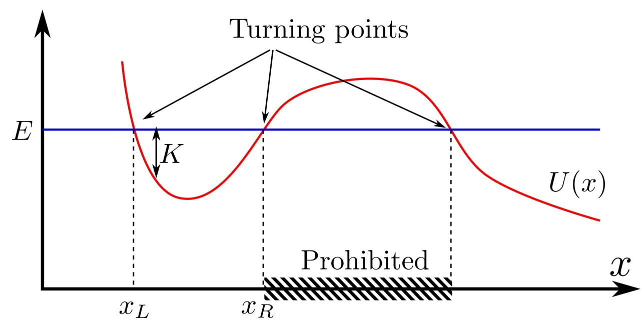
14.1.1. Conservative forces.

- So on a trajectory: $dK = \delta W = -dU$, or

$$d\left(\frac{mv^2}{2} + U\right) = 0, \qquad K + U = \text{const.}$$

- Examples.

14.2. $1D$ motion.



- In $1D$ the force that depends only on the coordinate is always conservative.
- In $1D$ in the case when the force depends only on coordinates the equation of motion can be solved in quadratures.
- The number of conservation laws is enough to solve the equations.
- If the force depends on the coordinate only $F(x)$, then there exists a function — potential energy — with the following property

$$F(x) = -\frac{\partial U}{\partial x}$$

  Such function is not unique as one can always add an arbitrary constant to the potential energy.
- The total energy is then conserved

$$K + U = \text{const.}, \qquad \frac{m\dot{x}^2}{2} + U(x) = E$$

- Energy E can be calculated from the initial conditions: $E = \frac{mv_0^2}{2} + U(x_0)$
- As $\frac{mv^2}{2} > 0$ the allowed areas where the particle can be are given by $E - U(x) > 0$.
- Picture. Turning points — the solutions of the equation $E = U(x)$. Prohibited regions.
- Notice, that the equation of motion depends only on the difference $E - U(x) = \frac{mv_0^2}{2} + U(x_0) - U(x)$ of the potential energies in different points, so the zero of the potential energy (the arbitrary constant that was added to the function) does not play a role.
- We thus found that

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

- Energy conservation law cannot tell the direction of the velocity, as the kinetic energy depends only on absolute value of the velocity. In $1D$ it cannot tell which sign to use "+" or "−". You must not forget to figure it out by other means.

- We then can solve the equation

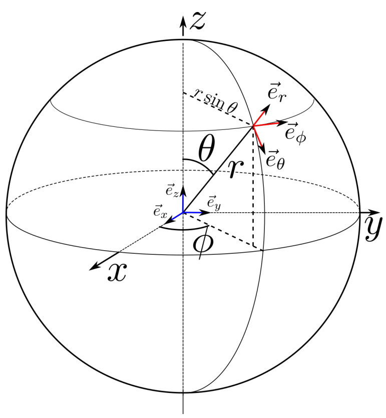
$$\pm\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E-U(x)}} = dt, \qquad t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E-U(x')}}$$

- Examples:
  - Motion under a constant force.
  - Oscillator.
  - Pendulum.
- Periodic motion. Period between two turning points x_L and x_R.

$$T = 2\sqrt{\frac{m}{2}}\int_{x_L}^{x_R}\frac{dx'}{\sqrt{E-U(x')}}$$

LECTURE 15
Spherical coordinates. Central forces.

15.1. Spherical coordinates.



15.1.1. Coordinate vectors of spherical coordinates.

- The spherical coordinates are given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \quad .$$

- The coordinates r, θ, and ϕ can be used to denote any point.
- There are corresponding unit vectors $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ at each point (r, θ, ϕ).
  - The vector $\vec{e}_r$ is the unit vector along the direction where our point shifts if we change the coordinate r, while keeping θ and ϕ constant.
  - The vector $\vec{e}_\theta$ is the unit vector along the direction where our point shifts if we change the coordinate θ, while keeping r and ϕ constant.
  - The vector $\vec{e}_\phi$ is the unit vector along the direction where our point shifts if we change the coordinate ϕ, while keeping θ and r constant.
- With such definitions of $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ we see, that
  - If we change only coordinate r to $r + dr$, then the position vector $\vec{r}$ changes by $d\vec{r} = \vec{e}_r dr$.

  - If we change only coordinate θ to $\theta + d\theta$, then the position vector $\vec{r}$ changes by $d\vec{r} = \vec{e}_\theta r\theta$.
  - If we change only coordinate ϕ to $\phi + d\phi$, then the position vector $\vec{r}$ changes by $d\vec{r} = \vec{e}_\phi r \sin\theta d\phi$.
- The vector $d\vec{r}$ then is expressed through the dr, $d\theta$ and $d\phi$ as
$$d\vec{r} = \vec{e}_r dr + \vec{e}_\theta r d\theta + \vec{e}_\phi r \sin\theta d\phi.$$
- In Cartesian coordinates the similar expression is
$$d\vec{r} = \vec{e}_x dx + \vec{e}_y dy + \vec{e}_z dz.$$
- Notice, that in this formulation we do not need to have the position vector $\vec{r}$. We can do everything with the coordinate vectors defined locally.

15.1.2. Connecting Spherical and Cartesian.

Here I show how to connect $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$, to $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$ using only local relations.

- Using the definition of the spherical coordinates we have locally
$$\begin{aligned} dx &= dr \sin\theta\cos\phi + d\theta r\cos\theta\cos\phi - d\phi r\sin\theta\sin\phi \\ dy &= dr\sin\theta\sin\phi + d\theta r\cos\theta\sin\phi + d\phi r\sin\theta\cos\phi \\ dz &= dr\cos\theta - d\theta r\sin\theta \end{aligned}$$
- Using these expressions in $d\vec{r}$ is Cartesian coordinates we find
$$d\vec{r} = (\vec{e}_x\sin\theta\cos\phi + \vec{e}_y\sin\theta\sin\phi + \vec{e}_z\cos\theta)\,dr + (\vec{e}_x r\cos\theta\cos\phi + \vec{e}_y r\cos\theta\sin\phi - \vec{e}_z r\sin\theta)\,d\theta$$
$$+ (\vec{e}_y r\sin\theta\cos\phi - \vec{e}_x r\sin\theta\sin\phi)\,d\phi$$
- Comparing this to the $d\vec{r}$ in spherical coordinates we get
$$\begin{aligned} \vec{e}_r &= \vec{e}_x\sin\theta\cos\phi + \vec{e}_y\sin\theta\sin\phi + \vec{e}_z\cos\theta \\ \vec{e}_\theta &= \vec{e}_x\cos\theta\cos\phi + \vec{e}_y\cos\theta\sin\phi - \vec{e}_z\sin\theta \\ \vec{e}_\phi &= -\vec{e}_x\sin\phi + \vec{e}_y\cos\phi \end{aligned}$$

15.1.3. Coordinate independent definition of the gradient.

- Imagine now a function of coordinates U. We want to find the components of a vector $\vec{\nabla}U$ in the spherical coordinates.
- Consider a function U as a function of Cartesian coordinates: $U(x, y, z)$. Then
$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \vec{\nabla}U \cdot d\vec{r}.$$
Notice, that we have a *coordinate independent* definition of the vector gradient. The vector of gradient $\vec{\nabla}U$ is such a vector that for *any* vector $d\vec{r}$ we have:
$$dU = \vec{\nabla}U \cdot d\vec{r} \quad \text{— definition of } \vec{\nabla}U.$$
It is coordinate independent as it is a scalar/dot product which doe not depend on coordinates.
- I want to make a few points about this definition.
  - This definition is constructive – it allows on to find the vector of gradient in any system of coordinates. For this it is important that $d\vec{r}$ is arbitrary infinitesimal vector.

  - It connects calculus dU with geometry — the scalar product of two vectors.
  - It thus gives the geometrical meaning/picture to calculus. In particular one can see that if one chooses a vector $d\vec{r}_\perp$ which is perpendicular to the vector of the gradient at some particular point, then the function U will not change along the direction of $d\vec{r}_\perp$ (in the infinitesimal neighborhood of that point).
- Let's see how this definition works in Cartesian coordinates.
  - In particular, if we use the standard Cartesian coordinates and write the vector of gradient as
$$\vec{\nabla}U = (\vec{\nabla}U)_x\vec{e}_x + (\vec{\nabla}U)_y\vec{e}_y + (\vec{\nabla}U)_z\vec{e}_z,$$
where $(\vec{\nabla}U)_x$, $(\vec{\nabla}U)_y$, and $(\vec{\nabla}U)_z$ are the components of the vector $\vec{\nabla}U$ in Cartesian coordinates. These are the components which we want to find.
  - Using the vector $d\vec{r}$ in Cartesian coordinate we find
$$dU = \vec{\nabla}U \cdot d\vec{r} = (\vec{\nabla}U)_x dx + (\vec{\nabla}U)_y dy + (\vec{\nabla}U)_z dz$$
  - Consider a function U as the function of Cartesian coordinates $U(x, y, z)$, we know from the standard calculus
$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$
  - Comparing these to results for dU (both are valid for arbitrary infinitesimal dx, dy, and dz) we find
$$(\vec{\nabla}U)_x = \frac{\partial U}{\partial x}, \quad (\vec{\nabla}U)_y = \frac{\partial U}{\partial y}, \quad (\vec{\nabla}U)_z = \frac{\partial U}{\partial z}.$$
  - This is our standard formulas for the gradient in Cartesian coordinates.
- Now we can use this procedure for any other system of coordinates, as long as we know how to express $d\vec{r}$ in the corresponding coordinate vectors.

15.1.4. Gradient in spherical coordinates.

- Let's write the vector $\vec{\nabla}U$ in the spherical coordinates.
$$\vec{\nabla}U = (\vec{\nabla}U)_r\vec{e}_r + (\vec{\nabla}U)_\theta\vec{e}_\theta + (\vec{\nabla}U)_\phi\vec{e}_\phi,$$
where $(\vec{\nabla}U)_r$, $(\vec{\nabla}U)_\theta$, and $(\vec{\nabla}U)_\phi$ are the components of the vector $\vec{\nabla}U$ in the spherical coordinates. It is those components that we want to find.
- By the definition of the gradient vector, and using $d\vec{r}$ in spherical coordinates we get
$$dU = \vec{\nabla}U \cdot d\vec{r} = (\vec{\nabla}U)_r dr + (\vec{\nabla}U)_\theta r d\theta + (\vec{\nabla}U)_\phi r \sin\theta d\phi$$
- On the other hand if we now consider U as a function of the spherical coordinates $U(r, \theta, \phi)$, then
$$dU = \frac{\partial U}{\partial r}dr + \frac{\partial U}{\partial \theta}d\theta + \frac{\partial U}{\partial \phi}d\phi$$
- Comparing the two expressions for dU we find
$$\begin{array}{l} (\vec{\nabla}U)_r = \frac{\partial U}{\partial r} \\ (\vec{\nabla}U)_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta} \\ (\vec{\nabla}U)_\phi = \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} \end{array}.$$

- The vector of gradient in spherical coordinates is then written as
$$\vec{\nabla}U = \frac{\partial U}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\vec{e}_\phi$$
- In particular if U is the potential energy, then
$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\vec{e}_r - \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\vec{e}_\phi.$$

15.2. Central force

- Consider a motion of a body under central force. Take the origin in the center of force.
- A central force is given by
$$\vec{F} = F(r)\vec{e}_r.$$
- Such force is always conservative: $\vec{\nabla} \times \vec{F} = 0$, so there is a potential energy:
$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\vec{e}_r, \quad \frac{\partial U}{\partial \theta} = 0, \quad \frac{\partial U}{\partial \phi} = 0,$$
so that potential energy depends only on the distance r, $U(r)$.

LECTURE 16
Effective potential. Kepler orbits.

16.1. Central force. General.

- Last lecture we started to consider a motion of a body under central force.
- A central force is given by
$$\vec{F} = F(r)\vec{e}_r.$$
- Such force is always conservative: $\vec{\nabla} \times \vec{F} = 0$,
- The potential energy $U(r)$ is a function of the distance r only.
$$F(r) = -\frac{\partial U}{\partial r}$$
- The torque of the central force $\tau = \vec{r} \times \vec{F} = 0$, so the angular momentum is conserved: $\vec{J} = \text{const}$.

16.2. Motion in under central force.

Consider now a particle of mass m which is moving in the central force field. The field is completely described by the potential energy function $U(r)$. We set this function such, that $U(r \to \infty) \to 0$.

In order to set up the problem we must also specify the initial conditions. So we know that at some time $t = 0$ the velocity of the particle is $\vec{v}_0$ and the position is $\vec{r}_0$.

- The vector of angular momentum can be computed from initial conditions $\vec{J} = \vec{r}_0 \times \vec{p}_0$.
- The energy can also be computed from the initial conditions $E = \frac{m\vec{v}_0^2}{2} + U(r_0)$.
- Both $\vec{J}$ and E are conserved. They are also independent conserved quantities.
- The direction of $\vec{J}$ is perpendicular to the initial momentum and initial coordinate.
- During the motion its direction will not change — it is conserved.
- So during the motion at any moment the momentum and position vectors will be in the same plane perpendicular to $\vec{J}$.
- The motion is all in one plane! The plane which contains the vector of the initial velocity and the initial radius vector.
- We take the direction of $\vec{J}$ as our z axis. The plane of motion is then $x - y$ plane.

- The angular momentum is $\vec{J} = J\vec{e}_z$, where $J = |\vec{J}| = \text{const.}$. This constant is given by initial conditions $J = m|\vec{r}_0 \times \vec{v}_0|$.
- In the $x - y$ plane $\theta = \pi/2$ we can use only r and ϕ coordinates — the polar coordinates.
- Writing the value of the angular momentum in the polar coordinates we get

$$mr^2\dot{\phi} = J, \qquad \dot{\phi} = \frac{J}{mr^2}$$

- The velocity in these polar coordinates is

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi = \dot{r}\vec{e}_r + \frac{J}{mr}\vec{e}_\phi$$

- The kinetic energy then is

$$K = \frac{m\vec{v}^2}{2} = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2}$$

- The total energy then is

$$E = K + U = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + U(r).$$

- If we introduce the effective potential energy

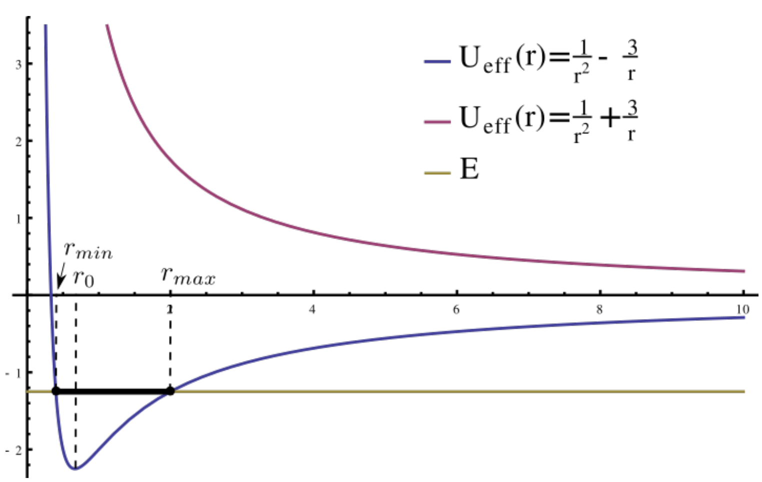
$$U_{eff}(r) = \frac{J^2}{2mr^2} + U(r),$$

then we have

$$\frac{m\dot{r}^2}{2} + U_{eff}(r) = E, \qquad m\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

- This is a one dimensional motion which was solved before.

16.3. Kepler orbits.

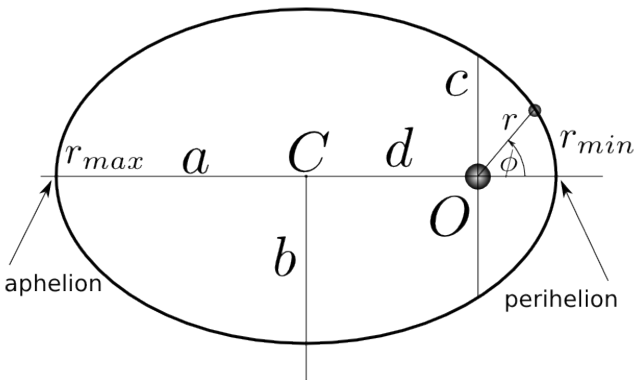


Historically, the Kepler problem — the problem of motion of the bodies in the Newtonian gravitational field — is one of the most important problems in physics. It is the solution of the problems and experimental verification of the results that convinced the physics community in the power of Newton's new math and in the correctness of his mechanics. For the first time people could understand the observed motion of the celestial bodies and make accurate predictions. The whole theory turned out to be much simpler than what existed before.

- In the Kepler problem we want to consider the motion of a body of mass m in the gravitational central force due to much larger mass M.

- As $M \gg m$ we ignore the motion of the larger mass M and consider its position fixed in space (we will discuss what happens when this limit is not applicable later)
- The force that acts on the mass m is given by the Newton's law of gravity:
$$\vec{F} = -\frac{GmM}{r^3}\vec{r} = -\frac{GmM}{r^2}\vec{e}_r$$
where $\vec{e}_r$ is the direction from M to m.
- The potential energy is then given by
$$U(r) = -\frac{GMm}{r}, \qquad -\frac{\partial U}{\partial r} = -\frac{GmM}{r^2}, \qquad U(r \to \infty) \to 0$$
- The effective potential is
$$U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GMm}{r},$$
where J is the angular momentum.
- For the Coulomb potential we will have the same r dependence, but for the like charges the sign in front of the last term is different — repulsion.
- In case of attraction for $J \neq 0$ the function $U_{eff}(r)$ always has a minimum for some distance r_0. It has no minimum for the repulsive interaction.
- Looking at the graph of $U_{eff}(r)$ we see, that
  - for the repulsive interaction there can be no bounded orbits. The total energy E of the body is always positive. The minimal distance the body may have with the center is given by the solution of the equation $U_{eff}(r_{min}) = E$.
  - for the attractive interaction there is a minimum of the effective potential energy at some $r_0 = \frac{J^2}{Gm^2M}$ (this is the solution of the equation $\partial U_{eff}/\partial r = 0$), and $U(r_0) < 0$, where $U(r \to \infty) \to 0$. Then, from the graph $U(r)$ we see
    * if $E > 0$, then the motion is not bounded. The minimal distance the body may have with the center is given by the solution of the equation $U_{eff}(r_{min}) = E$.
    * if $U_{eff}(r_0) < E < 0$, then the motion is bounded between the two real solutions of the equation $U_{eff}(r) = E$. One of the solution is larger than r_0, the other is smaller.
    * if $U_{eff}(r_0) = E$, then the only solution is $r = r_0$. So the motion is around the circle with fixed radius r_0. For such motion we must have
$$\frac{mv^2}{r_0} = \frac{GmM}{r_0^2}, \qquad \frac{J^2}{mr_0^3} = \frac{GmM}{r_0^2}, \qquad r_0 = \frac{J^2}{Gm^2M}.$$
Notice, that this is exactly r_0 that we found before. Also
$$U_{eff}(r_0) = E = \frac{mv^2}{2} - \frac{GmM}{r_0} = -\frac{1}{2}\frac{GmM}{r_0}.$$

LECTURE 17
Kepler orbits continued.



- In the motion the angular momentum and the energy are conserved
- All motion happens in one plane.
- In that plane we describe the motion by two time dependent polar coordinates $r(t)$ and $\phi(t)$. The dynamics is given by the angular momentum conservation and the effective equation of motion for the r coordinate

$$\dot{\phi} = \frac{J}{mr^2}, \qquad m\ddot{r} = -\frac{\partial U_{eff}(r)}{\partial r},$$

where $U_{eff}(r)$ is given by

$$U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GMm}{r}.$$

- For now I am not interested in the time evolution and only want to find the trajectory of the body. This trajectory is given by the function $r(\phi)$.
- However, if we know $r(\phi)$, we can solve $\dot{\phi} = \frac{J}{mr^2(\phi)}$ and find $\phi(t)$. Then we will also have $r(\phi(t))$. Thus one can consider finding of $r(\phi)$ as the first step in full solution.
- In order to find $r(\phi)$ I will use the trick we used before

$$\dot{r} = \frac{dr}{dt} = \frac{d\phi}{dt}\frac{dr}{d\phi} = \frac{J}{mr^2}\frac{dr}{d\phi} = -\frac{J}{m}\frac{d(1/r)}{d\phi}, \qquad \frac{d^2r}{dt^2} = \frac{d\phi}{dt}\frac{d\dot{r}}{d\phi} = -\frac{J^2}{m^2r^2}\frac{d^2(1/r)}{d\phi^2}$$

- On the other hand

$$\frac{\partial U_{eff}}{\partial r} = -\frac{J^2}{m}(1/r)^3 + GMm\,(1/r)^2.$$

- Now I denote $u(\phi) = 1/r(\phi)$ and get
$$-\frac{J^2}{m}u^2\frac{d^2u}{d\phi^2} = \frac{J^2}{m}u^3 - GMmu^2$$
or, denoting $\frac{d^2u}{d\phi^2} \equiv u''$
$$u'' = -u + \frac{GMm^2}{J^2}.$$
- The general solution of this equation is
$$u = \frac{GMm^2}{J^2} + A\cos(\phi - \phi_0),$$
where A and ϕ_0 are arbitrary constants.
- We can put $\phi_0 = 0$ by redefinition.
- Before I do that, I want to point out that this is cheating. The constants A and ϕ_0 should be obtained from the initial conditions. So unless we know how to get ϕ_0 from the initial conditions we cannot redefine our system of coordinates to measure the angle from the direction of ϕ_0. However, we know that such redefinition exists. We will discuss the issue of finding ϕ_0 from the initial conditions later and now we just go ahead and redefine ϕ.
- So by setting $\phi_0 = 0$ we have
$$\frac{1}{r} = \gamma + A\cos\phi, \qquad \gamma = \frac{GMm^2}{J^2}$$
If $\gamma = 0$ this is the equation of a straight line in the polar coordinates.
- A more conventional way to write the trajectory is
$$\frac{1}{r} = \frac{1}{c}\left(1 + \epsilon\cos\phi\right), \qquad c = \frac{J^2}{GMm^2} = \frac{1}{\gamma}$$
where $\epsilon > 0$ is dimentionless number. This is the equation of ellipse in polar coordinates. ϵ is called the eccentricity of the ellipse, it controls the "shape" of the ellipse, while c has a dimension of length and it controls the "size" of the ellipse.
- We see that
  - If $\epsilon < 1$ the orbit is periodic.
  - If $\epsilon < 1$ the minimal and maximal distance to the center — the perihelion and aphelion are at $\phi = 0$ and $\phi = \pi$ respectively.
$$r_{min} = \frac{c}{1+\epsilon}, \qquad r_{max} = \frac{c}{1-\epsilon}$$
  - If $\epsilon > 1$, then the trajectory is unbounded.
  - If $\epsilon \to \infty$ the trajectory is the straight line. (the only way to make this limit meaningful is to also take $c \to \infty$, which means $J \to \infty$. So the planet is either moving too far, or moving too fast.)
- If we know c and ϵ we know the orbit, so we must be able to find out J and E from c and ϵ. By definition of c we find $J^2 = cGMm^2$. In order to find E, we notice, that at $r = r_{min}$, $\dot{r} = 0$, so at this moment $v = r_{min}\dot{\phi} = J/mr_{min}$, so the kinetic energy $K = mv^2/2 = J^2/2mr_{min}^2$, the potential energy is $U = -GmM/r_{min}$. So the total

energy is

$$E = K + U = -\frac{1-\epsilon^2}{2}\frac{GmM}{c}, \qquad J^2 = cGMm^2,$$

Indeed we see, that if $\epsilon < 1$, $E < 0$ and the orbit is bounded.

