

Lorentz-Violating Supergravity

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Abstract

The standard forms of supersymmetry and supergravity are inextricably wedded to Lorentz invariance. Here a Lorentz-violating form of supergravity is proposed. The superpartners have exotic properties that are not possible in a theory with exact Lorentz symmetry and microcausality. For example, the bosonic sfermions have spin $1/2$ and the fermionic gauginos have spin 1. The theory is based on a phenomenological action that is shown to follow from a simple microscopic and statistical picture.

1 Introduction

There is no spin-statistics connection in nonrelativistic quantum mechanics [1], for a simple reason: Although causality is a meaningful concept in a nonrelativistic picture, microcausality is not. I.e., there is no requirement that commutators vanish outside a light cone. For the same reason, a fundamental theory which violates Lorentz invariance does not necessarily lead to the usual spin-statistics connection [2]. Such a theory may contain, e.g., bosons with spin $1/2$ and fermions with spin 1, and these particles will have very unusual properties. Here we will discuss a Lorentz-violating theory which contains spin $1/2$ sfermions (bosonic superpartners of the Standard Model fermions), and spin 1 gauginos (fermionic superpartners of the Standard Model gauge bosons).

It is worthwhile first to clarify the vocabulary, in which the definition of terms like “supersymmetry” is broadened in the interest of clarity [3]: An action or theory is defined to be supersymmetric if it is invariant under a transformation which converts fermions to bosons and vice-versa. Also, “super” is used as a generic prefix for any objects that involve both commuting and anticommuting variables [4]. This usage parallels that for “complex”, which signifies that a quantity contains both real and imaginary parts. The most standard forms of supersymmetry are inextricably wedded to Lorentz invariance [5], whereas the broader usage permits a Lorentz-violating theory to be supersymmetric. The fermionic superpartners of Standard Model bosons still have the suffix “ino” (as in gaugino) and the bosonic superpartners of fermions still have the prefix “s” (as in squark), although the “s” now stands for “superpartner” rather than “scalar”.

Just as a theory can have Lorentz symmetry without supersymmetry, it can have supersymmetry without Lorentz symmetry. Of course, a true theory of nature must exhibit near-perfect Lorentz symmetry in those regimes where this symmetry has been tested by experiment and observation.

There are several motivations for Lorentz-violating theories, ranging from proposed *ad hoc* solutions of problems in astrophysics [6] to the anticipation that some mechanism may lead to a breaking of Lorentz invariance that is experimentally detectable [7, 8]. The present theory has a different motivation: Namely, the goal is to understand *why* Lorentz invariance is a fundamental symmetry, how it emerges from a still more fundamental picture, and under what circumstances it may be violated. In the second part of this paper (Sections 3 and 4), we will see that a simple microscopic and statistical picture leads to the following phenomenological and supersymmetric action:

$$S = \int d^D x \left[\frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b (\Psi^\dagger \Psi)^2 \right] \quad (1)$$

$$\Psi = \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix} \quad (2)$$

where the notation is explained in Refs. 2 and 4. In the first part of the paper (Section 2), we will start with this action and demonstrate that it leads to a Lorentz-violating form of supergravity. We will, however, emphasize only the action for Standard Model fermions and their superpartners. This action involves the coupling of the fermions and sfermions to the gauge fields and gravity, and to gauginos and gravitinos, but the complete action for the Standard Model bosons and their superpartners will be deferred for future discussion elsewhere.