- The ellipse can be written as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with

$$a = \frac{c}{1-\epsilon^2}, \quad b = \frac{c}{\sqrt{1-\epsilon^2}}, \quad d = a\epsilon, \quad b^2 = ac.$$

- One can check, that the position of the large mass M is one of the focuses of the ellipse — NOT ITS CENTER!
- This is the **first Kepler's law**: all planets go around the ellipses with the sun at one of the foci.

17.1. Kepler's second law

The conservation of the angular momentum reads

$$\frac{1}{2}r^2\dot{\phi} = \frac{J}{2m}.$$

We see, that in the LHS rate at which a line from the sun to a comet or planet sweeps out area:

$$\frac{dA}{dt} = \frac{J}{2m}.$$

This rate is constant! So

- **Second Kepler's law:** A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

17.2. Kepler's third law

Consider now the closed orbits only. There is a period T of the rotation of a planet around the sun. We want to find this period.

The total area of an ellipse is $A = \pi ab$, so as the rate dA/dt is constant the period is

$$T = \frac{A}{dA/dt} = \frac{2\pi abm}{J},$$

Now we square the relation and use $b^2 = ac$ and $c = \frac{J^2}{GMm^2}$ to find

$$T^2 = 4\pi^2\frac{m^2}{J^2}a^3c = \frac{4\pi^2}{GM}a^3$$

Notice, that the mass of the planet and its angular momentum canceled out! so

- **Third Kepler's law:** For all bodies orbiting the sun the ratio of the square of the period to the cube of the semimajor axis is the same.

This is one way to measure the mass of the sun. For all planets one plots the cube of the semimajor axes as y and the square of the period as x. One then draws a straight line through all points. The slope of that line is $GM/4\pi^2$.

LECTURE 18
Another derivation. Conserved Laplace-Runge-Lenz vector.

18.1. Another way.

- Another way to solve the problem is starting from the following equations:

$$\dot{\phi} = \frac{J}{mr^2(t)}, \qquad \frac{m\dot{r}^2}{2} + U_{eff}(r) = E, \qquad U_{eff}(r) = \frac{J^2}{2mr^2} + U(r).$$

- For now we am not interested in the time evolution and only want to find the trajectory of the body. This trajectory is given by the function $r(\phi)$. In order to find it we express $\dot{r}$ from the second equation and divide it by $\dot{\phi}$ from the first. We then find

$$\frac{\dot{r}}{\dot{\phi}} = \frac{dr}{d\phi} \quad \text{and} \quad \frac{\dot{r}}{\dot{\phi}} = r^2\sqrt{\frac{2m}{J^2}}\sqrt{E - U_{eff}(r)},$$

or

$$\frac{J}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E - U_{eff}(r)}} = d\phi, \qquad \frac{J}{\sqrt{2m}}\int_{r(\phi_0}^{r}\frac{dr'}{r'^2\sqrt{E - U_{eff}(r')}} = \phi - \phi_0,$$

where E, $\vec{J}$, ϕ_0, and $r(\phi_0)$ (total 6) are given by initial conditions.
- These formulas give the trajectory for any central potential $U(r)$.
- For the $\sim 1/r$ potential the integral becomes a standard one after substitution $x = 1/r$.

18.2. A hidden symmetry.

Let's assume, that we have some central attractive potential $U(r)$, which decays to zero at infinity.

- The problem is mapped to a one dimensional problem for the coordinate r and effective potential energy $U_{eff}(r) = \frac{J^2}{2mr^2} + U(r)$.
- For total energy $E < 0$ we have bounded motion for r between r_{min} and r_{max}.

- We can compute the time T_r for a particle to go from r_{min} to r_{max} and back

$$T_r = \sqrt{2m}\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{E-U_{eff}(r)}}, \quad \text{where } r_{min} \text{ and } r_{max} \text{ are the solutions of } E = U_{eff}(r).$$

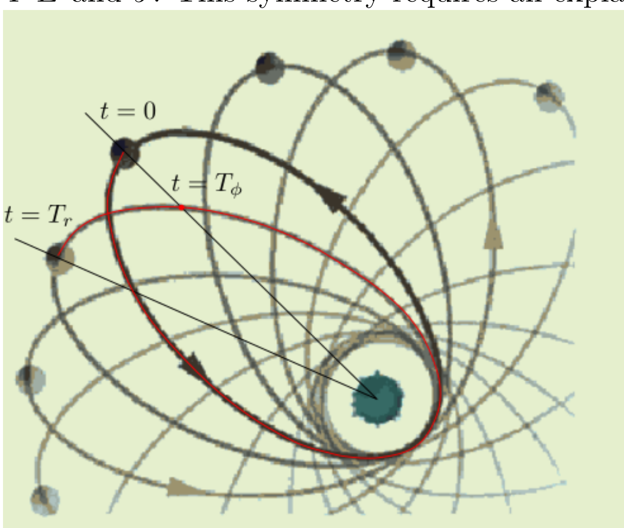
- We can also compute $r(t)$.
- We then can compute the time T_ϕ it takes for the angle ϕ to change by 2π

$$2\pi = \frac{J}{m}\int_0^{T_\phi} \frac{dt}{r^2(t)}.$$

- The two times T_r and T_ϕ do not necessarily coincide.
- It is also a very special condition that T_r and T_ϕ coincide for ANY E and J!

If $T_r \neq T_\phi$ the orbit is bounded, but not closed — this is the general situation.

It is a very special property of the gravitational (or Coulomb) potential that $T_r = T_\phi$ for ANY E and J. This symmetry requires an explanation.



If $U(r)$ is the gravitation potential energy with a small correction this discrepancy between T_r and T_ϕ is small. The orbit is almost closed, or one can say that it precesses.

18.3. Conserved vector $\vec{A}$.

The Kepler problem has an interesting additional symmetry. This symmetry ensures that $T_r = T_\phi$ (for any E and J). As usual this symmetry also leads to the conservation of the Laplace-Runge-Lenz vector $\vec{A}$. If the gravitational force is $\vec{F} = -\frac{k}{r^2}\vec{e}_r$, then we define:

$$\vec{A} = \vec{p}\times\vec{J} - mk\vec{e}_r,$$

where $\vec{J} = \vec{r}\times\vec{p}$. This vector can be defined for both gravitational and Coulomb forces: $k>0$ for attraction and $k<0$ for repulsion.

An important feature of the "inverse square force" is that this vector is conserved. Let's check it. First we notice, that $\dot{\vec{J}} = 0$, so we need to calculate:

$$\dot{\vec{A}} = \dot{\vec{p}}\times\vec{J} - mk\dot{\vec{e}}_r$$

Now using

$$\dot{\vec{p}} = \vec{F}, \qquad \dot{\vec{e}}_r = \vec{\omega} \times \vec{e}_r = \frac{1}{mr^2} \vec{J} \times \vec{e}_r$$

We then see

$$\dot{\vec{A}} = \vec{F} \times \vec{J} - \frac{k}{r^2} \vec{J} \times \vec{e}_r = \left(\vec{F} + \frac{k}{r^2} \vec{e}_r \right) \times \vec{J} = 0$$

So this vector is indeed conserved.

The question is: Is this conservation of vector $\vec{A}$ an independent conservation law? There are three components of the vector $\vec{A}$ are there three new conservation laws?
The answer is that not all of them are independent.

- As $\vec{J} = \vec{r} \times \vec{p}$ is orthogonal to $\vec{e}_r$, we see, that $\vec{J} \cdot \vec{A} = 0$. So the component of $\vec{A}$ perpendicular to the plane of the planet rotation is always zero.
- Now let's calculate the magnitude of this vector

  $$\vec{A} \cdot \vec{A} = \vec{p}^2 \vec{J}^2 - (\vec{p} \cdot \vec{J})^2 + m^2 k^2 - 2mk \vec{e}_r \cdot [\vec{p} \times \vec{J}] = \vec{p}^2 \vec{J}^2 + m^2 k^2 - \frac{2mk}{r} \vec{J} \cdot [\vec{r} \times \vec{p}]$$

  $$= 2m \left(\frac{\vec{p}^2}{2m} - \frac{k}{r} \right) \vec{J}^2 + m^2 k^2 = 2mE\vec{J}^2 + m^2 k^2.$$

  So we see, that the magnitude of $\vec{A}$ is not an independent conservation law.
- Using the relation between the eccentricity ϵ with $\vec{J}^2$ and E from the last lecture we find, that

  $$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} = \epsilon km$$
- We are left with only the direction of $\vec{A}$ within the orbit plane. Let's check this direction. As the vector is conserved we can calculate it in any point of orbit.
- So let's consider the perihelion. At perihelion $\vec{p}_{per} \perp \vec{r}_{per} \perp \vec{J}$, where the subscript *per* means the value at perihelion.
- Simple examination shows that $\vec{p}_{per} \times \vec{J} = p_{per} J \vec{e}_{per}$. Then at the perihelion $\vec{A} = (p_{per} J - mk) \vec{e}_{per}$.
- However, vector $\vec{A}$ is a constant of motion, so if it has this magnitude and direction in one point it will have the same magnitude and direction at all points!
- We computed its magnitude before $|\vec{A}| = \epsilon km$, so

  $$\vec{A} = mk\epsilon \vec{e}_{per}.$$

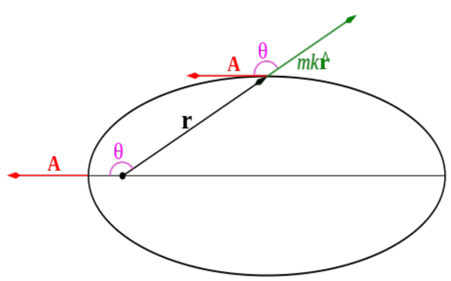
  We see, that for Kepler orbits $\vec{A}$ points to the point of the trajectory where the planet or comet is the closest to the sun.
- So we see, that $\vec{A}$ provides us with only one new independent conserved quantity.
- It also means, that if we know the velocity and the position of a planet or a comet at any time, we can compute the vector $\vec{A}$ at this moment of time and immediately know the position of the perihelion. And this position is constant — no precession.

We can also compute r_{min}, so we will know close, say, a comet will come to the sun and where the point of the closest approach will be. We can compute this from just the initial conditions and without solving any differential equations.
But we can do more!

LECTURE 19

Change of orbits. Virial theorem. Kepler orbits for comparable masses.

19.1. Kepler orbits from $\vec{A}$.



Last lecture we showed, that for the central force $\vec{F} = -\frac{k}{r^2}\vec{e}_r$ the vector

$$\vec{A} = \vec{p} \times \vec{J} - mk\vec{e}_r,$$

is conserved.

The existence of an extra conservation law simplifies many calculations. For example we can derive equation for the trajectories without solving any differential equations. Let's do just that.

Let's derive the equation for Kepler orbits (trajectories) from our new knowledge of the conservation of the vector $\vec{A}$. For this we consider $\vec{r} \cdot \vec{A}$.

$$\vec{r} \cdot \vec{A} = \vec{r} \cdot [\vec{p} \times \vec{J}] - mkr = J^2 - mkr$$

On the other hand

$$\vec{r} \cdot \vec{A} = rA\cos\theta, \quad \text{so} \quad rA\cos\theta = J^2 - mkr$$

Or

$$\frac{1}{r} = \frac{mk}{J^2}\left(1 + \frac{A}{mk}\cos\theta\right), \qquad c = \frac{J^2}{mk}, \qquad \epsilon = \frac{A}{mk}.$$

19.2. Change of orbits.

Consider a problem to change from an circular orbit Γ_1 of a radius R_1 to an orbit Γ_2 with a radius $R_2 > R_1$.

- For the transition we will use an elliptical orbit γ with $r_{min} = R_1$ and $r_{max} = R_2$.
- We need two boosts. One to go from Γ_1 to γ, and the second one to go from γ to Γ_2.
- The final speed on Γ_2 will be less than that on Γ_1.

19.3. Spreading of debris after a satellite explosion.

19.4. Virial theorem

Let's consider a collection of N particles interacting with each other. Let's assume that they undergo some motion with a period T — it also means that we are in the center of mass frame of reference. Then we can define an averaged quantities as follows: Let's imagine that we have a quantity $P(\vec{r}_i, \dot{\vec{r}}_i)$ which depends on the coordinates and the velocities of all particles. Then we define an average

$$\langle P\rangle = \frac{1}{T}\int_0^T P(\vec{r}_i, \dot{\vec{r}}_i)dt$$

Now let's calculate average total kinetic energy $K = \sum_i \frac{m_i \dot{\vec{r}}_i^2}{2}$

$$\langle K\rangle = \frac{1}{T}\int_0^T \sum_i \frac{m_i \dot{\vec{r}}_i^2}{2}dt = \sum_i \frac{m_i}{2}\frac{1}{T}\int_0^T \dot{\vec{r}}_i^2 dt = \sum_i \frac{m_i}{2}\frac{1}{T}\int_0^T \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i dt$$

Taking the last integral by parts and using the periodicity to cancel the boundary terms we get

$$\langle K\rangle = -\frac{1}{2}\sum_i \frac{1}{T}\int_0^T \vec{r}_i \cdot m_i \ddot{\vec{r}}_i dt = -\frac{1}{2}\sum_i \frac{1}{T}\int_0^T \vec{r}_i \cdot \vec{F}_i dt = -\frac{1}{2}\frac{1}{T}\int_0^T \sum_i \vec{r}_i \cdot \vec{F}_i dt,$$

where $\vec{F}_i$ is the total force which acts on the particle i.

So we find

$$2\langle K\rangle = -\left\langle \sum_i \vec{r}_i \cdot \vec{F}_i \right\rangle.$$

So far it was all very general. Now lets assume that all the forces are the forces of Coulomb/Gravitation interaction between the particles.

$$\vec{F}_i = \sum_{j\neq i}\vec{F}_{ij}, \qquad \vec{F}_{ij} = -\frac{k}{r_{ij}^2}\vec{e}_{ij},$$

where $\vec{e}_{ij}$ is a unit vector pointing from j to i and $r_{ij} = |\vec{r}_i - \vec{r}_j|$. We then have for any moment of time

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \sum_{i\neq j}\vec{r}_i \cdot \vec{F}_{ij} = \sum_{i>j}(\vec{r}_i - \vec{r}_j)\cdot \vec{F}_{ij} = -\sum_{i>j} r_{ij}\frac{k}{r_{ij}^2} = U,$$

where U is the total potential energy of the system of the particles at the given moment of time. So we have

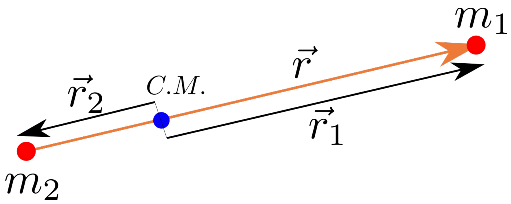
$$2\langle K\rangle = -\langle U\rangle$$

This is called the virial theorem. It also can be written as $E = -\langle K\rangle$.

It is important, that the above relation is stated for the AVERAGES only. for example in the perihelion of a Kepler orbit we know that $2K_{per}(1+\epsilon) = -U_{per}$.

On the other hand for the circular orbit kinetic and potential energies are constant in time, so the averages are just the values.

19.5. Kepler orbits for comparable masses.



If the bodies interact only with one another and no external force acts on them, then the center of mass has a constant velocity. We then can attach our frame of reference to the center of mass and work there. This way we will only be studying the relative motion of the bodies.

Let's now consider two bodies with masses m_1 and m_2 interacting by a gravitational force. We will use center of mass system of reference and place our coordinate origin at the center of mass. If the position of m_1 is given by $\vec{r}_1$ and the position of m_2 is given by $\vec{r}_2$, then as the center of mass is in the origin we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0,$$

then the vector $\vec{r}$ from the mass m_2 to the mass m_1 is

$$\vec{r} = \vec{r}_1 - \vec{r}_2 = \frac{m_1 + m_2}{m_2}\vec{r}_1.$$

Then the equation of motion for the mass m_1 is

$$m_1\ddot{\vec{r}}_1 = -\frac{k}{r^2}\vec{e}_r, \qquad \frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} = -\frac{k}{r^2}\vec{e}_r, \qquad \mu\ddot{\vec{r}} = -\frac{k}{r^2}\vec{e}_r,$$

where μ is a "reduced mass"

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

We then see, that the problem has reduced to a motion of a single body of a "reduced mass" μ under the same force. This is our standard problem, that we have solved before.

In the case of gravitation we can go further and us $k = Gm_1m_2 = G\frac{m_1m_2}{m_1+m_2}(m_1 + m_2) = G\mu M$, where $M = m_1 + m_2$ — the total mass. So the equation of motion is

$$\mu\ddot{\vec{r}} = -\frac{G\mu M}{r^2}\vec{e}_r,$$

Or just a motion of a particle of mass μ in the gravitational field of a fixed (immovable) mass M.

What one must not forget, though, is that after $\vec{r}(t)$ is found one still need to find $\vec{r}_1(t) = \frac{\mu}{m_1}\vec{r}(t)$ and $\vec{r}_2(t) = -\frac{\mu}{m_2}\vec{r}(t)$ to know the positions and motions of the real bodies.

LECTURE 20
Panegyric to Newton. Functionals.

20.1. How to see $F = \frac{GMm}{r^2}$ from Kepler's laws.

Here I will show how the Newton's gravity could be derived from the Kepler's laws. Kepler found Kepler's laws from the observations of the planet's motion. It is clear that there should be some attraction between the planets and the sun. How do we find the force of this attraction if we only know the Kepler's laws/observations and the Newton's laws of mechanics. In other words how could Newton figure out that the force of gravity is $F = \frac{GMm}{r^2}$?

The crucial observations made by Kepler were

- All planets move along ellipses with the sun in the focus. Different planet's ellipses have different eccentricity and different size.
- The ratio of the square of the period of orbit T to the cube of the large semi-axis a of the ellipses is the same for all planets — this ration does not depend on the mass of the planet or the eccentricity of the planet's orbit.

The argument, then is the following:

- As the ratio T^2/a^3 does not depend on eccentricity, it must be the same if a planet had a perfectly circular orbit. The radius of this orbit r will play the role of the large semi-axis.
- Let's consider this orbit of radius r. There is a force that acts on the planet $F(r)$, and we must have
$$m\frac{v^2}{r} = F(r),$$
where m is the mass of the planet and v is its velocity.
- The period of rotation is
$$T = \frac{2\pi r}{v}.$$
- So
$$\frac{T^2}{r^3} = (2\pi)^2\frac{r^2}{v^2}\frac{1}{r^3} = m(2\pi)^2\frac{1}{r^2F(r)}.$$
- As this ratio must not depend neither on mass m nor on the radius r, we then must have
$$F(r) \sim \frac{m}{r^2}$$

- If the sun attracts the planet with such a force, then the planet must attract the sun with the same force. But then, according to the above formula the force must be proportional to the mass of the sun. So we have

$$F(r) = G\frac{Mm}{r^2},$$

where G is just some constant.

This is not the complete proof. We need to take the force we found, compute the arbitrary orbits, and show, that they are ellipses — just as Kepler observed.

20.2. Difference between functions and functionals.

- A *function* establishes a correspondence/map between elements of one set with elements of another. Usually for a number x it gives back a (single) number y according to some rule: $y = f(x)$, where f denotes this rule. So a function is a rule according to which if I give it a number it returns back a number. For example the function $f(x) = x^2$ — it is a rule, according to which if I have a number x, I need to square it and return the result back. Two different x' may return back the same number. For the previous example the numbers x and $-x$ will return the same value of $f(x)$.

$$f: \text{ number} \longrightarrow \text{ number}.$$

A function of many variables is a rule by which it takes a few numbers and returns one number.

- A *functional* establishes a correspondence/map between functions and numbers. Normally one has to restrict the space of functions. So a functional is a rule which one applies to a function from established space receives back a number. Or if you give a function to a functional it returns back a number. In order to define a functional we must define the space of functions it can act on and a rule by which it returns/computes a number if we give it a function from that space.

$$F: \text{ function} \longrightarrow \text{ number}.$$

A functional can take more than one function as an argument.

- An *operator* takes a function from defined subspace and returns back a function (from the same subspace).

$$\hat{O}: \text{ function} \longrightarrow \text{ function}.$$

We will not be dealing with operators.

20.3. Examples of functionals.

- Everyday examples.
- Area under the graph: for a (integrable) functions on interval $[a, b]$ we can define a functional

$$A[f(x)] = \int_a^b f(x)dx.$$

That means, that if you have a function $f(x)$ which belongs to our space (it is integrable on the interval $[a, b]$) we can construct the number — the area under the graph. This is the rule which defines out functional.

- Length of a path.
  - Our space is the space of smooth functions on the interval $[a, b]$.
  - For any graph $y(x)$ we can compute its length
$$\mathcal{L}[y(x)] = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$
  - Let's now take a path $x(t)$, $y(t)$, where $t \in [a, b]$ is a parameter. Both $x(t)$, $y(t)$ are smooth. Then the length of this path is
$$\mathcal{L}[x(t), y(t)] = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

It is important to specify the space of functions.

LECTURE 21
More on functionals.

21.1. Examples of functionals. Continued.

- Length of a path. Invariance under reparametrization.
  - In the last lecture we considered a path $x(t)$, $y(t)$, where $t \in [a, b]$ is a parameter. Both $x(t)$, $y(t)$ are smooth. Then the length of this path is

$$\mathcal{L}[x(t), y(t)] = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

  - Let's now change this parameter. Namely we take t to be a function of another parameter τ: $t(\tau)$. The very same graph is given by $x(\tau) = x(t(\tau))$ and $y(\tau) = y(t(\tau))$. Then the length is

$$\mathcal{L}[x(\tau), y(\tau)] = \int_{a_\tau}^{b_\tau} \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2} d\tau,$$

    where $t(a_\tau) = a$, $t(b_\tau) = b$. Using the chain rule we get $\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau}$ and the same for $\frac{dx}{d\tau}$, as well as $d\tau = \frac{d\tau}{dt}dt$ we will get exactly the same expression as before. So the length – the functional – is invariant under reparametrization.
  - In N dimensional space a curve is given by smooth functions $x_i(t)$, $i = 1 \ldots N$. The (Euclidean) length of this curve is given by

$$\mathcal{L}[x_i(t)] = \int_a^b \sqrt{\frac{dx_i}{dt}\frac{dx_i}{dt}} dt.$$

    It is a functional on N functions.
- Energy of a horizontal string in the gravitational field.
  - Consider a rope linear density ρ and length L. We attach it to two nails distance $l < L$ apart which are on the same height. What is the potential energy of the rope which has a shape given by a function $y(x)$? (y-vertical, x-horizontal)

  - Consider a small piece of the rope. It has a mass $\rho\sqrt{(dx)^2+(dy)^2}$. The potential energy of this piece is $\rho g y\sqrt{(dx)^2+(dy)^2}$. So the total potential energy is
$$U[y(x)] = \rho g \int_0^l y(x)\sqrt{1+(y')^2}dx.$$
  - It is a functional on a space of smooth functions $y(x)$ in the interval $[0,l]$ which satisfy the constraint
$$L = \mathcal{L}[y(x)]$$
- Value at a point as functional. The functional which for any function returns the value of the function at a given point.
- Functions of many variables. Area of a surface. Invariance under reparametrization.

It is important to specify the space of functions.

21.2. General form of the functionals.

- We need to establish a rule which will allow to compute a number for a function.
- General form $\int_{x_1}^{x_2} L(x,y,y',y'',\dots)dx$. Important: In function L the y, y', y'' and so on are independent variables. It means that we consider a function $L(x,z_1,z_2,z_3,\dots)$ of normal variables x, z_1, z_2, z_3, …and for any function $y(x)$ at some point x we calculate $y(x)$, $y'(x)$, $y''(x)$, …and plug x and these values instead of z_1, z_2, z_3, …in $L(x,z_1,z_2,z_3,\dots)$. We do that for all points x, and then do the integration.

21.3. Discretization. Fanctionals as functions.

Let's consider a functional $A[f(x)]$ acting on the functions from some well defined space, let's say on smooth functions on the interval $[a,b]$. We can do the following trick.

- Consider the variable x to be discretized: instead of thinking of x as a continuous variable we will select N points x_i in the interval $[a,b]$. Lat's also take $x_1 = a$, $x_N = b$.
- Eventually we will need to take a limit $N\to\infty$. This limit should be taken in such a way, that $\max(\Delta x_i)\to 0$.
- A function $f(x)$ is then represented by its values f_i at x_i: $f_i = f(x_i)$.
- Then the functional $A[f(x)]$ can be thought as a function of the values f_i: $A[f(x)] = A(f_1,\dots,f_N)$.
- We then can deal with the functional as a with the function of many variables.
- At the end we must take the limit $N\to\infty$ as described above, and make sure, that such limit does exist.

In many non-trivial cases this procedure allows one to make sense out of the calculations.

If you are to compute the value of a functional numerically, then this procedure is exactly what you have to do.

LECTURE 22
Euler-Lagrange equation

22.1. Minimization problem

What kind of problems can we state with the functionals?

One of the most important problem (but not the only one) is stated as following: given a functional $A[f(x)]$ (remember, that the space the functional works on is a part of its definition) which function (from the defined space) will give the smallest (or the largest) value of the functional? How do we find this function?

For an arbitrary functional such function may not exist. Moreover, generally if you change the space you will find a different answer. In many cases, if you change the space the question will not have an answer.

Notice, that this is exactly the same situation as with functions. A function may or may not have minimum or maximum on a given interval. This statement depends on the interval. For example a function $1/x$ has no maximum or minimum in the interval $[-1, 1]$, but it has a minimum and a maximum in the interval $[1, 2]$. The position of the maximum and minimum depends on the interval boundaries.

In the following examples notice the importance of defining the space of functions.

- Minimal distance between two points.
- Minimal time of travel. Ferma Principe.
- Minimal potential energy of a string.
- etc.

22.2. Minimum of a function.

Before we derive the equation for the function which minimizes a functional. Let's remember how it is done for functions.

The questions is: if we have a function $f(x)$ how do we find the position x_0 of its minimum?

There are different ways to think about it. I want to emphasize the following line of arguments:

- Let's assume, that we know the position of the minimum x_0.
- Let's consider x which is very close to x_0.

- We know that if x is close enough to x_0 the value of the function at x_0 can be represented as a series ($\delta x \equiv x - x_0$)

$$f(x) = f(x_0) + a_1\delta x + a_2(\delta x)^2 + \dots$$

where the coefficients a_1, a_2, etc. are the coefficients of the Taylor expansion. They are some fixed numbers!
- In this series for δx small enough the term $a_1\delta x$ is dominant. And it's dominance is the larger then smaller δx is.
- So for very small δx we can write

$$\delta f = f(x) - f(x_0) \approx a_1\delta x.$$

- As $f(x_0)$ is the minimum, for small enough δx we must have $\delta f > 0$. This must be true for both positive and negative δx!
- The only way to have ensure this inequality is to have

$$a_1 = 0.$$

- Then the Taylor expansion starts with the term $a_2(\delta x)^2$ which is positive if $a_2 > 0$ for any δx.
- According to Taylor expansion $a_1 = \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$. So to find the minimum we need to solve the equation

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = 0$$

Notice, that the condition for leads to the equation above is that the change of the function in the first order in δx is zero!

22.3. The Euler-Lagrange equations

- The functional $A[y(x)] = \int_{x_1}^{x_2} L(y(x), y'(x), x)dx$ with the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.
- The problem is to find a function $y(x)$ which is the stationary "point" of the functional $A[y(x)]$.
- The stationary "point" of a functional $A[y(x)] = \int_{x_1}^{x_2} L(x, y(x), y'(x))dx$ for the functions satisfying $y(x_1) = y_1$, $y(x_2) = y_2$ is given by the solution of Euler-Lagrange equation.
- Euler-Lagrange equation is the second order differential equation with boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$.
- Derivation of the Euler-Lagrange equation.
  - Let's assume, that we found the function $y_0(x)$ which gives as a minimum of the functional $A[y(x)] = \int_{x_1}^{x_2} L(x, y(x), y'(x))dx$ for the functions satisfying $y(x_1) = y_1$, $y(x_2) = y_2$.
  - Lets shift this function a little and consider the function $y(x) = y_0(x) + \delta y(x)$, where $\delta y(x)$ is small/infinitesimal.
  - The new function $y(x)$ must be from the same space, so me must have

$$\delta y(x_1) = 0, \quad \delta y(x_2) = 0. \tag{22.1}$$

– The value of our functional on the new function is
$$A[y_0(x)+\delta y(x)] = \int_{x_1}^{x_2} L(x, y_0(x)+\delta y(x), y_0'(x)+\delta y'(x))dx$$
– Let's compute $A[y_0(x)+\delta y(x)]$ up to the linear order in $\delta y(x)$:
$$A[y_0(x)+\delta y(x)] \approx \int_{x_1}^{x_2} \left[L(x, y_0(x), y_0'(x)) + \left.\frac{\partial L}{\partial y}\right|_{y=y_0(x)} \delta y + \left.\frac{\partial L}{\partial y'}\right|_{y'=y_0'(x)} \delta y' \right] dx$$
Here I treated $L(x,y,y')$ as just a function of its INDEPENDENT variables x, y, and y', differentiated it with respect to these variables and then plugged y_0 instead of y and y_0' instead of y'.
– To shorten notations I will use $\frac{\partial L}{\partial y_0}$ to mean $\left.\frac{\partial L}{\partial y}\right|_{y=y_0(x)}$, and the same for the primed term.
– Notice, that after this substitution $y=y_0(x)$ the functions $\frac{\partial L}{\partial y_0}$ and $\frac{\partial L}{\partial y_0'}$ are the functions of x only!
– The first term under the integral is what is in $A[y_0(x)]$ — the value of the functional at the minimum.
$$A[y_0(x)+\delta y(x)] \approx A[y_0(x)] + \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial y_0}\delta y + \frac{\partial L}{\partial y_0'}\delta y' \right] dx$$
– Let's call $\delta A = A[y_0(x)+\delta y(x)] - A[y_0(x)]$. It is called variation of the functional.
$$\delta A \approx \int_{x_1}^{x_2} \frac{\partial L}{\partial y_0}\delta y(x)dx + \int_{x_1}^{x_2} \frac{\partial L}{\partial y_0'}\frac{d\delta y(x)}{dx}dx$$
– Notice, that in the last term in $\frac{d\delta y(x)}{dx}$ it is a full derivative over x. The function $\frac{\partial L}{\partial y_0'}$ is a function of x only, as we already plugged $y_0(x)$ instead of y and $y_0'(x)$ instead of y'.
– I will use the partial integration on that term
$$\delta A \approx \int_{x_1}^{x_2} \frac{\partial L}{\partial y_0}\delta y(x)dx + \delta y(x)\left.\frac{\partial L}{\partial y_0'}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y(x)\frac{d}{dx}\frac{\partial L}{\partial y_0'}dx.$$
Notice that in this step $\frac{d}{dx}\frac{\partial L}{\partial y'}$ assumes full differentiation over x.
– Now we use the boundary conditions (22.1) and see that
$$\delta y(x)\left.\frac{\partial L}{\partial y_0'}\right|_{x_1}^{x_2} = 0.$$
– So we have
$$\delta A \approx \int_{x_1}^{x_2} \delta y(x)\left[\frac{\partial L}{\partial y_0} - \frac{d}{dx}\frac{\partial L}{\partial y_0'} \right] dx.$$
– This equation tells us how the value of the functional $A[y(x)]$ changes, when we change the function from the minimum $y_0(x)$ by an ARBITRARY infinitesimal function δy (subject, of course to (22.1)).
– As the function $\delta y(x)$ is arbitrary, the value of the integral $\int_{x_1}^{x_2} \delta y(x) \left[\frac{\partial L}{\partial y_0} - \frac{d}{dx}\frac{\partial L}{\partial y_0'} \right] dx$ can be either positive or negative.

– But the function $y_0(x)$ is the minimum! If we shift from the minimum we can only go up, so the value δA must always be positive! (or non-negative in the linear order – it will become positive in the quadratic order)
– The only way to ensure that δA is non-negative for ARBITRARY $\delta y(x)$ is to demand, that

$$\frac{\partial L}{\partial y_0} - \frac{d}{dx}\frac{\partial L}{\partial y_0'} = 0.$$

The statement then is that the function $y_0(x)$ must be such as to satisfy this equation.

- The Euler-Lagrange equation reads

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}, \qquad y(x_1) = y_1, \quad y(x_2) = y_2$$

This is the second order differential equation with boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$.

22.4. Example

- Shortest path $\int_{x_1}^{x_2} \sqrt{1+(y')^2}dx$, $y(x_1) = y_1$, and $y(x_2) = y_2$.

$$L(y(x), y'(x), x) = \sqrt{1+(y')^2}, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}.$$

the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = 0, \qquad \frac{y'}{\sqrt{1+(y')^2}} = \text{const.}, \qquad y'(x) = \text{const.}, \qquad y = ax + b.$$

The constants a and b should be computed from the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

LECTURE 23
Euler-Lagrange equation continued.

23.1. Example

- Shortest time to fall – Brachistochrone.
  - What path the rail should be in order for the car to take the least amount of time to go from point A to point B under gravity if it starts with zero velocity.
  - Lets take the coordinate x to go straight down and y to be horizontal, with the origin in point A.
  - The boundary conditions: for point A: $y(0) = 0$; for point B: $y(x_B) = y_B$.
  - The time of travel is

$$T = \int \frac{ds}{v} = \int_0^{x_B} \frac{\sqrt{1+(y')^2}}{\sqrt{2gx}} dx.$$

  - We have

$$L(y, y', x) = \frac{\sqrt{1+(y')^2}}{\sqrt{2gx}}, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial y'} = \frac{1}{\sqrt{2gx}} \frac{y'}{\sqrt{1+(y')^2}}.$$

  - The Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}} \frac{y'}{\sqrt{1+(y')^2}}\right) = 0, \qquad \frac{1}{x} \frac{(y')^2}{1+(y')^2} = \frac{1}{2a}, \qquad y'(x) = \sqrt{\frac{x}{2a-x}}$$

  - So the path is given by

$$y(x) = \int_0^x \sqrt{\frac{x'}{2a-x'}} dx'$$

  - The integral is taken by substitution $x = a(1-\cos\theta)$. It then becomes $a\int(1 - \cos\theta)d\theta = a(\theta - \sin\theta)$. So the path is given by the parametric equations

$$x = a(1-\cos\theta), \qquad y = a(\theta - \sin\theta).$$

  the constant a must be chosen such, that the point x_B, y_B is on the path.

23.2. Reparametrization

The *form* of the Euler-Lagrange equation does not change under the reparametrization.

Consider a functional and corresponding E-L equation

$$A = \int_{x_1}^{x_2} L(y(x), y'_x(x), x)dx, \qquad \frac{d}{dx}\frac{\partial L}{\partial y'_x} = \frac{\partial L}{\partial y(x)}$$

Let's consider a new parameter ξ and the function $x(\xi)$ converts one old parameter x to another ξ. The functional

$$A = \int_{x_1}^{x_2} L(y(x), y'_x(x), x)dx = \int_{\xi_1}^{\xi_2} L\left(y(\xi), y'_\xi \frac{d\xi}{dx}, x\right)\frac{dx}{d\xi}d\xi,$$

where $y(\xi) \equiv y(x(\xi))$. So that

$$L_\xi = L\left(y(\xi), y'_\xi \frac{d\xi}{dx}, x\right)\frac{dx}{d\xi}$$

The E-L equation then is

$$\frac{d}{d\xi}\frac{\partial L_\xi}{\partial y'_\xi} = \frac{\partial L_\xi}{\partial y(\xi)}$$

Using

$$\frac{\partial L_\xi}{\partial y'_\xi} = \frac{dx}{d\xi}\frac{\partial L}{\partial y'_x}\frac{d\xi}{dx} = \frac{\partial L}{\partial y'_x}, \qquad \frac{\partial L_\xi}{\partial y(\xi)} = \frac{dx}{d\xi}\frac{\partial L}{\partial y(x)}$$

we see that E-L equation reads

$$\frac{d}{d\xi}\frac{\partial L}{\partial y'_x} = \frac{dx}{d\xi}\frac{\partial L}{\partial y(x)}, \qquad \frac{d}{dx}\frac{\partial L}{\partial y'_x} = \frac{\partial L}{\partial y(x)}.$$

So we return back to the original form of the E-L equation.

What we found is that E-L equations are invariant under the parameter change.

23.3. The Euler-Lagrange equations, for many variables.

If we have a functional of two functions $y(x)$ and $z(x)$

$$\mathcal{A} = \int_{x_1}^{x_2} L(x, y(x), z(x), y'(x), z'(x))dx$$

then, as we derived the Euler-Lagrange equation working with the functional variations only in the linear order, we have simply the E-L equation for each of the function

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}$$
$$\frac{d}{dx}\frac{\partial L}{\partial z'} = \frac{\partial L}{\partial z}$$

And so on.

LECTURE 24
Lagrangian mechanics.

24.1. Problems of Newton laws.

- Not invariant when we change the coordinate system:

$$\text{Cartesian:} \begin{cases} m\ddot{x} = F_x \\ m\ddot{y} = F_y \end{cases}, \qquad \text{Cylindrical:} \begin{cases} m\left(\ddot{r} - r\dot{\phi}^2\right) = F_r \\ m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = F_\phi \end{cases}.$$

- Too complicated, too tedious. Consider two pendulums.
- Difficult to find conservation laws.
- Symmetries are not obvious.
- Cannot be used in non-classical world.

24.2. Newton second law as Euler-Lagrange equations

Second order differential equation.

24.3. Hamilton's Principle. Action.

For each conservative mechanical system there exists a functional, called action, which is minimal on the solution of the equation of motion

This functional — Action — has the following form:

$$\mathcal{A}[\{q_i(t)\}] = \int_{t_i}^{t_f} L(t, \{q_i(t)\}, \{\dot{q}_i(t)\})dt.$$

Let's see what it means.

- $\{q_i\}$ — a set of numbers which describes the configuration/position of our system. These numbers are called generalized coordinates.
  - A set of numbers which ambiguously describe the configuration of the system.
  - These numbers must be independent.
  - These numbers must provide the complete description.
- During the motion these generalized coordinates change as functions of time t. I collectively denoted the full set of these functions as $\{q_i(t)\}$.
- Correspondingly, there are generalized velocities: $\dot{q}_i = \frac{dq_i}{dt}$ for each of the coordinates. I collectively denote them as $\{\dot{q}_i(t)\}$.

- t_i is initial moment of time, t_f is the final moment.
- The function $L(t, \{q_i(t)\}, \{\dot{q}_i(t)\})$ of time t, generalized coordinates $\{q_i(t)\}$, and generalized velocities $\{\dot{q}_i(t)\})$ is called the Lagrangian of the system.
- The integration is done over time t.

The Hamilton's principle is not constructive. It states that such functional — Action $\mathcal{A}[\{q_i(t)\}]$ — exists. We still need to construct this functional. This means, that for any system, after we have chosen the coordinates $\{q_i\}$, we need to be able to construct the Lagrangian $L(t, \{q_i(t)\}, \{\dot{q}_i(t)\})$.

24.4. Lagrangian.

Before I show how to construct the Lagrangian, I want to emphasize two important points:

- Lagrangian is not energy. We do not minimize energy. We do not even minimize the Lagrangian. We minimize action!
- Lagrangian is a function of generalized coordinates $\{q_i\}$ and generalized velocities $\{\dot{q}_i\}$. There must be no momenta in Lagrangian.

The Lagrangian is constructed bu the following procedure:

- After we have chosen the generalized coordinates $\{q_i\}$ and assuming, that we know the generalized velocities $\{\dot{q}_i(t)\}$ we compute the kinetic energy of our system: $K(t, \{q_i\}, \{\dot{q}_i\})$ — it may or may not explicitly depend on time.
- We also compute the potential energy $U(t, \{q_i\})$ — it also may or may not explicitly depend on time.
- The Lagrangian then is given by:

$$L(t, \{q_i\}, \{\dot{q}_i\}) = K(t, \{q_i\}, \{\dot{q}_i\}) - U(t, \{q_i\}).$$

After we constructed the Lagrangian, we can write the equation of motion for each of generalized coordinates:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

24.5. Examples.

- Free fall.
  - We chose our standard y vertical coordinate, to describe the position of the body.
  - The kinetic energy is $K = \frac{m\dot{y}^2}{2}$.
  - The potential energy is $U = mgy$.
  - The Lagrangian is

    $$L(y, \dot{y}) = K - U = \frac{m\dot{y}^2}{2} - mgy.$$
  - The Lagrange equation is

    $$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}.$$

    or

    $$m\ddot{y} = -mg.$$
- Motion of a particle in an arbitrary potential $U(\vec{r})$.

- We chose our standard Cartesian coordinates.
- The kinetic energy is $K = \frac{m\dot{\vec{r}}^2}{2}$.
- The potential energy is $U(\vec{r})$.
- The Lagrangian is

$$L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r}).$$

- The Lagrange equation for the component x is

$$m\ddot{x} = -\frac{\partial U}{\partial x}.$$

- The same are for the other components, so we can write

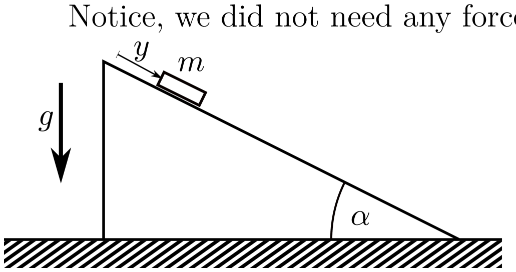
$$m\ddot{\vec{r}} = -\vec{\nabla}U.$$

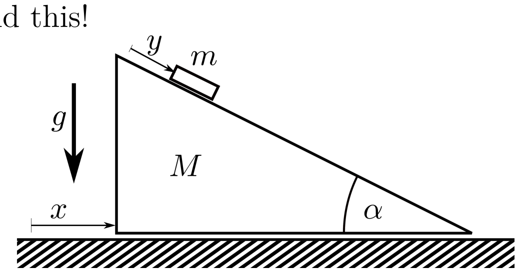
This is Newton's equation $\vec{F} = m\vec{a}$! So we indeed reproduced the Newtonian dynamics!