2 Lorentz-Violating Supersymmetry and Supergravity

After a transformation to Lorentzian spacetime, the Euclidean action of (1) becomes

$$S_L = - \int d^D x \left[-\frac{1}{2m} \Psi^\dagger \partial^M \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b (\Psi^\dagger \Psi)^2 \right]. \quad (3)$$

Here, as in Refs. 2 and 4, the following convention is used for arbitrary D-dimensional vectors a_M : in the Euclidean formulation, $a^M = \delta^{MN} a_N$, but in the Lorentzian formulation, $a^M = \eta^{MN} a_N$, where $\eta^{MN} = \text{diag}(-1, 1, \dots, 1)$ is the Minkowski metric tensor in D dimensions ($M = 0, 1, 2, \dots, D-1$). (In both the Euclidean and Lorentzian formulations, x^0 is the coordinate corresponding to the physical time evolution of field densities like (51), and only the mathematical description is different.) The metric tensor associated with gravity and local Lorentz invariance, g_{MN} or g^{MN} , is always shown explicitly in expressions like $g^{MN} a_N$.

As in Refs. 2 and 4, it is assumed that the physical vacuum contains a GUT-scale condensate which forms in the very early universe, and which exhibits rotations in both external spacetime x^μ ($\mu = 0, 1, 2, 3$) and an internal space x^m ($m = 4, 5, \dots, D-1$). These rotations are described by unitary matrices U_{ext} and U_{int} , so that the order parameter

$$\Psi_{cond} = \langle \Psi_b \rangle_{vac} \quad (4)$$

has the form

$$\Psi_{cond} = \Psi_{ext}(x^\mu) \Psi_{int}(x^m, x^\mu) \quad (5)$$

$$\Psi_{ext}(x^\mu) = U_{ext}(x^\mu) n_{ext}^{1/2}(x^\mu) \eta_{ext} \quad (6)$$

$$\Psi_{int} = U_{int}(x^m, x^\mu) n_{int}^{1/2}(x^m, x^\mu) \eta_{int} \quad (7)$$

where η_{ext} and η_{int} are constant vectors. External and internal ‘‘superfluid velocities’’ are defined by

$$mv^\mu = -iU_{ext}^{-1}\partial^\mu U_{ext} - iU_{int}^{-1}\partial^\mu U_{int} \quad (8)$$

$$mv_m = -iU_{int}^{-1}\partial_m U_{int}. \quad (9)$$

Let us write

$$v^M = v_\alpha^M \sigma^\alpha + v_c^M \sigma^c, \quad v_M = v_{M\alpha} \sigma^\alpha + v_{Mc} \sigma^c \quad (10)$$

where $\alpha = 0, 1, 2, 3$ and $c > 3$. Also, σ^k ($k = 1, 2, 3$) is a Pauli matrix, σ^0 is the 2×2 identity matrix, and the σ^c are generators associated with an initial internal symmetry group G_{int} . As in Refs. 2 and 4, v_α^μ is interpreted as the gravitational vierbein e_α^μ :

$$e_\alpha^\mu = v_\alpha^\mu \quad (11)$$

$$g^{\mu\nu} = \eta^{\alpha\beta} e_\alpha^\mu e_\beta^\nu. \quad (12)$$

Also, $v_{\mu c}$ is interpreted as giving the gauge potentials A_μ^i through

$$e_{\mu c} = A_\mu^i K_i^n v_{nc} \quad (13)$$

with

$$e_{\mu c} = -v_{\mu c} \quad (14)$$

where K_i^n corresponds to the Killing vectors of the internal space, just as in classic Kaluza-Klein theories.

In the present paper we wish to generalize the above treatment by allowing for more general rotations which mix bosonic and fermionic degrees of freedom. First consider the global supersymmetry transformation

$$\Psi \rightarrow \Psi' = \mathcal{U} \Psi \quad (15)$$

or, with bosonic and fermionic fields shown separately,

$$\begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix} \rightarrow \begin{pmatrix} \Psi'_b \\ \Psi'_f \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{bb} & \mathcal{U}_{bf} \\ \mathcal{U}_{fb} & \mathcal{U}_{ff} \end{pmatrix} \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix}. \quad (16)$$

If

$$\mathcal{U}^\dagger \mathcal{U} = 1 \quad (17)$$

(3) is invariant under this transformation, so the theory is supersymmetric according to the definition given above. The elements of \mathcal{U}_{bb} and \mathcal{U}_{ff} are ordinary commuting variables, like the components of Ψ_b . The elements of \mathcal{U}_{bf} and \mathcal{U}_{fb} are anticommuting Grassmann variables, like the components of Ψ_f .