- A mass on a stationary wedge. No friction.
  - There is only one coordinate here y.
  - The kinetic energy is $\frac{m\dot{y}^2}{2}$.
  - The potential energy is $-mgy\sin\alpha$.
  - The Lagrangian is $L = \frac{m\dot{y}^2}{2} + mgy\sin\alpha$.
  - The Lagrange equation is

$$m\ddot{y} = mg\sin\alpha.$$

Notice, we did not need any forces to find this!





- A mass on a moving wedge. No friction.
  - The coordinates are x and y – see figure.
  - The kinetic energy of the wedge is $\frac{M\dot{x}^2}{2}$.
  - Let's compute the kinetic energy of the mass m. Its horizontal position is $x + y\cos\alpha$, it's vertical position is $-y\sin\alpha$, so its horizontal velocity component $v_{hor} = \dot{x} + \dot{y}\cos\alpha$, its vertical velocity component is $v_{ver} = -\dot{y}\sin\alpha$. So its velocity squared is given $v^2 = v_{hor}^2 + v_{ver}^2 = \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha$.
  - So the total kinetic energy is

$$K = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha\right).$$

  - Total potential energy is $-mgy\sin\alpha$.
  - The Lagrangian is

$$L = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 + m\dot{x}\dot{y}\cos\alpha + mgy\sin\alpha.$$

– There are two Lagrange equations, for x and y

$$(M+m)\ddot{x} + m\ddot{y}\cos\alpha = 0$$
$$m\ddot{y} + m\ddot{x}\cos\alpha = mg\sin\alpha$$

LECTURE 25
Lagrangian mechanics.

25.1. General strategy.

ONLY IF ALL THE FORCES ARE CONSERVATIVE!!!

- Choose generalized coordinates $\{q_i\}$.
- Generalized coordinates:
  - A set of numbers which ambiguously describe the configuration of the system.
  - These numbers must be independent.
  - These numbers must provide the complete description.
- Write the total kinetic energy K of the system in terms of the generalized coordinates and their time derivatives: $\{q_i\}$ and $\{\dot{q}_i\}$.
- Write the total potential energy U in terms of the generalized coordinates $\{q_i\}$.
- Both kinetic and potential energy may or may not depend on time explicitly.
- Define the Lagrangian $L = K(\{\dot{q}_i\}, \{q_i\}) - U(\{q_i\})$.
- Write down the Lagrange equations for all/every generalized coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

- Set up the initial conditions for all generalized coordinates $\{q_i\}$ and generalized velocities $\{\dot{q}_i\}$.
- Solve the equations.

25.2. Examples.

25.2.1. A pendulum.

- The coordinate is ϕ — the angle the pendulum makes with the vertical line.
- The Lagrangian is

$$L = \frac{ml^2\dot{\phi}^2}{2} - mgl(1 - \cos\phi).$$

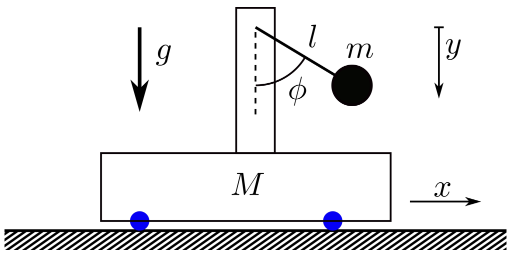
  - The first term is the rotational kinetic energy $\frac{I\omega^2}{2}$, where $I = ml^2$ — the moment of inertia, and $\omega = \dot{\phi}$ — angular velocity.
  - The second term is simply the potential energy.

- The Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{\partial L}{\partial \phi}, \qquad \frac{\partial L}{\partial \dot\phi} = I\dot\phi, \quad \frac{\partial L}{\partial \phi} = -mgl\sin\phi$$

$$\ddot\phi = -\frac{g}{l}\sin\phi.$$

25.2.2. A pendulum on a cart.



- The coordinate x — the position of the Cart and ϕ — the angle of the pendulum are good generalized coordinates.
- The kinetic energy of the cart is $M\dot x^2/2$.
- To find the kinetic energy of the pendulum we need to find the velocity of the ball m through our generalized coordinates. The x position of the ball is $x_m = x + l\sin\phi$, the y position of the ball is $y_m = l\cos\phi$. Then for the ball we have $v_x = \dot x_m = \dot x + \dot\phi l\cos\phi$, and $v_y = \dot y_m = -\dot\phi l\sin\phi$. So $v^2 = v_x^2 + v_y^2 = (\dot x + \dot\phi l\cos\phi)^2 + \dot\phi^2 l^2\sin^2\phi$. And the total kinetic energy is

$$K = \frac{M\dot x^2}{2} + \frac{m}{2}\left(\dot x^2 + 2\dot x\dot\phi l\cos\phi + l^2\dot\phi^2\right).$$

- The potential energy is $U = -mgy_m = -mgl\cos\phi$.
- The Lagrangian is

$$L = K - U = \frac{M\dot x^2}{2} + \frac{m}{2}\left(\dot x^2 + 2\dot x\dot\phi l\cos\phi + l^2\dot\phi^2\right) + mgl\cos\phi.$$

- We need to write two equations for x and ϕ.
  - For x we have:

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot x} = M\dot x + m\dot x + m\dot\phi l\cos\phi, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot x} = M\ddot x + m\ddot x + m\ddot\phi l\cos\phi - m\dot\phi^2 l\sin\phi.$$

  - The first Lagrange equation is

$$M\ddot x + m\ddot x + m\ddot\phi l\cos\phi - m\dot\phi^2 l\sin\phi = 0.$$

  - For ϕ we have

$$\frac{\partial L}{\partial \phi} = -m\dot x\dot\phi l\sin\phi - mgl\sin\phi, \quad \frac{\partial L}{\partial \dot\phi} = m\dot x l\cos\phi + ml^2\dot\phi, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = m\ddot x l\cos\phi - m\dot x\dot\phi l\sin\phi + l^2\ddot\phi.$$

  - The second Lagrange equation is

$$m\ddot x l\cos\phi - m\dot x\dot\phi l\sin\phi + ml^2\ddot\phi = -m\dot x\dot\phi l\sin\phi - mgl\sin\phi$$

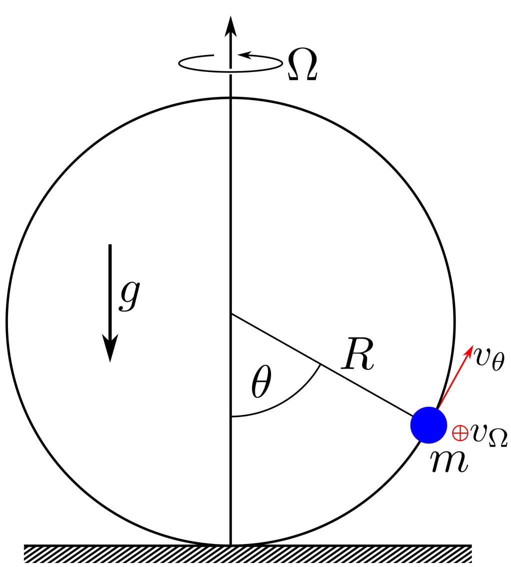
- So the Lagrange equations are

$$M\ddot x + m\ddot x + m\ddot\phi l\cos\phi - m\dot\phi^2 l\sin\phi = 0$$
$$m\ddot x l\cos\phi + ml^2\ddot\phi = -mgl\sin\phi$$

LECTURE 26
Lagrangian mechanics.

26.1. Examples.

26.1.1. A bead on a vertical rotating hoop.



We have a loop of radius R rotating with a constant and fixed(!) angular velocity Ω around a diameter in the vertical direction, see figure. There is a bead of mass m which can freely — without friction — move along the loop. There is gravity acting on the bead. We want to write the equations of motion for the system, analyze them, and see if we can learn something interesting.

"Something interesting" means that we want to learn some universal aspects. The aspects which do not depend on the details of the problem and can be used in developing intuition about more general and more complicated physical effects.

In particular, this problem illustrates a very general idea of spontaneous symmetry breaking. This idea is used very widely in physics. It is central for the Landau theory of the second order phase transitions. Such diverse phenomena as Higgs boson, magnetization in magnets, superfluidity, superconductivity, etc are all in the realm of this theory.

The phenomena mentioned above are quantum and as such requires a different machinery, but, remarkably this simple problem shows one of the most important aspects of all of them.

26.1.1.1. *Equation of motion.*

- The loop is rotating with the constant/fixed angular velocity Ω, so its motion is known and no equation required for it (Notice, that this would be different should the loop rotate freely, then its motion would be influenced by the motion of the bead and we would have to write the equations of motion for both the loop and the bead.)
- Ω is a parameter of the problem. We have full control over it.

- The position of the bead at any moment of time is then fully described by just one generalized coordinate — the angle θ.
- Lagrangian. We need potential and kinetic energies:
  - The potential energy $U(\theta) = mgR(1 - \cos\theta)$.
  - For the kinetic energy we notice, that the total vector velocity of the bead has to components v_θ — the velocity along the loop, and v_Ω — the velocity perpendicular to the plane of the loop, see figure. We also see that $v_\theta = R\dot{\theta}$, and $v_\Omega = \Omega R \sin\theta$. The two components are perpendicular to each other, the total velocity of the bead is $v^2 = R^2\dot{\theta}^2 + \Omega^2 R^2 \sin^2\theta$. The kinetic energy then is $K(\theta, \dot{\theta}) = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta$.

  So the Lagrangian is
$$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}\Omega^2 R^2 \sin^2\theta - mgR(1 - \cos\theta).$$
- Equation of motion.
$$R\ddot{\theta} = (\Omega^2 R \cos\theta - g)\sin\theta.$$

26.1.1.2. *Analysis of the motion.* The motion of the bead depends on the initial conditions. If one wants to know the full solution one has to set up initial conditions and then solve the equation of motion. This exact solution is fairly complicated and not very illuminating.

Instead we want to consider the motion around the equilibrium positions of the bead. We expect this motion to be a harmonic motion and have some universal features.

- There are four equilibrium points — points where $\ddot{\theta} = 0$ — the bead can remain stationary on the loop.
$$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R}$$
- Critical Ω_c. The second two equilibriums are possible only if
$$\frac{g}{\Omega^2 R} < 1, \qquad \Omega > \Omega_c = \sqrt{g/R}.$$
- The most interesting regime is $\Omega \sim \Omega_c$ and θ small.
- Effective potential energy for $\Omega \sim \Omega_c$. From the Lagrangian we can read the effective potential energy:
$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1 - \cos\theta).$$
  Assuming $\Omega \sim \Omega_c$ we are interested only in small θ. So
$$U_{eff}(\theta) \approx \frac{1}{2}mR^2(\Omega_c^2 - \Omega^2)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4$$
$$U_{eff}(\theta) \approx mR^2\Omega_c(\Omega_c - \Omega)\theta^2 + \frac{3}{4!}mR^2\Omega_c^2\theta^4,$$
  One should notice, that there are two terms: one of the order of $(\Omega - \Omega_c)\theta^2$ and the other is of the order of θ^2. It seems unreasonable to keep only these terms and drop the rest. However, we will see below, that $\theta^2 \sim (\Omega - \Omega_c)$, so in fact both terms are of the same order $(\Omega - \Omega_c)^2$ and the rest of them are of the higher order.
- Spontaneous symmetry breaking. Plot the function $U_{eff}(\theta)$ for $\Omega < \Omega_c$, $\Omega = \Omega_c$, and $\Omega > \Omega_c$. Discuss universality.

- Small oscillations around $\theta = 0$, $\Omega < \Omega_c$

$$mR^2\ddot{\theta} = -mR^2(\Omega_c^2 - \Omega^2)\theta, \qquad \omega = \sqrt{\Omega_c^2 - \Omega^2}.$$

- Small oscillations around θ_0, $\Omega > \Omega_c$.

$$U_{eff}(\theta) = -\frac{m}{2}\Omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta),$$

$$\frac{\partial U_{eff}}{\partial \theta} = -mR(\Omega^2 R\cos\theta - g)\sin\theta, \qquad \frac{\partial^2 U_{eff}}{\partial \theta^2} = mR^2\Omega^2\sin^2\theta - mR\cos\theta(\Omega^2 R\cos\theta - g)$$

$$\left.\frac{\partial U_{eff}}{\partial \theta}\right|_{\theta=\theta_0} = 0, \qquad \left.\frac{\partial^2 U_{eff}}{\partial \theta^2}\right|_{\theta=\theta_0} = mR^2(\Omega^2 - \Omega_c^2)$$

So the Tylor expansion gives

$$U_{eff}(\theta \sim \theta_0) \approx \text{const} + \frac{1}{2}mR^2(\Omega^2 - \Omega_c^2)(\theta - \theta_0)^2$$

The frequency of small oscillations then is

$$\omega = \sqrt{\Omega^2 - \Omega_c^2}.$$

26.1.1.3. *Universality.*

- The effective potential energy for small θ and $|\Omega - \Omega_c|$

$$U_{eff}(\theta) = \frac{1}{2}a(\Omega_c - \Omega)\theta^2 + \frac{1}{4}b\theta^4.$$

- θ_0 for the stable equilibrium is given by $\partial U_{eff}/\partial\theta = 0$

$$\theta_0 = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \sqrt{\frac{a}{b}(\Omega - \Omega_c)} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

  Plot $\theta_0(\Omega)$. Non-analytic behavior at Ω_c.

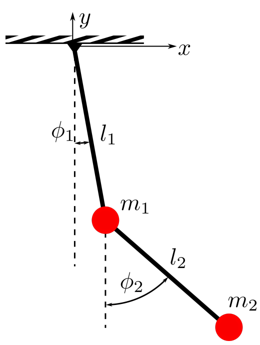
- Response: how θ_0 responses to a small change in Ω.

$$\frac{\partial\theta_0}{\partial\Omega} = \begin{cases} 0 & \text{for} \quad \Omega < \Omega_c \\ \frac{1}{2}\sqrt{\frac{a}{b}}\frac{1}{\sqrt{(\Omega - \Omega_c)}} & \text{for} \quad \Omega > \Omega_c \end{cases}$$

  Plot $\frac{\partial\theta_0}{\partial\Omega}$ vs Ω. The response *diverges* at Ω_c.

LECTURE 27
Lagrangian mechanics.

27.1. Example.



Here we consider one more example — a double pendulum. The strategy is same as always

- Choosing the generalized coordinates.
- Write the potential energy.
- Kinetic energy. Normally, most trouble for students.

Here the most natural choice of coordinates are the angles ϕ_1 and ϕ_2. It is also convenient to use the auxiliary x and y for the intermediate steps. So that we have

$$\begin{aligned} x_1 &= l_1 \sin\phi_1, & x_2 &= l_1 \sin\phi_1 + l_2 \sin\phi_2 \\ y_1 &= -l_1 \cos\phi_1, & y_2 &= -l_1 \cos\phi_1 - l_2 \cos\phi_2 \end{aligned}$$

- Now we can write the potential energy

$$U = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

- In order to find the kinetic energy we need velocities

$$\begin{aligned} v_{1x} &= \dot{x}_1 = l_1 \dot{\phi}_1 \cos\phi_1, & v_{2x} &= \dot{x}_2 = l_1 \dot{\phi}_1 \cos\phi_1 + l_2 \dot{\phi}_2 \cos\phi_2 \\ v_{1y} &= \dot{y}_1 = l_1 \dot{\phi}_1 \sin\phi_1, & v_{2y} &= \dot{y}_2 = l_1 \dot{\phi}_1 \sin\phi_1 + l_2 \dot{\phi}_2 \sin\phi_2 \end{aligned}$$

so

$$\begin{aligned} v_1^2 &= v_{1x}^2 + v_{1y}^2 = l_1^2 \dot{\phi}_1^2 \\ v_2^2 &= v_{2x}^2 + v_{2y}^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \end{aligned}$$

and the kinetic energy

$$K = \frac{(m_1 + m_2) l_1^2}{2} \dot{\phi}_1^2 + \frac{m_2 l_2^2}{2} \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

- The Lagrangian then is

$$L = \frac{(m_1 + m_2) l_1^2}{2} \dot{\phi}_1^2 + \frac{m_2 l_2^2}{2} \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + (m_1 + m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2.$$

- Now we write the Lagrangian equations. We first compute the partial derivatives:

$$\frac{\partial L}{\partial \dot\phi_1} = (m_1+m_2)l_1^2\dot\phi_1 + m_2 l_1 l_2 \dot\phi_2\cos(\phi_1-\phi_2), \qquad \frac{\partial L}{\partial \phi_1} = -m_2 l_1 l_2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - (m_1+m_2)gl_1\sin\phi_1$$
$$\frac{\partial L}{\partial \dot\phi_2} = m_2 l_2^2\dot\phi_2 + m_2 l_1 l_2 \dot\phi_1\cos(\phi_1-\phi_2), \qquad \frac{\partial L}{\partial \phi_2} = +m_2 l_1 l_2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - m_2 g l_2\sin\phi_2$$

and then the full derivative for each

$$(m_1+m_2)l_1^2\ddot\phi_1 + m_2 l_1 l_2\ddot\phi_2\cos(\phi_1-\phi_2) + m_2 l_1 l_2\dot\phi_2^2\sin(\phi_1-\phi_2) = -(m_1+m_2)gl_1\sin\phi_1$$
$$m_2 l_2^2\ddot\phi_2 + m_2 l_1 l_2\ddot\phi_1\cos(\phi_1-\phi_2) - m_2 l_1 l_2\dot\phi_1^2\sin(\phi_1-\phi_2) = -m_2 g l_2\sin\phi_2$$

(some terms which appeared originally have canceled each other)

These are the equations of motion. They are second order coupled nonlinear differential equations. In order to complete them we need to supply also the initial conditions for both variables.

Such equations are hard to solve or analyze. Typically we are mainly interested in the small oscillations around the equilibrium position. In this case the equilibrium position is obvious: $\phi_{1,eq} = \phi_{2,eq} = 0$. So we need to linearize our equations around this point.

Linearizaton means that you only keep the linear terms in $\phi_1 - \phi_{1,eq}$ and in $\phi_2 - \phi_{2,eq}$ and their derivatives. In our case we then have

$$(m_1+m_2)l_1^2\ddot\phi_1 + m_2 l_1 l_2\ddot\phi_2 = -(m_1+m_2)gl_1\phi_1$$
$$m_2 l_2^2\ddot\phi_2 + m_2 l_1 l_2\ddot\phi_1 = -m_2 g l_2\phi_2$$

These are much simpler — they are still coupled, but at least they are linear! They can be solved by a simple Fourier transform.

27.2. Small Oscillations.

We will study the problem of small oscillation in the next semester. Here is just an overview.

A system will always have some dissipation. In many cases the dissipation can be considered to be very small. However, no matter how small it is if one waits long enough the system will find one of its equilibrium positions (there can be several.) Such equilibrium positions are the minimums of the potential energy. If $\{q_i\}$ are the set of N generalized coordinates and $U(\{q_i\})$ is the potential energy, then the equilibrium positions $\{q_{i,eq}\}$ are the solutions of N algebraic equations

$$\left.\frac{\partial U}{\partial q_i}\right|_{\{q_i=q_{i,eq}\}} = 0.$$

After one solves these equations, then for each solution one must make sure, that this is indeed the minimum, not the maximum or a saddle point.

In many cases the equilibrium position can be guessed form the problem itself, but not always!!! One has to be careful.

The Lagrangian equations of motion contain the derivative of the Lagrangian $\frac{\partial L}{\partial \dot q_i}$ and $\frac{\partial L}{\partial q_i}$. So in order for the equations of motion to be linear in the displacement of the generalized coordinates from the equilibrium positions $\{q_i - q_{i,eq}\}$ and generalized velocities one needs to write the Lagrangian in quadratic order in displacement of generalized coordinates and generalized velocities.

For example, for the problem of the double pendulum the equilibrium position is obvious $\phi_{1,eq} = \phi_{2,eq} = 0$. We can write the Lagrangian in the quadratic order in $\phi_1 - \phi_{1,eq} = \phi_1$, $\phi_2 - \phi_{2,eq} = \phi_2$ and in $\dot{\phi}_1$, $\dot{\phi}_2$.

$$L = \frac{(m_1 + m_2)l_1^2}{2}\dot{\phi}_1^2 + \frac{m_2 l_2^2}{2}\dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 - \frac{1}{2}(m_1 + m_2) g l_1 \phi_1^2 - \frac{1}{2} m_2 g l_2 \phi_2^2.$$

(I dropped the constant terms from the Lagrangian.) One can see, that our linearized equations can be obtained from this Lagrangian right away, by the standard procedure.

LECTURE 28
Lagrangian mechanics.

28.1. Generalized momentum.

- Definition: For a coordinate q the generalized momentum is defined as

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

- Examples:
  - For a particle in a potential field $L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r})$ we have

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

    The generalized momentum is just the usual momentum.
  - For a rotation around a fixed axis $L = \frac{I\dot{\phi}^2}{2} - U(\phi)$, then

$$p = \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi} = J.$$

    The generalized momentum is just an angular momentum.

28.2. Ignorable coordinates. Conservation laws.

If one chooses the coordinates in such a way, that the Lagrangian does not depend on say one of the coordinates q_1 (but it still depends on $\dot{q}_1$, then the corresponding generalized momentum $p_1 = \frac{\partial L}{\partial \dot{q}_1}$ is conserved as

$$\frac{d}{dt}p_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} = 0$$

- Problem of a freely horizontally moving cart of mass M with hanged pendulum of mass m and length l.

$$L = \frac{M\dot{x}^2}{2} + \frac{m}{2}\left(\dot{x}^2 + 2\dot{x}\dot{\phi}l\cos\phi + l^2\dot{\phi}^2\right) + mgl\cos\phi.$$

  We see right away, that there is no x (remember x and $\dot{x}$ are different variables for the Lagrangian) in the Lagrangian. So x is ignorable variable. It means, that the corresponding generalized momentum $p_x = \frac{\partial L}{\partial \dot{x}}$ is conserved. So we can write one of the equations of motion as

$$p_x = (M+m)\dot{x} + m\dot{\phi} l \cos\phi = \text{const}.$$

This constant should be obtained from the initial conditions.

28.3. Momentum conservation. Translation invariance

Let's consider a translationally invariant problem. For example all interactions depend only on the distance between the particles. The Lagrangian for this problem is $L(\vec{r}_1, \ldots \vec{r}_i, \dot{\vec{r}}_1, \ldots \dot{\vec{r}}_i)$. Then we add a constant vector ϵ to all coordinate vectors and define

$$L_\epsilon(\vec{r}_1, \ldots \vec{r}_i, \dot{\vec{r}}_1, \ldots \dot{\vec{r}}_i, \vec{\epsilon}) \equiv L(\vec{r}_1 + \vec{\epsilon}, \ldots \vec{r}_i + \vec{\epsilon}, \dot{\vec{r}}_1, \ldots \dot{\vec{r}}_i)$$

It is clear, that in the translationally invariant system the Lagrangian will not change under such a transformation. So we find

$$\frac{\partial L_\epsilon}{\partial \vec{\epsilon}} = 0.$$

But according to the definition

$$\left.\frac{\partial L_\epsilon}{\partial \vec{\epsilon}}\right|_{\vec{\epsilon}=0} = \sum_i \frac{\partial L}{\partial \vec{r}_i}.$$

Hence

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = 0.$$

On the other hand the Lagrange equations tell us that

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = \frac{d}{dt} \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = \frac{d}{dt} \sum_i \vec{p}_i,$$

so

$$\frac{d}{dt} \sum_i \vec{p}_i = 0, \qquad \sum_i \vec{p}_i = \text{const}.$$

We see, that the total momentum of the system is conserved!

28.4. Non uniqueness of the Lagrangian.

For any problem and any given set of generalized coordinates the Lagrangian is not uniquely defined. This is similar to the fact that the potential energy is not uniquely defined — one can always add a constant to it.

In the same way as two potential energy functions which differ only by a constant give the same equations of motion, two Lagrangians for the same problem must give the same equations of motion. So two Lagrangians are equivalent if the resulting Lagrangian equations are the same.

- Let's take a Lagrangian $L(\dot{q}, q, t)$.
- Let's take an arbitrary function $G(q, t)$.
- Let's construct a new Lagrangian $\tilde{L}(\dot{q}, q, t) = L + \dot{q}\frac{\partial G}{\partial q} + \frac{\partial G}{\partial t}$.
- The statement is that the two Lagrangians L and $\tilde{L}$ are equivalent.

28.4.1. Proof of equivalence.

- The Lagrange equation
$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}} = \frac{\partial \tilde{L}}{\partial q}.$$
Let's use our definition of $\tilde{L}$ and see how it works
$$\frac{\partial \tilde{L}}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial G}{\partial q}, \qquad \frac{\partial \tilde{L}}{\partial q} = \frac{\partial L}{\partial q} + \dot{q}\frac{\partial^2 G}{\partial^2 q} + \frac{\partial^2 G}{\partial t \partial q}$$
then
$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{\partial^2 G}{\partial^2 q} + \frac{\partial^2 G}{\partial q \partial t}$$
and we see
$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.$$
- So we see, that the equation we obtain using $\tilde{L}$ is exactly the same as the equation we obtain using L.
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

28.4.2. The reason.

We want to understand <u>why</u> the above transformation of the Lagrangian does not change the equations of motion.

- The reason for this is the following: the expression I added to the Lagrangian $\dot{q}\frac{\partial G}{\partial q} + \frac{\partial G}{\partial t}$ is a full derivative $\dot{q}\frac{\partial G}{\partial q} + \frac{\partial G}{\partial t} = \frac{dG}{dt}$ as can be seen using the chain rule. So $\tilde{L} = L + \frac{dG}{dt}$. But then the Action changes by
$$\tilde{\mathcal{A}} = \int_{t_i}^{t_f} \tilde{L} dt = \int_{t_i}^{t_f} L dt + \int_{t_i}^{t_f} \frac{dG}{dt} dt = \int_{t_i}^{t_f} L dt + G(q(t_f), t_f) - G(q(t_i), t_i) = \mathcal{A} + \text{const}.$$
So the variation of the Action does not change, and thus the condition for the extremum — the Euler-Lagrange equation — also does not change.

So one can always add a full time derivative to a Lagrangian.

The last statement is correct only in the classical mechanics. In quantum mechanics the Action itself has its own meaning (unlike the classical mechanics where we are only interested in its minimum.) and addition of a constant to the Action is not necessarily harmless.

LECTURE 29
Lagrangian's equations for magnetic forces.

The equation of motion is

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

The question is what Lagrangian gives such equation of motion?

29.1. Electric and magnetic fields.

In order to answer the question above we need to know a bit more about electric and magnetic fields. Classically these fields are completely described by the Maxwell equations. There are four of thes equations, but we will need only two of them: magnetic Gauss law and Faraday's law

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

Notice, that these are the two Maxwell equations which do not have matter — charge or current densities.

Consider first magnetic Gauss law. Which is the statement that there are no magnetic charges.

$$\nabla \cdot \vec{B} = 0$$

This equation is satisfied by the following solution

$$\vec{B} = \nabla \times \vec{A},$$

for any vector field $\vec{A}(\vec{r}, t)$.

The Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

then gives

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t},$$

where ϕ is the electric potential and is again an arbitrary function.

The vector potential $\vec{A}$ and the potential ϕ are not uniquely defined. One can always choose another potential

$$\vec{A}' = \vec{A} + \nabla F, \qquad \phi' = \phi - \frac{\partial F}{\partial t}$$

and obtain exactly the same electric and magnetic fields. As the other two Maxwell equations contain only electric and magnetic fields (and not the potentials) they will also not fix this freedom.

Moreover, in any experiment we can only measure electric $\vec{E}$ and magnetic $\vec{B}$ fields. This means that the potentials — vector potential $\vec{A}$ and scalar potential ϕ cannot be measured by itself.

Such fields are called **gauge fields**. The freedom of choice is called **gauge freedom**. The transformation from one set of fields $\vec{A}$ and ϕ to $\vec{A}'$ and ϕ' is called **gauge transformation**. The fact, that no physical results must depend on the choice of gauge (physical quantities must be invariant under the gauge transformation) is called **gauge symmetry**.

Such gauge symmetries are extremely important in physics. A lot of constructions in modern physics involve some sort of gauge symmetry.

As any continuous symmetry gauge symmetry leads to conservation laws. In the case of electromagnetism it leads to the charge conservation law (we will not discuss it any further in this class).

Notice, that if $\vec{B}$ and $\vec{E}$ are zero, the gauge fields do not have to be zero. For example if $\vec{A}$ and ϕ are constants, $\vec{B} = 0$, $\vec{E} = 0$.

29.2. The Lagrangian.

Now we can write the Lagrangian:

$$L = \frac{m\dot{\vec{r}}}{2} - q(\phi - \dot{\vec{r}} \cdot \vec{A})$$

I note, that this Lagrangian has a simpler and more transparent form in the notations adopted in the special and general relativity — four dimensional space-time with Minkovskii metric.

- It is impossible to write the Lagrangian in terms of the physical fields $\vec{B}$ and $\vec{E}$!
- The expression
$$\phi dt - d\vec{r} \cdot \vec{A}$$
is a full differential if and only if
$$-\nabla\phi - \frac{\partial \vec{A}}{\partial t} = 0, \qquad \nabla \times \vec{A} = 0,$$
which means that it is full differential, and hence can be thrown out, only if the physical fields are zero!

The generalized momenta are

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$

(Notice, that the generalized momentum is not the same as usual momentum. Moreover, it is <u>not</u> gauge invariant!)

The Lagrange equations are:

$$\frac{d}{dt}\vec{p} = \frac{\partial L}{\partial \vec{r}}$$

Let's consider the x component

$$\frac{d}{dt}p_x = \frac{\partial L}{\partial x},$$

$$m\ddot{x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_x}{\partial y} + q\dot{z}\frac{\partial A_x}{\partial z} + q\frac{\partial A_x}{\partial t} = -q\frac{\partial \phi}{\partial x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x}$$

$$m\ddot{x} = q\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \dot{z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\right)$$

$$m\ddot{x} = q\left(E_x + \dot{y}B_z - \dot{z}B_y\right)$$

LECTURE 30
Energy conservation.

30.1. Energy conservation.

We also have the time translation invariance in many systems. It means that the Lagrangian does not explicitly depend on time. So we have $L(q, \dot{q})$, and not $L(q, \dot{q}, \mathbf{t})$. However, the coordinate $q(t)$ does depend on the time. So let's see how the Lagrangian on a trajectory depends on time.

Let me clarify the question. Assume that we have a Lagrangian $L(q, \dot{q})$. We then write Lagrangian equation of motion $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ with some initial conditions $q(t=0) = q_0$, $\dot{q}(t=0) = v_0$. Then we solve this equation an obtained $q(t)$ and hence we also obtained $\dot{q}(t) = \frac{dq(t)}{dt}$. We then take these functions and plug them into the Lagrangian $L(q(t), \dot{q}(t))$. Now the Lagrangian becomes a function of time on the trajectory. We want to see how it depends on time.

In our standard definition it means that we are interested in the full time derivative of the Lagrangian.

$$\frac{d}{dt}L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\partial L}{\partial q}\dot{q} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) - \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) + \dot{q}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)$$

But as we are looking at the real trajectory — the function $q(t)$ is the solution of the Lagrange equation. So according to the Lagrange equation the last term is zero, so we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q})\right) = 0$$

or

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q}) = \text{const} = E$$

Using generalized momentum we can write

$$p\dot{q} - L = E, \qquad \text{Constant on trajectory.}$$

If we have many variables q_i, then

$$E = \sum_i p_i \dot{q}_i - L$$

This is another conserved quantity.

Examples:

- A particle in a potential field.
  - The Lagrangian
$$L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r})$$
  - The momenta
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}.$$
  - The Energy
$$E = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L = \frac{m\dot{\vec{r}}^2}{2} + U(\vec{r})$$
- A particle on a circle.
  - The Lagrangian
$$L = \frac{mR^2}{2}\dot{\phi}^2 - U(\phi).$$
  - Generalized momentum
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}.$$
  - The Energy
$$E = \dot{\phi}p_\phi - L = \frac{mR^2}{2}\dot{\phi}^2 + U(\phi)$$
- A cart (mass M) with a pendulum (mass m, length l).
  - The Lagrangian:
$$L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 - mgl(1-\cos\phi).$$
  - The generalized momenta:
$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\dot{\phi}l\cos\phi, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\dot{x}l\cos\phi + ml^2\dot{\phi}.$$
  - The Energy
$$E = \dot{x}p_x + \dot{\phi}p_\phi - L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 + mgl(1-\cos\phi)$$
- A string with tension and gravity:
  - The Functional
$$\int_0^L (\rho g y - T)\sqrt{1+(y')^2}dx.$$
  - One can thin of it as an Action of some mechanical system. Then for this system we identify the “Lagrangian”
$$L = (\rho g y - T)\sqrt{1+(y')^2}.$$
    We also use the letter x to denote the time in that mechanical system.
  - So the “generalized momentum” is
$$p = \frac{\partial L}{\partial y'} = \frac{\rho g y - T}{\sqrt{1+(y')^2}}y'.$$

– And conserved “energy”

$$E = y'p - L = \frac{\rho g y - T}{\sqrt{1 + (y')^2}}.$$

This is now a first order differential equation which can be solved much easier, than the second order Euler-Lagrange equation.

– This conserved quantity has a physical meaning for the initial problem of the rope. It is the x component of the tension force.

LECTURE 31
Hamiltonian.

In this lecture we will construct a function of generalized momenta and coordinates, which is called Hamiltonian. In this lecture I will not describe how it is used — this will be done later. Here we just construct this function and consider a few examples.

31.1. Hamiltonian.

Given a Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ the energy

$$E = \sum_i p_i \dot{q}_i - L, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is a number defined on a trajectory! One can say that it is a function of initial conditions.

We can construct a function **a function** of p and q in the following way: we first solve the set of equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

with respect to $\dot{q}_i$, we then have these functions

$$\dot{q}_i = \dot{q}_i(\{q_j\}, \{p_j\})$$

and define a function $H(\{q_i\}, \{p_i\})$

$$H(\{q_i\}, \{p_i\}) = \sum_i p_i \dot{q}_i(\{q_j\}, \{p_j\}) - L(\{q_i\}, \{\dot{q}_i(\{q_j\}, \{p_j\})\}),$$

- Notice, that in this construction we have never used the equations of motion! we have treated q, $\dot{q}$ and p simply as variables, not as some functions of time.

This function is called a Hamiltonian! The Hamiltonian is a function of coordinates and momenta! THERE MUST BE NO VELOCITIES IN THE HAMILTONIAN!

- Hamiltonian is NOT energy. Energy is a number on a trajectory. Hamiltonian is a function of p and q — it, by itself, knows nothing about trajectories.
- Hamiltonian and energy are related to each other. The value of the Hamiltonian on a trajectory is energy.

The importance of variables:

- We have three kinds of variables:

generalized coordinates — q_i, generalized velocities — $\dot{q}_i$, generalized momenta — p_i.

- A Lagrangian is a function of generalized coordinates and velocities: q_i and $\dot{q}_i$.

- A Hamiltonian is a function of the generalized coordinates and **momenta**: q_i and p_i.

Here are the steps to get a Hamiltonian from a Lagrangian

(a) Write down a Lagrangian $L(\{q_i\},\{\dot{q}_i\})$ – it is a function of generalized coordinates and velocities q_i, $\dot{q}_i$

(b) Find generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

(c) Treat the above definitions as equations and solve them for all $\dot{q}_i$, so for each velocity $\dot{q}_i$ you have an expression $\dot{q}_i = \dot{q}_i(\{q_j\},\{p_j\})$.

(d) Substitute these function $\dot{q}_i = \dot{q}_i(\{q_j\},\{p_j\})$ into the expression

$$\sum_i p_i \dot{q}_i - L(\{q_i\},\{\dot{q}_i\}).$$

The resulting function $H(\{q_i\},\{p_i\})$ of generalized coordinates and momenta is called a Hamiltonian.

31.2. Examples.

- A particle in a potential field.
  - The Lagrangian
    $$L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r})$$
  - The momenta
    $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}.$$
  - The velocity
    $$\dot{\vec{r}} = \frac{\vec{p}}{m}$$
  - The Hamiltonian
    $$H(\vec{r},\vec{p}) = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L = \frac{\vec{p}^2}{m} - L = \frac{\vec{p}^2}{2m} + U(\vec{r})$$
- A particle on a circle.
  - The Lagrangian
    $$L = \frac{mR^2}{2}\dot{\phi}^2 - U(\phi).$$
  - Generalized momentum
    $$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}.$$
  - The velocity
    $$\dot{\phi} = \frac{p_\phi}{mR^2}$$
  - The Hamiltonian
    $$H(\phi, p_\phi) = \dot{\phi}p_\phi - L = \frac{p_\phi^2}{2mR^2} + U(\phi)$$
- A cart (mass M) with a pendulum (mass m, length l).

- The Lagrangian:

$$L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 - mgl(1-\cos\phi).$$

- The generalized momenta:

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\dot{\phi}l\cos\phi, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\dot{x}l\cos\phi + ml^2\dot{\phi}.$$

- The generalized velocities

$$\dot{x} = \frac{1}{l}\frac{lp_x - p_\phi\cos\phi}{M+m\sin^2\phi}, \qquad \dot{\phi} = \frac{1}{ml^2}\frac{(M+m)p_\phi - mlp_x}{M+m\sin^2\phi}.$$

- The Hamiltonian

$$H = \dot{x}p_x + \dot{\phi}p_\phi - L = \frac{1}{2ml^2}\frac{ml^2p_x^2 - 2mlp_xp_\phi\cos\phi + (m+M)p_\phi^2}{M+m\sin^2\phi} + mgl(1-\cos\phi)$$

- Central symmetric potential in $3D$. (Kepler problem)
  - We need to write the Lagrangian in spherical coordinates. We know

$$d\vec{r} = \vec{e}_r dr + \vec{e}_\theta r d\theta + \vec{e}_\phi r\sin\theta d\phi.$$

Dividing this by dt we get

$$\vec{v} = \vec{e}_r\dot{r} + \vec{e}_\theta r\dot{\theta} + \vec{e}_\phi r\dot{\phi}\sin\theta,$$

so

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta.$$

The Lagrangian is

$$L = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{m}{2}r^2\dot{\phi}^2\sin^2\theta - U(r)$$

  - The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta.$$

  - The generalized velocities

$$\dot{r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}.$$

  - The Hamiltonian

$$H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + U(r)$$

LECTURE 32
Hamiltonian equations.

32.1. Hamiltonian equations.

- If Lagrangian explicitly depends on time...
- New notation for the partial derivatives. What do we keep fixed?
  - The notation explicitly keeps the notion of what is kept fixed.
  - The definition of momentum then is
$$p = \left(\frac{\partial L}{\partial \dot{q}}\right)_q .$$
  - The Lagrangian equation of motion
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)_q = \left(\frac{\partial L}{\partial q}\right)_{\dot{q}}$$
    etc.
- Derivation of the Hamiltonian equations.
  - Let's differentiate the Hamiltonian $H(p, q)$ with respect to momentum p, while keeping the coordinate q fixed.
  - We will use $H = p\dot{q} - L(q, \dot{q})$, but we will remember, that $\dot{q}$ is the function of p and q, i.e. $\dot{q}(p, q)$.
  - So we differentiate the function $H(p, q) = p\dot{q}(p, q) - L(q, \dot{q}(p, q))$.
  - We will also remember, that by definition $p = \left(\frac{\partial L}{\partial \dot{q}}\right)_q$.
  - So we have:
$$\left(\frac{\partial H}{\partial p}\right)_q = \dot{q} + p\left(\frac{\partial \dot{q}}{\partial p}\right)_q - \left(\frac{\partial L}{\partial \dot{q}}\right)_q \left(\frac{\partial \dot{q}}{\partial p}\right)_q = \dot{q} + p\left(\frac{\partial \dot{q}}{\partial p}\right)_q - p\left(\frac{\partial \dot{q}}{\partial p}\right)_q = \dot{q}$$
    This is the first Hamiltonian equation.
  - Now lets differentiate the Hamiltonian with respect to q, while keeping p fixed.
  - Again we must remember that $\dot{q}(p, q)$ is the function of p and q.
  - So using $H(p, q) = p\dot{q}(p, q) - L(q, \dot{q}(p, q))$ we have

$$\left(\frac{\partial H}{\partial q}\right)_p = p\left(\frac{\partial \dot{q}}{\partial q}\right)_p - \left(\frac{\partial L}{\partial q}\right)_{\dot{q}} - \left(\frac{\partial L}{\partial \dot{q}}\right)_q \left(\frac{\partial \dot{q}}{\partial q}\right)_p = -\left(\frac{\partial L}{\partial q}\right)_{\dot{q}} + \left(p - \left(\frac{\partial L}{\partial \dot{q}}\right)_q\right)\left(\frac{\partial \dot{q}}{\partial q}\right)_p .$$

- Using the definition of momentum $p = \left(\frac{\partial L}{\partial \dot{q}}\right)_q$, we see, that the last term is zero. So we have
$$\left(\frac{\partial H}{\partial p}\right)_p = -\left(\frac{\partial L}{\partial q}\right)_{\dot{q}}.$$
- According to the Lagrangian equation of motion $\left(\frac{\partial L}{\partial q}\right)_{\dot{q}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)_q = \dot{p}$. The last equality comes from the definition of momentum. So we have
$$\left(\frac{\partial H}{\partial q}\right)_p = -\dot{p}.$$
This is the second Hamiltonian equation.

- The two Hamiltonian equation together are
$$\dot{q} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial q}.$$
- Notice, that the equations are "self-contained" there is no notion of the generalized velocities. Everything is written in terms of the coordinates, momenta and their time dependence.
- If we have many degrees of freedom, then this pair of equations is written for each degree of freedom.

32.2. Examples.

- Energy conservation.
  - Energy is the value of the Hamiltonian on the trajectory!!!!
  - What it means, is that we take a Hamiltonian, write the Hamilton equations, solve them for some initial conditions $q(t=0) = q_0$, $p(t=0) = p_0$ (and so forth if we have more degrees of freedom). We then have two functions $q(t)$ and $p(t)$.
  - We now substitute these functions into the Hamiltonian $H(p,q,t)$ and obtain a function of time $H(p(t), q(t), t)$.
  - Now lets differentiate this function with respect to time. This is a full derivative now
$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial t} = -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$
where we used the Hamilton equations for the functions $p(t)$ and $q(t)$.
  - So we see, that if the Hamiltonian does not explicitly depend on time (in less words $\frac{\partial H}{\partial t} = 0$), then
$$\frac{dH}{dt} = 0.$$
or the value of the Hamiltonian on a trajectory is constant.
  - Notice the importance of the minus sign in the second of the Hamilton equations!
- Velocity.
  - In many cases the Hamiltonian is the starting point.