Now let us replace the picture of a rotating GUT-scale condensate by a more general picture in which all the fields of the vacuum contain a rotation described by a supermatrix \mathcal{U} which varies as a function of the spacetime coordinates. With a possible redefinition of the fermion fields, we can choose $\mathcal{U}_{ff} = \mathcal{U}_{bb}$ and write

$$\Psi^{vac} = \mathcal{U} n_{vac}^{1/2} \Psi^0 \quad (18)$$

where Ψ^0 is constant and

$$\Psi^{vac} = \langle \Psi \rangle_{vac} = \begin{pmatrix} \langle \Psi_b \rangle_{vac} \\ \langle \Psi_f \rangle_{vac} \end{pmatrix} \quad (19)$$

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_{bb} & \mathcal{U}_{bf} \\ \mathcal{U}_{fb} & \mathcal{U}_{bb} \end{pmatrix} \quad (20)$$

$$\Psi^{0\dagger} \Psi^0 = 1. \quad (21)$$

The generalizations of (8) - (10) and (13) - (14) are

$$mV^\mu = -i\mathcal{U}^{-1} \partial^\mu \mathcal{U} \quad (22)$$

$$\partial_m \mathcal{U} = \partial_m U = i U m v_m \quad (23)$$

$$V^M = V_\alpha^M \sigma^\alpha + V_c^M \sigma^c, \quad V_M = V_{M\alpha} \sigma^\alpha + V_{Mc} \sigma^c \quad (24)$$

$$E_{\mu c} = \mathcal{A}_\mu^i K_i^n v_{nc} \quad (25)$$

$$E_{\mu c} = -V_{\mu c} \quad (26)$$

where the last two expressions in (23) implicitly multiply a 2×2 identity matrix, and it is assumed that the internal coordinate space contains no supersymmetric rotations.

The fact that \mathcal{U} is unitary implies that $\partial_M \mathcal{U}^\dagger \mathcal{U} = -\mathcal{U}^\dagger \partial_M \mathcal{U}$ with $\mathcal{U}^\dagger = \mathcal{U}^{-1}$, or

$$mV_M = i\partial_M \mathcal{U}^\dagger \mathcal{U} \quad (27)$$

so that

$$V_M^\dagger = V_M, \quad V^{M\dagger} = V^M. \quad (28)$$

We can then write, e.g.,

$$V^M = \begin{pmatrix} V_{bb}^M & V_{bf}^M \\ V_{bf}^{M\dagger} & V_{bb}^M \end{pmatrix}. \quad (29)$$

At this point, the logic of Ref. 2 can be repeated with

$$v_M \rightarrow V_M, \quad v^M \rightarrow V^M \quad (30)$$

$$e_\alpha^\mu \rightarrow E_\alpha^\mu = \begin{pmatrix} e_\alpha^\mu & f_\alpha^\mu \\ f_\alpha^{\mu\dagger} & e_\alpha^\mu \end{pmatrix} \quad (31)$$

$$A_\mu^i \rightarrow \mathcal{A}_\mu^i = \begin{pmatrix} A_\mu^i & B_\mu^i \\ B_\mu^{i\dagger} & A_\mu^i \end{pmatrix}. \quad (32)$$