– The dependence of the velocity on momentum is then given by the Hamilton equation

$$\dot{q} = \frac{\partial H}{\partial p}.$$

– In particular if we have a normal “kinetic energy” $E(p) = \frac{p^2}{2m}$, then this equations gives

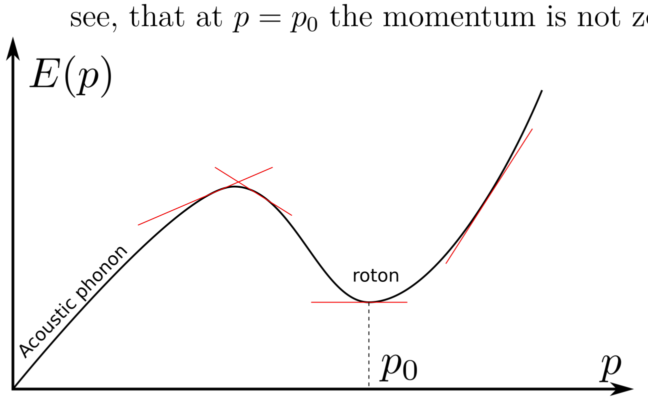
$$\dot{x} = \frac{\partial E}{\partial p} = p/m.$$

This is the usual $p = mv$.

– The kinetic energy as a function of momentum $E(p)$ is called dispersion relation.

– It is the dispersion relation which gives the relation between the velocity and momentum, by the Hamilton equation.

– There are cases where this is very nontrivial. For example in liquid Helium the dispersion of “exitations” is similar to the one shown in the picture. One can see, that at $p = p_0$ the momentum is not zero (it is p_0), but the velocity is zero!



LECTURE 33
Hamiltonian equations. Examples

The Hamiltonian and Lagrangian formulations of mechanics are equivalent to each other. Namely, if we know the Lagrangian we will know the Hamiltonian and if we know the Hamiltonian we will know the Lagrangian.

33.1. Lagrangian→Hamiltonian, Hamiltonian→Lagrangian.

33.1.1. $L \to H$

- We are given a Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ as a function of coordinates $\{q_i\}$ and velocities $\{\dot{q}_i\}$. There are no momenta in the Lagrangian!
- We write the definition of momenta
$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$
- We treat these equations as equations for all velocities $\{\dot{q}_i\}$ and solve them with respect to the velocities
$$\dot{q}_j = \dot{q}_j(\{p_i\}, \{q_i\}).$$
- We construct the Hamiltonian
$$H(\{p_i\}, \{q_i\}) = \sum_j p_j \dot{q}_j(\{p_{i'}\}, \{q_{i'}\}) - L(\{q_{j'}\}, \{\dot{q}_{j'}(\{p_{i'}\}, \{q_{i'}\})\}).$$

The Hamiltonian thus constructed is the function of all coordinates $\{q_i\}$ and all momenta $\{p_i\}$. There are must be no velocities in the Hamiltonian!

33.1.2. $H \to L$

- We are given a Hamiltonian $H(\{p_i\}, \{q_i\})$ as a function of all coordinates $\{q_i\}$ and all momenta $\{p_i\}$. There are no velocities in the Hamiltonian!
- We write the definition of velocity for each momentum
$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$
- We treat these equations as equations for all momenta $\{p_i\}$ and solve them with respect to the momenta
$$p_j = p_j(\{q_i\}, \{\dot{q}_i\}).$$

- We construct the Lagrangian

$$L(\{q_i\},\{\dot q_i\}) = \sum_j \dot q_j p_j(\{q_{i'}\},\{\dot q_{i'}\}) - H(\{p_{j'}(\{q_{i'}\},\{\dot q_{i'}\})\},\{q_{j'}\})$$

The Lagrangian thus constructed is the function of all coordinates $\{q_i\}$ and all velocities $\{\dot q_i\}$. There are must be no momenta in the Lagrangian!

33.1.3. Equations of motion.

If we have a Lagrangian and a Hamiltonian which are connected by the procedures described above, then the Lagrangian and Hamiltonian equations are equivalent — they describe the same motion!

$$\begin{array}{ccc} L(\{q_i\},\{\dot q_i\}) & \Longleftrightarrow & H(\{p_i\},\{q_i\}) \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q_i} = \dfrac{\partial L}{\partial q_i} & \Longleftrightarrow & \begin{array}{l} \dot q_i = \frac{\partial H}{\partial p_i} \\ \dot p_i = -\frac{\partial H}{\partial q_i} \end{array} \\ q_i(t=0) = q_{i0}, \quad \dot q_i(t=0) = v_{i0} & \Longleftrightarrow & q_i(t=0) = q_{i0}, \quad p_i(t=0) = p_{i0}. \end{array}$$

Given equivalent initial conditions these equations will give exactly the same $q_i(t)$!

33.2. Examples.

33.2.1. A particle in a potential field.

- Lagrangian→Hamiltonian, Hamiltonian→Lagrangian.
  - The Lagrangian
  $$L(\dot{\vec r},\vec r) = \frac{m\dot{\vec r}}{2} - U(\vec r).$$
  - The momentum
  $$\vec p = \frac{\partial L}{\partial \dot{\vec r}} = m\dot{\vec r}.$$
  - The velocity
  $$\dot{\vec r} = \frac{\vec p}{m}$$
  - The Hamiltonian
  $$H(\vec p,\vec r) = \vec p\cdot\dot{\vec r} - L = \frac{\vec p^2}{2m} + U(\vec r)$$
  - From the Hamiltonian
  $$\dot{\vec r} = \frac{\partial H}{\partial \vec p} = \frac{\vec p}{m}.$$
  - The momentum
  $$\vec p = m\dot{\vec r}.$$
  - The Lagrangian
  $$L(\dot{\vec r},\vec r) = \dot{\vec r}\cdot\vec p - H = \frac{m\dot{\vec r}}{2} - U(\vec r).$$
- We found the Hamiltonian from the Lagrangian and then from Hamiltonian we found the same Lagrangian.
- The equations of motion:

– The Lagrangian equations of motion

$$m\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}.$$

– The Hamiltonian equations of motion

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{m}, \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -\frac{\partial U}{\partial \vec{r}}.$$

– Taking the time derivative of the first equation we find $\dot{\vec{p}} = m\ddot{\vec{r}}$. Using this in the second equation we find

$$m\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}.$$

– We see, that the Hamiltonian and Lagrangian equations give the same $\vec{r}(t)$!

33.2.2. Rotation around a fixed axis.

- Lagrangian→Hamiltonian, Hamiltonian→Lagrangian.

$$L(\dot{\phi}, \phi) = \frac{I\dot{\phi}^2}{2} - U(\phi).$$

– The momentum

$$p_\phi = I\dot{\phi}.$$

– Velocity

$$\dot{\phi} = \frac{p_\phi}{I}$$

– The Hamiltonian

$$H(p_\phi, \phi) = p_\phi\dot{\phi} - L = \frac{p_\phi^2}{2I} + U(\phi).$$

– The velocity

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{I}.$$

– The momentum

$$p_\phi = I\dot{\phi}.$$

– The Lagrangian

$$L(\dot{\phi}, \phi) = \frac{I\dot{\phi}^2}{2} - U(\phi)$$

- The equations of motion
  – Lagrangian equation

$$I\ddot{\phi} = -\frac{\partial U}{\partial \phi}$$

– The Hamiltonian equations

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{I}, \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$

Differentiating the first equation with respect to time and using the result in the second equation we get

$$I\ddot{\phi} = -\frac{\partial U}{\partial \phi}$$

– We see, that the Hamiltonian and Lagrangian equations give the same $\phi(t)$!

An example of the system considered above is a pendulum.

33.2.3. Motion in a central symmetric field.

- Lagrangian→Hamiltonian, Hamiltonian→Lagrangian.
  - The Lagrangian is

$$L = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{m}{2}r^2\dot{\phi}^2\sin^2\theta - U(r)$$

  - The momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta.$$

  - The velocities

$$\dot{r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}.$$

  - The Hamiltonian

$$H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + U(r).$$

  - The velocities

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}.$$

  - The momenta

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta.$$

  - The Lagrangian

$$L = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - H = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{m}{2}r^2\dot{\phi}^2\sin^2\theta - U(r)$$

- The equations of motion
  - The Lagrangian equations of motion

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - \frac{\partial U}{\partial r} \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= mr^2\dot{\phi}^2\sin\theta\cos\theta \\ m\ddot{\phi}r^2\sin^2\theta + m\dot{\phi}r\dot{r}\sin^2\theta + 2r^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta &= 0 \end{aligned}$$

– The Hamiltonian equations of motion

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2\theta} - \frac{\partial U}{\partial r}$$
$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta}$$
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}. \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

– You are welcome to check that these equations are equivalent to the Lagrangian equations.

33.2.4. Relativistic particle.

Consider a Hamiltonian $H(p,x) = \epsilon(p) = c\sqrt{p^2 + m_0^2c^2}$ in $1D$. We do not consider any field so the Hamiltonian does not depend on x.

- Equations of motion.

$$\dot{p} = -\frac{\partial H}{\partial x} = 0, \qquad \dot{x} = \frac{\partial H}{\partial p} = \frac{cp}{\sqrt{p^2 + m_0^2c^2}}$$

  So we see, that the momentum is conserved, but the velocity has a nontrivial dependence on momentum. In particular if $p \to \infty$ we have $\dot{x} \to c$. Moreover, the velocity can never exceed c!
- The momentum. From the last equation

$$p = \frac{m_0 c\dot{x}}{\sqrt{c^2 - \dot{x}^2}}.$$

  Notice, if we introduce a "mass" as $m = \frac{m_0}{\sqrt{1 - \dot{x}^2/c^2}}$, then we have $p = m\dot{x}$ – the usual formula. Now if we use this notation to write the energy, then we get $\epsilon = mc^2$.
- Lagrangian.

$$L(\dot{x}, x) = \dot{x}p - H = -m_0c\sqrt{c^2 - \dot{x}^2}$$

- Action. It is very instructive to write the Action for this problem

$$\mathcal{S} = -m_0c\int\sqrt{c^2 - \dot{x}^2}dt = -m_0c\int\sqrt{(cdt)^2 - (dx)^2}.$$

- Geometrical meaning of Action. Notice, that the action above is the length of the interval in the space-time (ct, x) with the metric $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

$$(ds)^2 = (cdt, dx)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} cdt \\ dx \end{pmatrix}$$

  The Action then is

$$\mathcal{A} = -m_0c\int ds$$

- One now can easily extend this construction to the full $3+1$ space by using the Minkovskii metric
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
- Moreover, one is not restricted to the flat Minkovskii space and can write the Action for a particle in a curved space-time — the space-time with Einstein's gravity.

LECTURE 34
Hamiltonian equations. Examples. Phase space.

34.1. Examples.

- General case for the kinetic energy quadratic in velocities
$$L = \frac{1}{2}\dot{q}_i M_{ij}(\{q_k\})\dot{q}_j - U(\{q_k\}),$$
where $M_{ij}(\{q_k\})$ – a symmetric q-dependent positive definite matrix.
  - The momenta
$$p_i = \frac{\partial L}{\partial \dot{q}_i} = M_{ij}(\{q_k\})\dot{q}_j.$$
  - The velocities
$$\dot{q}_i = \left(M^{-1}(\{q_k\})\right)_{ij} p_j.$$
  - The Hamiltonian
$$H = \frac{1}{2}p_i \left(M^{-1}(\{q_k\})\right)_{ij} p_j + U(\{q_k\})$$
  - The Hamiltonian equations
$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \left(M^{-1}(\{q_k\})\right)_{ij} p_j$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = -\frac{1}{2}p_i \left(\frac{\partial M^{-1}(\{q_{k'}\})}{\partial q_k}\right)_{ij} p_j - \frac{\partial U}{\partial q_k}$$

    Derivative of a matrix $\left(\frac{\partial M^{-1}(\{q_{k'}\})}{\partial q_k}\right)_{ij}$ means simply the matrix where each matrix element is the derivative of the original matrix elements $\left(M^{-1}(\{q_k\})\right)_{ij}$.
- A cart (mass M) with a pendulum (mass m, length l).
$$L = \frac{M+m}{2}\dot{x}^2 + m\dot{\phi}\dot{x}l\cos\phi + \frac{m}{2}l^2\dot{\phi}^2 - mgl(1-\cos\phi).$$
This is a particular case of the example above.
  - The coordinates/velocities
$$\begin{pmatrix} \dot{x} \\ \dot{\phi} \end{pmatrix}$$

– The matrix $\hat{M}$

$$\hat{M} = \begin{pmatrix} M+m & ml\cos\phi \\ ml\cos\phi & ml^2 \end{pmatrix}$$

– The inverse M^{-1}

$$\hat{M}^{-1} = \frac{1}{ml^2}\frac{1}{M+m\sin^2\phi}\begin{pmatrix} ml^2 & -ml\cos\phi \\ -ml\cos\phi & m+M \end{pmatrix}$$

– The Hamiltonian

$$H = \frac{1}{2ml^2}\frac{ml^2p_x^2 - 2mlp_xp_\phi\cos\phi + (m+M)p_\phi^2}{M+m\sin^2\phi} + mgl(1-\cos\phi).$$

– etc.

34.2. Phase space. Hamiltonian vector field. Phase trajectories.

Hamiltonian equations are *the first order* differential equations! We double the number of variables and the number of equations, but each equation is now the first order differential equations. We still need two initial conditions for each degree of freedom.

- The space of all q and all p is called a *phase space* of the Hamiltonian system.

Let's consider a one dimensional problem with time independent Hamiltonian. So we have only one generalized coordinate q. The phase space is then two dimensional: (q,p). For a given Hamiltonian the equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

Let's assume that a system had a phase space coordinates (q_t, p_t) at time t. The equations of motion show that at time $t+dt$ the system will be at the point

$$q_{t+dt} = q_t + \frac{\partial H}{\partial p}dt$$
$$p_{t+dt} = p_t - \frac{\partial H}{\partial q}dt$$

Let's now define the Hamiltonian vector field by

$$\vec{\mathcal{H}} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}.$$

Then we see, that a point (q_t, p_t) after time dt shifts to

$$\begin{pmatrix} q_{t+dt} \\ p_{t+dt} \end{pmatrix} = \begin{pmatrix} q_t \\ p_t \end{pmatrix} + \vec{\mathcal{H}}dt$$

So the vector $\vec{\mathcal{H}}$ is a vector of *velocity in the phase space.*

- We can plot the vector field $\vec{\mathcal{H}}$ at every point of the phase space.
- Notice, that we do not need to solve any differential equations for that. We just need to differentiate the Hamiltonian!
- This vector field will show the *velocity in the phase space* for our system.

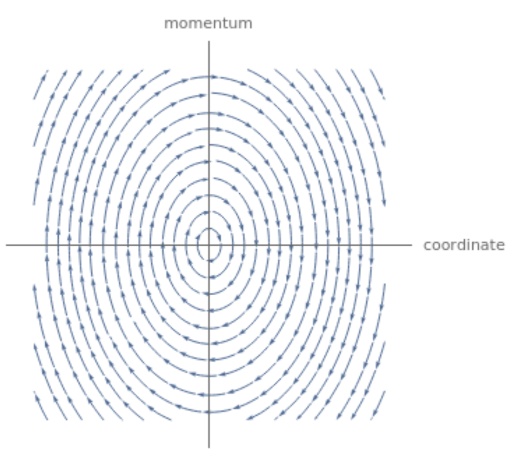
A trajectories of the system in the phase space are simply the lines which are tangential to the Hamiltonian vector field at every point of the line. Different trajectories correspond to different initial conditions.

This construction is very similar to the electric field and electric field lines.

- Motion in the phase space: we can consider the motion of a system in the phase space: we start from an initial point (q_i, p_i) and continue along the Hamiltonian vector field — along phase space trajectories.
- Trajectories do not intersect (except in isolated singular points). This is the same as for electric field lines. The phase space trajectories (electric field lines) can have one tangential vector at each point, except the points where $\vec{\mathcal{H}} = 0$ — the singular points — all the derivatives of the Hamiltonian are zero.
- On the phase trajectories the Hamiltonian is constant — the energy is conserved!

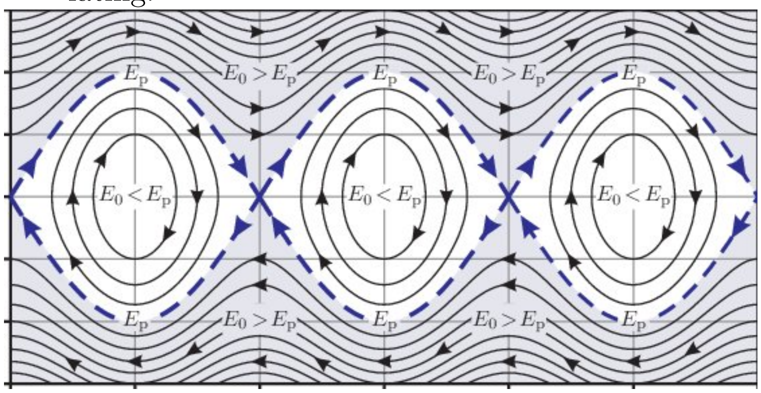
These simple rules allow one to construct the phase space trajectories for many (usually $1D$) systems. Here are the couple of examples.

- Harmonic oscillator.
  - The Hamiltonian of the Harmonic oscillator is
$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$
  - On the phase space trajectories the Hamiltonian is constant. The lines in (x, p) space given by
$$\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = E$$
are ellipses with the semiaxes $\sqrt{2mE}$ and $\sqrt{2E/m\omega^2}$. (The area of these ellipses is $2\pi E/\omega = ET$, where T is the period.)
  - The Hamiltonian vector field is
$$\vec{\mathcal{H}} = \begin{pmatrix} p/m \\ -m\omega^2 x \end{pmatrix}$$



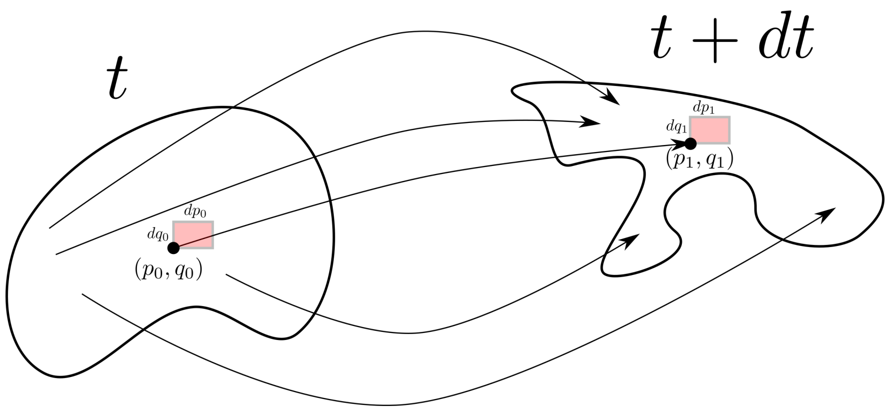
- Pendulum.
  - When energy is small the pendulum is a harmonic oscillator, so for small energies the trajectories are ellipses.
  - When energy grows the ellipses grow.

– Eventually the ellipse must hit a singular point – this is when the energy of the pendulum is enough to reach the highest point.
– If we increase the energy further the pendulum starts to rotate instead of oscillating.



LECTURE 35
Liouville's theorem. Poincaré recurrence theorem. Area law.

35.1. Liouville's theorem.



Theorem: The phase space volume is conserved under the Hamiltonian flow.
Proof: As the trajectories do not intersect, we can consider the Hamiltonian flow as a map of the phase space on itself: any initial point (q_0, p_0) is mapped to a point $(q(t), p(t))$ after time t, where $q(t)$ and $p(t)$ are the solutions of the Hamiltonian equations with (q_0, p_0) as initial conditions.

For a small time interval dt the map is given by

$$q_1 = q_0 + \frac{\partial H}{\partial p_0} dt, \qquad p_1 = p_0 - \frac{\partial H}{\partial q_0} dt$$

We can consider these equations as the equations for the change of variables from (q_0, p_0) to (q_1, p_1).

Consider a piece of volume at time $t = 0$: $\mathcal{A}_0 = \int dq_0 dp_0$. After time dt, this volume becomes $\mathcal{A}_1 = \int dq_1 dp_1$. We want to compute the change of this volume $d\mathcal{A} = \mathcal{A}_1 - \mathcal{A}_0$.

$$d\mathcal{A} = \int dq_1 dp_1 - \int dq_0 dp_0 = \int \left(\frac{\partial q_1}{\partial q_0} \frac{\partial p_1}{\partial p_0} - \frac{\partial q_1}{\partial p_0} \frac{\partial p_1}{\partial q_0} \right) dq_0 dp_0 - \int dq_0 dp_0.$$

Using the formulas for our change of variables we find

$$\frac{\partial q_1}{\partial q_0} = 1 + \frac{\partial^2 H}{\partial p_0 \partial q_0} dt, \qquad \frac{\partial p_1}{\partial p_0} = 1 - \frac{\partial^2 H}{\partial p_0 \partial q_0} dt, \qquad \frac{\partial q_1}{\partial p_0} = \frac{\partial^2 H}{\partial p_0^2} dt, \qquad \frac{\partial p_1}{\partial q_0} = -\frac{\partial^2 H}{\partial q_0^2} dt.$$

Or

$$\left(\frac{\partial q_1}{\partial q_0}\frac{\partial p_1}{\partial p_0} - \frac{\partial q_1}{\partial p_0}\frac{\partial p_1}{\partial q_0}\right) = 1 - \left(\frac{\partial^2 H}{\partial p_0 \partial q_0}\frac{\partial^2 H}{\partial p_0 \partial q_0} - \frac{\partial^2 H}{\partial p_0^2}\frac{\partial^2 H}{\partial q_0^2}\right)(dt)^2$$

so that $d\mathcal{A} \sim (dt)^2$. It means, that $\frac{d\mathcal{A}}{dt} \sim dt$, so when we take the limit $dt \to 0$ we get

$$\frac{d\mathcal{A}}{dt} = 0, \qquad \mathcal{A} = \text{const}.$$

- This is Liouville's theorem. It states, that the volume of phase space is unchanged under the map on itself induced by the equations of motion.
- It is also correct for any number of degrees of freedom.
- Notice the importance of the minus sign in the Hamiltonian equations.

35.2. Poincaré recurrence theorem.

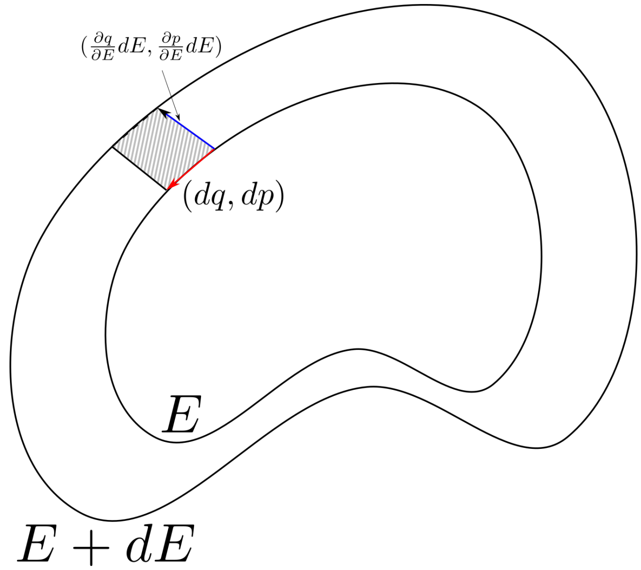
If the available phase space for the system is finite. Let's starts the motion at some point of the phase space. Let's consider an evolution of some finite but small neighborhood of this point. The volume of the neighborhood is constant, so eventually it will cover all of the available volume. Then the tube of the trajectories must intersect itself. But it cannot, as trajectories do not intersect. It means that it must return to the starting neighborhood (or intersect it at least partially.)

It means that under Hamiltonian dynamics the system will always return arbitrary close to the initial starting point.

The time it will take for the system to return is another matter.

35.3. Area law.

This law is valid only in $1D$. Let's consider a Hamiltonian motion in $1D$. We will assume, that the motion is periodic — in $1D$ the motion is either periodic, or unbounded. In the phase space picture the periodic motion means that the phase space trajectory is a closed loop (without self-crossings). We then can compute the area of the phase space $\mathcal{A} = \int dpdq$ of the loop inside the phase space trajectory of a motion with energy E. We thus will have a function $\mathcal{A}(E)$.



If we change the energy by dE the area will change. Consider two trajectories one with the energy E and the other with the energy $E + dE$. We want to compute the difference between the areas for the two trajectories.

The change of the area is the sum of the vector product of the vectors (dq, dp) and $(\frac{\partial q}{\partial E}dE, \frac{\partial p}{\partial E}dE)$, so

$$d\mathcal{A} = -dE \oint \left(\frac{\partial q}{\partial E} dp - \frac{\partial p}{\partial E} dq \right) = -dE \oint \left(\frac{\partial q}{\partial E} \dot{p} - \frac{\partial p}{\partial E} \dot{q} \right) dt = dE \oint \left(\frac{\partial q}{\partial E} \frac{\partial H}{\partial q} + \frac{\partial p}{\partial E} \frac{\partial H}{\partial p} \right) dt$$
$$= dE \oint \left(\frac{\partial H}{\partial q} \frac{\partial q}{\partial E} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial E} \right) dt.$$

The Hamiltonian is the function of coordinate q and momentum p. The Hamiltonian in the last formula is the Hamiltonian on the trajectory. The trajectory depends on the energy E, so we have $H(q(t, E), p(t, E))$. This function does not depend on time t, as it is conserved. So we can look at the last formula above as if it is a chain rule for the derivative dH/dE. But this derivative is 1, as the value of the Hamiltonian on a trajectory is energy! So we have.

$$d\mathcal{A} = dE \oint \frac{dH}{dE} dt = dE \oint dt = T dE.$$

Thus we have our Area Law:

$$\frac{d\mathcal{A}}{dE} = T(E).$$

- In particular for oscillator we saw that $\mathcal{A} = 2\pi E/\omega = ET$.
- Connection to Bohr's quantum mechanics.

LECTURE 36
Adiabatic invariants.

We want to consider the following problem:

- We have a conservative $1D$ system with slowly varying parameter.
- The system is described by a Hamiltonian $H(p, q; \lambda)$, where λ is a parameter, say a spring constant, etc.
- The system undergoes a periodic motion with some period T which depends on energy E and the value of the parameter λ.
- We now start to slowly change the parameter λ as a function of time.
- What can we say about the motion?

Before we do anything we need to understand what does it mean to change the parameter "slowly". The natural definition is that the change of the parameter $\Delta\lambda$ during one period T is small in comparison to the value of the parameter itself:

$$T\frac{d\lambda}{dt} \ll \lambda.$$

Rewriting this as

$$T \ll \lambda/\dot{\lambda}$$

we see, that there are two vastly different time scales: T — typical time for the motion; $\lambda/\dot{\lambda}$ — typical time of change of the parameter λ.

What do we expect:

- If the parameter is a function of time the energy is no longer conserved.
- The rate of change of the energy averaged over the period of the motion will be very slow.
- The averaged rate of change of the energy will be proportional to $\dot{\lambda}$. If $\dot{\lambda} = 0$ — the parameter is constant — then the energy does not change, it is conserved.

So we have rapid oscillations and slow change of the parameter. Let's compute how the energy is changing. Energy is the value of the Hamiltonian on a trajectory.

$$\frac{dE}{dt} = \left(\frac{\partial H}{\partial t}\right)_{p,q} = \left(\frac{\partial H}{\partial \lambda}\right)_{p,q} \frac{d\lambda}{dt}.$$

Where in the RHS in $\left(\frac{\partial H}{\partial \lambda}\right)_{p,q}$ we must substitute the solution of the equation of motion $p(t)$ and $q(t)$. The p and q are changing rapidly with time — the typical time of change is the period T. We want to average the above expression over the period T. As $\frac{d\lambda}{dt}$ almost does not change during the period we can take it out of the averaging

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\left(\overline{\frac{\partial H}{\partial \lambda}}\right)_{p,q}$$

While λ in $\frac{\partial H}{\partial \lambda}$ is changing just a little during the period we can do the averaging in $\overline{\frac{\partial H}{\partial \lambda}}$ assuming λ to be constant.

- So from now on we can consider the Hamiltonian system with constant λ. Which also means constant energy E.

The averaging means

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \left(\frac{\partial H(p(t), q(t), \lambda)}{\partial \lambda}\right)_{p,q,E} dt.$$

According to the Hamiltonian equation (remember λ is fixed)

$$\dot{q} = \left(\frac{\partial H}{\partial p}\right)_{q,\lambda,E}, \quad \text{or} \quad dt = \frac{dq}{(\partial H/\partial p)_{q,\lambda,E}}.$$

so we have

$$T = \int_0^T dt = \oint \frac{dq}{(\partial H/\partial p)_{q,\lambda,E}}, \qquad \int_0^T \frac{\partial H}{\partial \lambda} dt = \oint \frac{(\partial H/\partial \lambda)_{p,q,E}}{(\partial H/\partial p)_{q,\lambda,E}} dq,$$

where $\oint$ means integrating there and back. We thus have:

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\frac{\oint \frac{(\partial H/\partial \lambda)_{p,q,E}}{(\partial H/\partial p)_{q,\lambda,E}} dq}{\oint \frac{dq}{(\partial H/\partial p)_{q,\lambda}}}.$$

In the RHS the integrals must be taken on some particular trajectory. The trajectory depends on the energy E and on the parameter λ.

- This is an important point. All the integrals in the RHS are taken along a trajectory at fixed E and fixed λ!
- So we solve the Hamiltonian equations for some fixed E and λ, and find $p(t; E, \lambda)$ and $q(t; E, \lambda)$ — this is a parametric form (t is a parameter) of a phase space trajectory for given E and λ.
- On this trajectory the momentum p can be considered to be a function of the coordinate q. The phase space trajectory is given by $p(q; E, \lambda)$.

Also on a trajectory, at fixed λ the energy is conserved and $E = H(q, p(q; E, \lambda), \lambda)$ Taking the derivative of this equation with respect to λ for fixed E and q we find

$$\left(\frac{\partial H}{\partial \lambda}\right)_{q,p,E} + \left(\frac{\partial H}{\partial p}\right)_{q,\lambda,E}\left(\frac{\partial p}{\partial \lambda}\right)_{q,E} = 0,$$

or

$$\frac{(\partial H/\partial \lambda)_{p,q,E}}{(\partial H/\partial p)_{q,\lambda,E}} = -\left(\frac{\partial p}{\partial \lambda}\right)_{q,E}.$$

Also on a trajectory $\frac{\partial H}{\partial p} = \frac{\partial E}{\partial p}$. So together we have

$$\frac{d\overline{E}}{dt} = -\frac{d\lambda}{dt}\frac{\oint \left(\frac{\partial p}{\partial \lambda}\right)_{q,E} dq}{\oint \left(\frac{\partial p}{\partial E}\right)_{q,\lambda} dq}$$

or

$$\oint \left(\left(\frac{\partial p}{\partial E}\right)_{q,\lambda} \frac{d\overline{E}}{dt} + \left(\frac{\partial p}{\partial \lambda}\right)_{q,E} \frac{d\lambda}{dt}\right) dq = 0.$$

Again, considering p as $p(q;\overline{E},\lambda)$, where the dependence of p on q comes from the solution of the Hamiltonian equations at FIXED E and λ we can write (for fixed q) $dp(q;E,\lambda) = \left(\frac{\partial p}{\partial E}\right)_{q,\lambda} d\overline{E} + \left(\frac{\partial p}{\partial \lambda}\right)_{q,E} d\lambda$. (As energy is conserved, there is no distinction between E and $\overline{E}$ in this procedure.) The above equation then is

$$\frac{d}{dt}\oint p(q;E,\lambda)dq = 0.$$

So the quantity

$$I = \frac{1}{2\pi}\oint pdq$$

is called *adiabatic invariant*. This quantity does not change during the adiabatic change of the parameters.

Let me repeat the story:

- We have a conservative $1D$ system with.
- The system is described by a Hamiltonian $H(p,q;\lambda)$, where λ is a parameter, say a spring constant, etc.
- The system undergoes a periodic motion.
- The Hamiltonian equations of motion for FIXED parameter λ conserve the energy E.
- From the equation $E = H(p,q,\lambda)$ we find $p(q;E,\lambda)$
- We compute the quantity

  $$I(E,\lambda) = \frac{1}{2\pi}\oint pdq = \frac{1}{2\pi}\oint p(q;E,\lambda)dq$$

  Notice, that all this is done at FIXED E and λ — we are solving the equations for a purely conservative system!
- If we now start to slowly change the parameter λ with time, the energy of the system will be changing in such a way, that

  $$I(E(t),\lambda(t)) = \text{const.}$$

  will remain constant.

36.1. Examples.

36.1.1. A particle in a box.

- A free $1D$ particle in a box of length L.
- We want to see how the energy of the particle depends on L if we slowly change L. Namely, we start with the particle of some particular energy E at some length L. We then slowly change the length L. How the energy of the particle will change?

We start at fixed E and L. At fixed E the momentum of the particle is $p = \sqrt{2mE}$. The adiabatic invariant then is

$$I = \frac{1}{2\pi} \oint \sqrt{2mE} dq = \frac{\sqrt{2mE}}{2\pi} \oint dq = \frac{\sqrt{2mE}}{2\pi} 2L.$$

So the combination $\sqrt{E}L$ will remain constant if we slowly change L. So will remain the combination EL^2.

In particular, lets assume, that we slowly changed L to $L + dL$. As $EL^2 = \text{const.}$, differentiating this with respect to L we find

$$dEL^2 + 2ELdL = 0,$$

or

$$dE = -2\frac{E}{L} dL.$$

Notice, that this also can be written as ($E = \frac{mv^2}{2} = \frac{1}{2}pv$)

$$dE = -2p\frac{v}{2L} dL = -\frac{2p}{T} dL.$$

but $2p/T$ is the average change of the particle's momentum during one period, so it is an average force f which the particle exerts on the wall. Then fdL is work which the system did while the wall was moving from L to $L + dL$. Accordingly, the energy of the particle has decreased by exactly the work the particle has done.

36.1.2. Oscillator.

The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2.$$

We want to see how the energy changes if we slowly change the frequency ω.

Considering the motion at fixed E and ω we write

$$p = \pm\sqrt{2mE - m^2\omega^2 x^2}$$

The adiabatic invariant is ($x_E = \sqrt{\frac{2E}{m\omega^2}}$)

$$I = \frac{1}{2\pi} 2 \int_{-x_E}^{x^E} \sqrt{2mE - m^2\omega^2 x^2} dx = \frac{E}{\omega}.$$

So if we slowly change ω the energy will always stay proportional to the frequency

$$E \sim \omega.$$

LECTURE 37
Poisson brackets. Change of Variables. Canonical variables.

- Students' evaluations.

37.1. Poisson brackets.

Consider a function of time, coordinates and momenta: $f(t, \{q\}, \{p\})$. We want to know how the value of this function changes with time on the solutions of the equations of motion. Namely, we have a Hamiltonian $H(\{p\}, \{q\})$ and the Hamiltonian equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

We want to solve them with some initial conditions and find the functions $q_i(t)$ and $p_i(t)$. We then plug these functions in the function f and get $f(t, \{q(t)\}, \{p(t)\})$, which is now a function of time – the value of the function f of the trajectory. We want to see how this value changes with time.

So we want to compute $\frac{df}{dt}$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = \frac{\partial f}{\partial t} + \{H, f\}$$

where we defined the Poisson brackets for any two functions g and f

$$\{g, f\} = \sum_i \left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right).$$

- Notice, that the Poisson brackets are defined for *any* two functions f and g.

In particular we see, that

$$\{p_i, q_k\} = \delta_{i,k}.$$

According to the definition Poisson brackets are

- Antisymmetric.
- Bilinear.
- For a constant c, $\{f, c\} = 0$.
- $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.
- Jacobi's identity. (we will talk about it later.)

37.2. Change of Variables.

We want to answer the following question. What change of variables will keep the Hamiltonian equations intact? Namely We have our original variables $\{p\}$ and $\{q\}$ and the original Hamiltonian $H(\{p\},\{q\})$. The Hamiltonian equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

We want to find the new variables $\{P\}$ and $\{Q\}$, such that the *form* of the Hamiltonian equations for the new variables is the same

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial Q_i}.$$

Let's consider an arbitrary transformation of variables: $P_i = P_i(\{p\},\{q\})$, and $Q_i = Q_i(\{p\},\{q\})$. We then have

$$\dot{P}_i = \{H, P_i\}, \qquad \dot{Q}_i = \{H, Q_i\}.$$

or

$$\dot{P}_i = \sum_k \left[\frac{\partial H}{\partial p_k}\frac{\partial P_i}{\partial q_k} - \frac{\partial H}{\partial q_k}\frac{\partial P_i}{\partial p_k}\right].$$

At this point I want to make the change of variables in the Hamiltonian. For that I invert/solve the equations for the change of variables to get $p_i = p_i(\{P\},\{Q\})$ and $q_i = q_i(\{P\},\{Q\})$ and substitute these functions into the original Hamiltonian $H(\{p\},\{q\})$

$$H(\{p(\{P\},\{Q\})\},\{q(\{P\},\{Q\})\}) \equiv H(\{P\},\{Q\})$$

we then have by the chain rule

$$\frac{\partial H}{\partial p_k} = \sum_\alpha \left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial p_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial p_k}\right), \qquad \frac{\partial H}{\partial q_k} = \sum_\alpha \left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial q_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial q_k}\right)$$

Substituting this into our equation for $\dot{P}_i$ we get

$$\begin{aligned}\dot{P}_i &= \sum_{k,\alpha}\left[\left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial p_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial p_k}\right)\frac{\partial P_i}{\partial q_k} - \left(\frac{\partial H}{\partial P_\alpha}\frac{\partial P_\alpha}{\partial q_k} + \frac{\partial H}{\partial Q_\alpha}\frac{\partial Q_\alpha}{\partial q_k}\right)\frac{\partial P_i}{\partial p_k}\right] \\ &= -\sum_\alpha \left[\frac{\partial H}{\partial P_\alpha}\{P_i, P_\alpha\} + \frac{\partial H}{\partial Q_\alpha}\{P_i, Q_\alpha\}\right]\end{aligned}$$

Analogously,

$$\dot{Q}_i = -\sum_\alpha \left[\frac{\partial H}{\partial Q_\alpha}\{Q_i, Q_\alpha\} + \frac{\partial H}{\partial P_\alpha}\{Q_i, P_\alpha\}\right]$$

We see, that the Hamiltonian equations keep their form if

$$\{P_i, Q_\alpha\} = \delta_{i,\alpha}, \qquad \{P_i, P_\alpha\} = \{Q_i, Q_\alpha\} = 0$$

- So in order for the Hamiltonian equation to have the same *form* in the new variables the Poisson brackets of the new variables must be the same as the Poisson brackets of the old variables.

37.3. Canonical variables.

The Poisson brackets

$$\{P_i, Q_\alpha\} = \delta_{i,\alpha}, \qquad \{P_i, P_\alpha\} = \{Q_i, Q_\alpha\} = 0$$

are called *canonical Poisson brackets*.

The variables that have such Poisson brackets are called the *canonical variables*, they are *canonically conjugated*. Transformations that keep the canonical Poisson brackets are called *canonical transformations*.

- Non-uniqueness of the Hamiltonian.
- Coordinates and momenta obtained from Lagrangian are always canonically conjugated.
- $L = p\dot{q} - H$ only if p and q are canonical variables.
- Canonical Poisson brackets are encoded in the $p\dot{q}$ term.

LECTURE 38
Hamiltonian equations. Jacobi's identity.

- Last week. The lecture time next Monday will be a help session.
- Evaluations.

38.1. Hamiltonian mechanics

- The Poisson brackets are property of the phase space and have nothing to do with the Hamiltonian.
- The Hamiltonian is just a function on the phase space.
- Given the phase space p_i, q_i, the Poisson brackets and the Hamiltonian. We can construct the equations of the Hamiltonian mechanics:
$$\dot{p}_i = \{H, p_i\}, \qquad \dot{q}_i = \{H, q_i\}.$$
- In this formulation there is no need to distinguish between the coordinates and momenta. So we can use $\xi_1 \ldots \xi_{2N}$ instead of $q_1 \ldots q_N$ and $p_1 \ldots p_N$, with given Poisson brackets $\{\xi_i, \xi_j\}$.
- The equations of motion are then
$$\dot{\xi}_i = \{H, \xi_i\}.$$
- Time evolution of any function $f(\{\xi\}, t)$ is given by the equation
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}.$$
difference between the full and the partial derivatives!

38.2. New formulation of the Hamiltonian mechanics.

Here is the new formulation of mechanics:

- We have a phase space with coordinates $\{\xi_i\}$.
- This phase space is equipped with Poisson brackets.
  Poisson brackets are
  - Antisymmetric.
  - Bilinear.
  - For a constant c, $\{f, c\} = 0$.
  - $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$.
  - Jacobi's identity. (we will talk about it later.)

- Any function on the phase space $H(\{\xi\}, t)$ can be a Hamiltonian (which function you take as a Hamiltonian depends on the problem you are solving.)
- Time evolution of any function $f(\{\xi\}, t)$ is given by the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}.$$

In this formulation the phase space and the Poisson brackets play the major role. They are independent of a Hamiltonian (the are defined before the Hamiltonian even introduced) If we know the Hamiltonian we can also construct the Hamiltonian equations of time evolution of any function.

In particular the time evolution of the Hamiltonian itself is given by

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t}$$

as $\{H, H\} = 0$ due to antisymmetry of the Poisson brackets. So if the Hamiltonian does not explicitly depend on time, then it is conserved on the trajectories.

38.3. How to compute Poisson brackets for any two functions.

In order to use our formulation we need a way to compute the Poisson bracket between any two functions f and g if we know all $\{\xi_i, \xi_j\}$. In general the Poisson bracket $\{\xi_i, \xi_j\}$ is the function of all the phase space coordinates. We only require that all the properties listed in definition hold.

The answer is:

$$\{f, g\} = \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}.$$

(summation over the repeated indexes is implied.) Notice the order of indexes. It is important.

Let's prove this formula.

- We start with the Poisson bracket of $\{\xi_j, g\}$.
- In order to compute it we consider ξ_j as our Hamiltonian. This Hamiltonian then gives a flow

$$\frac{dg}{dt} = \{\xi_j, g\}.$$

- On the other hand, by the chain rule

$$\frac{dg}{dt} = \frac{\partial g}{\partial \xi_i}\frac{d\xi_i}{dt} = \frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}.$$

- Comparing the two results we see, that

$$\{\xi_j, g\} = \frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$

- To compute the Poisson bracket $\{g, f\}$ we consider the function g as the Hamiltonian, then

$$\frac{df}{dt} = \{g, f\}.$$

- On the other hand, by the chain rule
$$\frac{df}{dt} = \frac{\partial f}{\partial \xi_j}\frac{d\xi_j}{dt} = \frac{\partial f}{\partial \xi_j}\{g, \xi_j\} = -\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}$$
so that
$$\{f, g\} = \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_i}\{\xi_j, \xi_i\}.$$
- Using this rule we see, that if all the requirements for the Poisson brackets are satisfied for all $\{\xi_i, \xi_j\}$, then these requrements are satisfied for any functions f and g.

There is one more identity the Poisson brackets must satisfy – the Jacobi's identity.