In particular, (3.38) - (3.41) of Ref. 2 become

$$\Psi(x^\mu, x^m) = \mathcal{U}(x^\mu, x^m) \Psi^r(x^\mu) \psi_r^{int}(x^m), \quad (33)$$

$$\partial_\mu \Psi = \mathcal{U}(x^\mu, x^m) (\partial_\mu + imV_{\mu\alpha}\sigma^\alpha + imV_{\mu c}\sigma^c) \Psi^r \psi_r^{int} \quad (34)$$

$$\begin{aligned} \int d^d x \Psi^\dagger \partial^\mu \partial_\mu \Psi &= \int d^d x \psi_r^{int\dagger} \Psi^{r\dagger} \eta^{\mu\nu} (\partial_\mu + imV_{\mu\alpha}\sigma^\alpha + imV_{\mu c}\sigma^c) \\ &\quad \times (\partial_\nu + imV_{\nu\beta}\sigma^\beta + imV_{\nu d}\sigma^d) \Psi^s \psi_s^{int} \\ &= \Psi^{r\dagger} \eta^{\mu\nu} \langle r | (\partial_\mu + imV_{\mu\alpha}\sigma^\alpha + imV_{\mu c}\sigma^c) \\ &\quad \times \sum_t |t\rangle \langle t| (\partial_\nu + imV_{\nu\beta}\sigma^\beta + imV_{\nu d}\sigma^d) |s\rangle \Psi^s \\ &= \Psi^{r\dagger} \eta^{\mu\nu} \left[\delta_{rt} (\partial_\mu + imV_{\mu\alpha}\sigma^\alpha) - i\mathcal{A}_\mu^i t_i^{rt} \right] \\ &\quad \times \left[\delta_{ts} (\partial_\nu + imV_{\nu\beta}\sigma^\beta) - i\mathcal{A}_\nu^j t_j^{ts} \right] \Psi^s \\ &= \Psi_{ext}^\dagger \eta^{\mu\nu} \left[(\partial_\mu - i\mathcal{A}_\mu^i t_i) + imV_{\mu\alpha}\sigma^\alpha \right] \\ &\quad \times \left[(\partial_\nu - i\mathcal{A}_\nu^j t_j) + imV_{\nu\beta}\sigma^\beta \right] \Psi_{ext} \end{aligned} \quad (35)$$

$$\begin{aligned} S_L &= \int d^4 x \Psi_{ext}^\dagger \times \\ &\quad \left(\frac{1}{2m} \mathcal{D}^\mu \mathcal{D}_\mu + \frac{1}{2} iV_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu + \frac{1}{2} \mathcal{D}_\mu iV_\alpha^\mu \sigma^\alpha - \frac{1}{2} mV_\alpha^\mu V_{\mu\alpha} + \mu_{ext} \right) \Psi_{ext} \end{aligned} \quad (36)$$

where

$$\mathcal{D}_\mu = \partial_\mu - i\mathcal{A}_\mu^i t_i. \quad (37)$$

Also, the generalization of (2.22) of Ref. 2 is

$$\Psi^{0\dagger} n_{vac}^{1/2} \left[\left(\frac{1}{2} mV^\mu V_\mu - \frac{1}{2m} \partial^\mu \partial_\mu - \mu_{ext} \right) - i \left(\frac{1}{2} \partial^\mu V_\mu + V^\mu \partial_\mu \right) \right] n_{vac}^{1/2} \Psi^0 = 0. \quad (38)$$

Adding this equation to its Hermitian conjugate gives a still more general Bernoulli equation

$$\frac{1}{2} m \Psi^{0\dagger} V^\mu V_\mu \Psi^0 + P_{ext} = \mu_{ext} \quad (39)$$

where

$$P_{ext} = -\frac{1}{2m} n_{vac}^{-1/2} \partial^\mu \partial_\mu n_{vac}^{1/2}. \quad (40)$$