38.4. The Jacobi's identity

Using the definition of the Poisson brackets in the canonical coordinates it is easy, but lengthy to prove, that for any three functions f, g, and h:
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$
As it holds for any functions this is the property of the phase space and the Poisson brackets.

Let's take $\xi_1 \ldots \xi_{2N}$, to be *canonical coordinates*, so that $\{\xi_i, \xi_j\} = \text{const}$. Then we can write
$$\{h, \{f, g\}\} = \frac{\partial h}{\partial \xi_p}\frac{\partial}{\partial \xi_l}\left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\right)\{\xi_p, \xi_l\}.$$
Taking the derivative, remembering that $\{\xi_i, \xi_j\} = \text{const}$ and cycle permuting the functions we get
$$\begin{aligned}
\{h, \{f, g\}\} &= \frac{\partial h}{\partial \xi_p}\frac{\partial^2 f}{\partial \xi_i \partial \xi_l}\frac{\partial g}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial h}{\partial \xi_p}\frac{\partial f}{\partial \xi_i}\frac{\partial^2 g}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} \\
\{g, \{h, f\}\} &= \frac{\partial g}{\partial \xi_p}\frac{\partial^2 h}{\partial \xi_i \partial \xi_l}\frac{\partial f}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial g}{\partial \xi_p}\frac{\partial h}{\partial \xi_i}\frac{\partial^2 f}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}, \quad p \to j, j \to l, l \to i, i \to p \\
\{f, \{g, h\}\} &= \frac{\partial f}{\partial \xi_p}\frac{\partial^2 g}{\partial \xi_i \partial \xi_l}\frac{\partial h}{\partial \xi_j}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\} + \frac{\partial f}{\partial \xi_p}\frac{\partial g}{\partial \xi_i}\frac{\partial^2 h}{\partial \xi_j \partial \xi_l}\{\xi_i, \xi_j\}\{\xi_p, \xi_l\}
\end{aligned}$$
Combining the terms with the same second derivatives, relabeling the indexes, and using antisymmetry of the Poisson brackets we see, that the Jacobi identity is satisfied.

- As it holds for any functions this is the property of the phase space and the Poisson brackets themselves.
- If the Poisson brackets are not constants, one can show (the same way as above) that if Jacoby's identity is satisfied by the phase space coordinates Poisson brackets$\{\xi_i, \xi_j\}$, then it is satisfied for any functions.

38.5. Commutation of Hamiltonian flows.

For a Hamiltonian H we can introduce the operator $\hat{\zeta}_H$ of the Hamiltonian flow by the following definition: for any function g
$$\hat{\zeta}_H g \equiv \{H, g\}$$

Let's now consider two Hamiltonians H_1 and H_2 and compute the commutator of their flows. Namely, for any function g we have (using Jacobi's identity)

$$\hat{\zeta}_{H_1}\hat{\zeta}_{H_2}g - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1}g = \{H_1, \{H_2, g\}\} - \{H_2, \{H_1, g\}\} = \{\{H_1, H_2\}, g\} = \hat{\zeta}_{\{H_1,H_2\}}g.$$

As this is true for *any* function g we have

$$\hat{\zeta}_{H_1}\hat{\zeta}_{H_2} - \hat{\zeta}_{H_2}\hat{\zeta}_{H_1} = \hat{\zeta}_{\{H_1,H_2\}}.$$

So the commutator of the Hamiltonian flows is also a Hamiltonian flow.

LECTURE 39
Integrals of motion. Angular momentum.

39.1. Time evolution of Poisson brackets.

Consider two arbitrary functions $f(q,p,t)$ and $g(q,p,t)$. We want to compute the full time derivative of their Poisson bracket

$$\frac{d}{dt}\{f,g\}.$$

It means, that we have a phase space with Poisson brackets. We also have a Hamiltonian. We solve the Hamiltonian equations of motion and find $p(t)$, $q(t)$ (or $\xi(t)$ if we do not distinguish between coordinates and momenta) We compute the Poisson bracket $\{f,g\}$ – it will be some function of all ξ. We substitute the solutions $\xi(t)$ in this function and then take the time derivative.

Our general procedure allows us to do it much simpler, but before we do that I want to compute

$$\frac{\partial}{\partial t}\{f,g\}.$$

This is partial derivative. So we just consider the explicit dependence of $\{f,g\}$ on time. We keep fixed all other variables except t, so I will leave them out to shorten the formulas

$$\{f(t+\Delta t),g(t+\Delta t)\}-\{f(t),g(t)\}=\{f(t+\Delta t),g(t+\Delta t)\}-\{f(t),g(t+\Delta t)\}+\{f(t),g(t+\Delta t)\}-\{f(t),g(t)\}=$$

so, dividing by Δt and taking the limit $\Delta t \to 0$ we get

$$\frac{\partial}{\partial t}\{f,g\}=\left\{\frac{\partial f}{\partial t},g\right\}+\left\{f,\frac{\partial g}{\partial t}\right\}.$$

Notice

- The only property of the Poisson brackets which we used is its bi-linearity.

Now Let's compute the full time evolution of the Poisson bracket $\{f,g\}$ under the Hamiltonian H.

$$\frac{d}{dt}\{f,g\}=\frac{\partial}{\partial t}\{f,g\}+\{H,\{f,g\}\}=\left\{\frac{\partial f}{\partial t},g\right\}+\left\{f,\frac{\partial g}{\partial t}\right\}+\{\{H,f\},g\}+\{f,\{H,g\}\}$$
$$=\left\{\frac{\partial f}{\partial t}+\{H,f\},g\right\}+\left\{f,\frac{\partial g}{\partial t}+\{H,g\}\right\}$$

Notice, that in this derivation we used

- the Jacobi's identity,
- the antisymmetry,
- and the bi-linearity

Of the Poisson brackets.

So we get

$$\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}.$$

- Notice, that these are the full derivatives, not partial!!

39.2. Integrals of motion.

A conserved quantity is such a function $f(q,p,t)$, that $\frac{df}{dt} = 0$ under the evolution of a Hamiltonian H. So we have if we have two conserved quantities $f(q,p,t)$ and $g(q,p,t)$, then

$$\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\} = 0$$

So if we have two conserved quantities we can construct a new conserved quantity! Sometimes it will turn out to be an independent conservation law!

39.3. Angular momentum.

This is an example of a case where the Poisson brackets do not have a global canonical form.

39.3.1. Poisson Brackets.

Let's calculate the Poisson brackets for the angular momentum: $\vec{M} = \vec{r} \times \vec{p}$.

Coordinate $\vec{r}$ and momentum $\vec{p}$ are canonically conjugated so

$$\{p^i, r^j\} = \delta_{ij}, \qquad \{p^i, p^j\} = \{r^i, r^j\} = 0.$$

As our coordinates and momenta are canonical, we can use the definition of the Poisson brackets through derivatives — the way they were introduced from the very beginning. However, I will show that we can compute the Poisson brackets between the angular momentum components algebraically — using only the properties of the Poisson brackets.

Using $M^i = \epsilon^{ijk} r^j p^k$ we write

$$\begin{aligned}
&\{M^i, M^j\} = \epsilon^{ilk}\epsilon^{jmn}\{r^l p^k, r^m p^n\} = \epsilon^{ilk}\epsilon^{jmn}\left(r^l\{p^k, r^m p^n\} + p^k\{r^l, r^m p^n\}\right) = \\
&\epsilon^{ilk}\epsilon^{jmn}\left(r^l p^n\{p^k, r^m\} + r^l r^m\{p^k, p^n\} + p^k p^n\{r^l, r^m\} + p^k r^m\{r^l, p^n\}\right) = \\
&\epsilon^{ilk}\epsilon^{jmn}\left(r^l p^n \delta_{km} - p^k r^m \delta_{ln}\right) = \left(\epsilon^{ilk}\epsilon^{jkn} - \epsilon^{ikn}\epsilon^{jlk}\right) p^n r^l = p^i r^j - r^i p^j = -\epsilon^{ijk} M^k
\end{aligned}$$

(I used $\epsilon^{ilk}\epsilon^{jnk} = \delta^{ij}\delta^{ln} - \delta^{in}\delta^{lj}$). In short the result is

$$\{M^i, M^j\} = -\epsilon^{ijk} M^k$$

Notice:

- The components of the angular momentum construct their own phase space closed under the Poisson brackets!
- Unlike the usual phase space this phase space looks odd (3) dimensional!

- This puzzle is resolved by the following observation:
$$\{M^i, \vec{M}^2\} = \{M^i, M^k M^k\} = 2\{M^i, M^k\}M^k = -2\epsilon^{ikj}M^j M^k = 0.$$
- So for any Hamiltonian which depends on $\vec{M}$ only, the $\vec{M}^2$ will be conserved!
- $$\frac{d\vec{M}^2}{dt} = \{H, \vec{M}^2\} = \frac{\partial H}{\partial M^i}\{M^i, \vec{M}^2\} = 0.$$
- So in $3D$ space of $\vec{M}$ all motion will happen on the spheres $\vec{M}^2 = \text{cons.}$.
- The sphere is $2D$ – even dimension.
- There is no way to construct global canonical coordinates on this space.

39.3.2. Spin in magnetic field.

We can now consider a Hamiltonian mechanics, say for the Hamiltonian
$$H = \vec{h} \cdot \vec{M}.$$
In this case
$$\dot{M}^i = \{H, M^i\} = h_j\{M^j, M^i\} = -h_j\epsilon^{jik}M^k,$$
or
$$\dot{\vec{M}} = \vec{h} \times \vec{M}.$$
Notice:

- $\dot{M}^2 = \vec{M} \cdot \dot{\vec{M}} = \vec{M} \cdot \left[\vec{h} \times \vec{M}\right] = 0.$
- This equation (Bloch equation) describes a vector $\vec{M}$ rotating with constant angular velocity around the direction of $\vec{h}$.

39.3.3. Euler equations

Consider a free rigid body with tensor of inertia $\hat{I}$. The Hamiltonian is just the kinetic energy.
$$H = \frac{1}{2}M^i(\hat{I}^{-1})_{ij}M^j.$$
The equations of motion then is
$$\dot{M}^k = \{H, M^k\} = \frac{1}{2}\{M^i, M^k\}(\hat{I}^{-1})_{ij}M^j + \frac{1}{2}M^i(\hat{I}^{-1})_{ij}\{M^j, M^k\} = \epsilon^{kil}M^l(\hat{I}^{-1})_{ij}M^j.$$
Let's write this equation in the system of coordinates of the principal axes of the body. Then the tensor of inertia is diagonal, and for x component we get
$$\dot{M}^x = M^z I_{yy}^{-1} M^y - M^y I_{zz}^{-1} M^z.$$
or, using that $M^x = I_{xx}\Omega^x$, etc we get
$$I_{xx}\dot{\Omega}^x = (I_{zz} - I_{yy})\Omega^z\Omega^y,$$
and two more equations under the cyclic permutations.

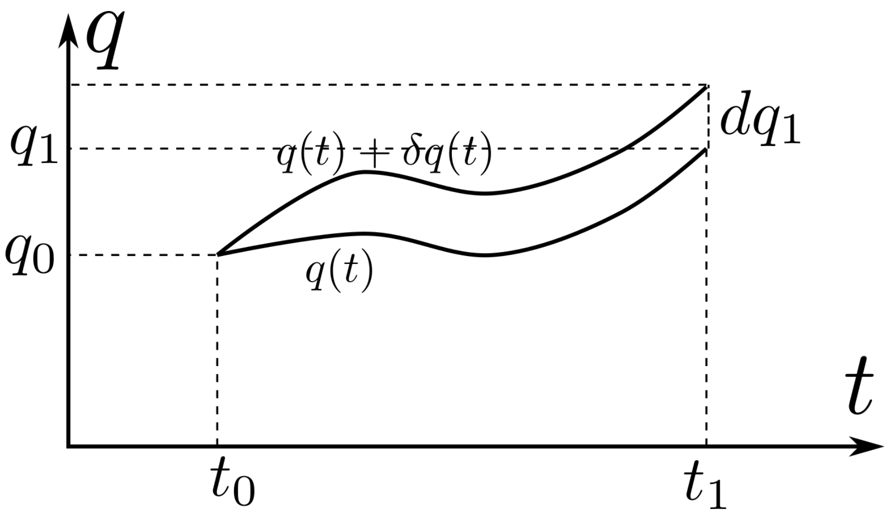
- Three degrees of freedom. We must have three second order differential equations for complete description. We have only three first order equations. Three more equations are missing.
- The equations are written for the components of $\vec{\Omega}$ in the non-internal system of coordinates which is rotating with $\vec{\Omega}$.

- In order to find the orientation of the rigid body as a function of time we need to write and solve three more first order differential equations.
- We will do it at some point next semester.

LECTURE 40
Hamilton-Jacobi equation.

This is the last lecture for the class. In this lecture we will tie together the concepts of Action, Lagrangian, and Hamiltonian.

40.1. Momentum.



Consider an action

$$\mathcal{S} = \int_{t_0}^{t_1} L(q, \dot{q}, t)dt, \qquad q(t_0) = q_0, \quad q(t_1) = q_1.$$

Consider the value of the action on the trajectory as a function of q_1. What it means is the following:

- We have an action and thus we have a Lagrangian.
- We write the Lagrangian equations of motion with the boundary conditions: $q(t_0) = q_0$ and $q(t_1) = q_1$.
- We solve these equation of motion (with the boundary conditions) and find the functions $q(t; t_0, q_0, t_1, q_1)$ – those are coordinates as function of time, the boundary conditions are the parameters the function depends on.
- We substitute those functions $q(t; t_0, q_0, t_1, q_1)$ into the action and take the integral over time t.
- The result will be a function — the value of the action on the trajectory. This function will depend on t_0, q_0, t_1, and q_1 .

- We are interested in how this function depends on q_1 with all other parameters fixed.

If we change the upper limit from q_1 to $q_1 + dq_1$ the trajectory will also change from $q(t)$ to $q(t) + \delta q(t)$, where $\delta q(t_0) = 0$, and $\delta q(t_1) = dq_1$. The change of the action then is

$$d\mathcal{S} = \int_{t_0}^{t_1} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_0}^{t_1} L(q, \dot{q}, t)dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}\delta q(t) + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}(t) \right) dt =$$

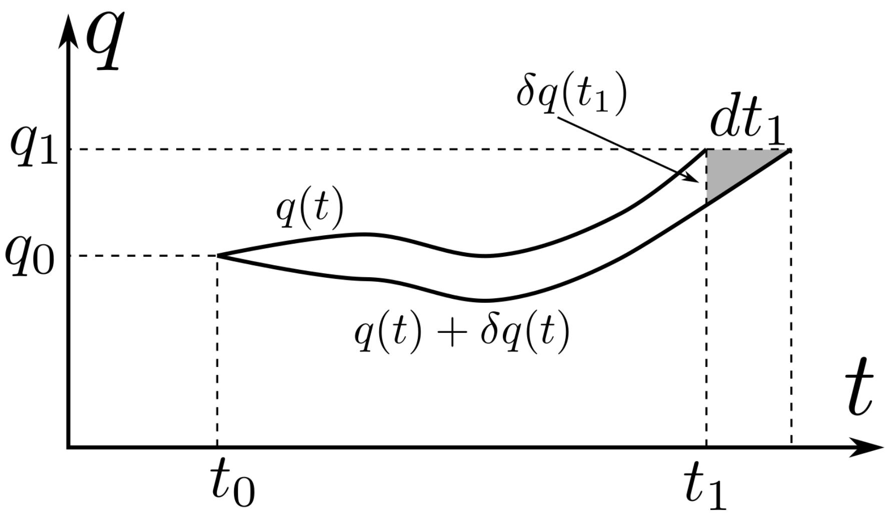
$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q(t)dt + \left.\frac{\partial L}{\partial \dot{q}}\delta q(t)\right|_{t_0}^{t_1} = \left.\frac{\partial L}{\partial \dot{q}}\delta q(t)\right|_{t_0}^{t_1} = pdq_1.$$

So we have

$$\frac{\partial \mathcal{S}}{\partial q} = p.$$

- I want to emphasize ones more: $\mathcal{S}$ here is <u>not</u> a functional! We already substituted the solution in. It is here the function of the upper (and lower) boundary conditions.

40.2. Energy.



Consider an action

$$\mathcal{S} = \int_{t_0}^{t_1} L(q, \dot{q}, t)dt, \qquad q(t_0) = q_0, \quad q(t_1) = q_1.$$

Consider <u>the value of the action on the trajectory</u> as a function of t_1.

Notice, that t_1 is there in two places: as the upper limit of integration and in the boundary condition. <u>We do not change the value of q at the upper limit! but the trajectory changes!</u>. So we have

$$\mathcal{S}(t_1 + dt_1) = \int_{t_0}^{t_1+dt_1} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt = Ldt_1 + \int_{t_0}^{t_1} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt.$$

Using the usual trick we will get

$$d\mathcal{S} = Ldt_1 + p\delta q|_{t_0}^{t_1} = Ldt_1 - p\dot{q}dt_1,$$

where I used $\delta q(t_1) = -\dot{q}(t_1)dt_1$ — see picture.

So we have

$$\frac{\partial \mathcal{S}}{\partial t} = -H.$$

- Notice, that this is all on a trajectory. So in the right hand side it is the value of the Hamiltonian on the trajectory. If energy is conserved, then it is just a number — energy.

40.3. Hamilton-Jacobi equation

We have on a trajectory

$$-\frac{\partial \mathcal{S}}{\partial t} = H(p, q, t),$$

but on a trajectory we also have $p = \frac{\partial \mathcal{S}}{\partial q}$, so we can write

$$-\frac{\partial \mathcal{S}}{\partial t} = H\left(\frac{\partial \mathcal{S}}{\partial q}, q, t\right).$$

This is a partial differential equation for the function $\mathcal{S}(q, t)$. This equation is called Hamilton-Jacobi equation.

The function $\mathcal{S}(q, t)$ at any moment of time defines a $N - 1$ dimensional hypersurface $\mathcal{S}(q, t) = \text{const.}$ in the N dimensional coordinate space — the space of all coordinates. With time this surface changes. One can imagine these as propagation of wave fronts — the Hamilton-Jacobi equation then is the non-linear wave equation. The rays corresponding to these hypersurfaces are different trajectories (for different initial conditions).

Let's imagine, that we solved this equation and found the function $\mathcal{S}(q, t, \alpha_1 \dots \alpha_N)$, where N is the number of the coordinates. Let's see how $\frac{\partial \mathcal{S}}{\partial \alpha_i}$ depends on time.

$$\frac{d}{dt}\frac{\partial \mathcal{S}}{\partial \alpha_i} = \dot{q}\frac{\partial^2 \mathcal{S}}{\partial q \partial \alpha_i} + \frac{\partial}{\partial t}\frac{\partial \mathcal{S}}{\partial \alpha_i} = \dot{q}\frac{\partial^2 \mathcal{S}}{\partial q \partial \alpha_i} - \frac{\partial}{\partial \alpha_i} H\left(\frac{\partial \mathcal{S}}{\partial q}, q, t\right) = \dot{q}\frac{\partial^2 \mathcal{S}}{\partial q \partial \alpha_i} - \frac{\partial H}{\partial p}\frac{\partial^2 \mathcal{S}}{\partial q \partial \alpha_i} = \dot{q}\frac{\partial^2 \mathcal{S}}{\partial q \partial \alpha_i} - \dot{q}\frac{\partial^2 \mathcal{S}}{\partial q \partial \alpha_i}.$$

Where we used the Hamilton-Jacobi equation and the Hamiltonian equation $\frac{\partial H}{\partial p} = \dot{q}$. So we see, that

$$\frac{d}{dt}\frac{\partial \mathcal{S}}{\partial \alpha_i} = 0.$$

So all $\frac{\partial \mathcal{S}}{\partial \alpha_i}$ do not change with time and are constants. Then the N equations

$$\frac{\partial \mathcal{S}}{\partial \alpha_i} = \beta_i$$

are implicit definitions of the solutions of the equations of motions $q(t, \alpha_i, \beta_i)$ that depend on $2N$ arbitrary constants, which are given by initial conditions.

40.4. Connection to quantum mechanics.

The quasiclassical approximation of quantum mechanics $\hbar \to 0$ transfers the Schrödinger equation into the Hamilton-Jacobi equation.

- $H(p, x)$ is a polynomial of p.
- Substitute $p \to \hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$ into the Hamiltonian and obtain the operator $\hat{H} = H(\hat{p}, x)$. This operator is called Hamiltonian operator.
- Consider a function $\Psi = e^{\frac{i}{\hbar}\mathcal{S}}$.
- Notice, $\hat{p}^2\Psi = -i\hbar\mathcal{S}''e^{\frac{i}{\hbar}\mathcal{S}} + \mathcal{S}'^2 e^{\frac{i}{\hbar}\mathcal{S}}$.
- Notice, that at $\hbar \to 0$ we have $\hat{p}^2\Psi \to \mathcal{S}'^2 e^{\frac{i}{\hbar}\mathcal{S}} = \Psi\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2$.

- It is clear, that the same will happen for any (positive integer) power of $\hat{p}$, namely at $\hbar \to 0$ we have $\hat{p}^n \Psi \to \Psi \left(\frac{\partial \mathcal{S}}{\partial x}\right)^n$.
- As $H(p, x)$ is a polynomial of p we will have $\hat{H}\Psi = \Psi H\left(\frac{\partial \mathcal{S}}{\partial x}, x\right)$.
- Also $i\hbar \frac{\partial}{\partial t}\Psi = -\Psi \frac{\partial \mathcal{S}}{\partial t}$.
- So we see that the equation

  $$i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

  at the limit $\hbar \to 0$ goes to

  $$-\frac{\partial \mathcal{S}}{\partial t} = H\left(\frac{\partial \mathcal{S}}{\partial x}, x\right).$$

- The first equation is the Schrödinger equation. The second one is the Hamilton-Jacobi equation.

Closing remarks

- No lecture on Monday.
- Student evaluation.
- How much you have learned.