As before, it is assumed that the basic texture of the vacuum field rotations is such that

$$V_0^k = V_a^0 = 0 \quad , \quad k, a = 1, 2, 3, \quad (41)$$

and that the nonzero gauge potentials are not coupled to Ψ^0 at energies well below the GUT scale [4], so that (39) reduces to

$$\frac{1}{2}mV_\alpha^\mu V_{\alpha\mu} + P_{ext} = \mu_{ext}. \quad (42)$$

When $\partial_\mu n_{vac}^{1/2}$ and $\partial^\mu V_\mu$ are neglected, (36) then simplifies to

$$S_L = \int d^4x \Psi_{ext}^\dagger \left(\frac{1}{2m} \mathcal{D}^\mu \mathcal{D}_\mu + iE_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu \right) \Psi_{ext}. \quad (43)$$

Since m is comparable to the Planck mass, it is reasonable to assume that the first term can be neglected, giving

$$S_L = \int d^4x \Psi_{ext}^\dagger iE_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu \Psi_{ext} \quad (44)$$

or, with e_α^μ again slowly varying,

$$S_L = \int d^4x e \bar{\Psi}_{ext}^\dagger iE_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu \bar{\Psi}_{ext} \quad (45)$$

$$\bar{\Psi}_{ext} = e^{-1/2} \Psi_{ext} \quad , \quad e = \det(e_{\alpha\mu}) . \quad (46)$$

According to (31) and (32), the bosonic fields play the same role as before. Namely, e_α^μ is the vierbein representing the gravitational field, and A_μ^i is the potential representing the gauge fields of a grand-unified theory – an $SO(10)$ theory [9, 10, 11] if the dimension of the internal space is 10. The fermionic fields can be interpreted in an equally simple way: namely, f_α^μ corresponds to a spin 2 gravitino and B_μ^i to spin 1 gauginos. Again, we have generalized the usual vocabulary, so that the superpartner of the graviton is defined to be the gravitino, and the superpartners of gauge bosons to be gauginos, even though these fermions have quite unconventional properties. The unconventional spin is required because, e.g., B_μ^i transforms as a vector.

Here we have emphasized the fermions of the Standard Model and their bosonic superpartners. We have shown that all of these particles are automatically coupled to gauge fields, the gravitational field, gauginos, and gravitinos, if one assumes the action (1) and gives A_μ^i , e_α^μ , B_μ^i , and f_α^μ the physical interpretation above. The Higgs sector has not been considered here; as will be discussed elsewhere, the action for Higgs fields is interpreted as a low-energy effective action. As will also be discussed elsewhere, the action for the gauge fields, gravitational field, gauginos, and gravitinos is now interpreted as arising from a “vacuum diamagnetism”, rather than just from the topological defects responsible for curvature [4]. Finally, a discussion of the detailed phenomenology (including the existence of a stable LSP) will again be presented elsewhere.

3 Statistical Origin of the Bosonic Action

In this section we turn to a different issue: the origin of the phenomenological action (1). We will show that this action follows from a simple microscopic and statistical picture. Our starting point is a single fundamental system which consists of N_w identical “whits”, with N_w variable. (“Whit”, whose meanings include “particle” and “least possible amount”, is an appropriate name for the irreducible objects that are postulated here.) Each whit can exist in any of M_w states, with the number of whits in the i th state represented by n_i . A *microstate* of the fundamental system is specified by the number of whits and the state of each whit. A *macrostate* is specified by only the occupancies n_i of the states.

As discussed below, D of the states are used to define D coordinates x^M in Euclidean spacetime, m_w of the states are used to define observable fields ϕ_k , and the remaining $(M_w - m_w - D)$ states may be regarded as corresponding to fields that are unobservable (at least at the energy scales considered here).

Let us begin by defining an initial set of coordinates X^M in terms of the occupancies n_M :

$$X^M = \pm n_M a_0 \quad (47)$$

where $M = 0, 1, \dots, D - 1$. It is convenient to include a fundamental length a_0 in this definition, so that we can later express the coordinates in conventional units. As one might expect, a_0 will eventually turn out to be comparable to the Planck length:

$$a_0 \sim \ell_P = (16\pi G)^{1/2} \quad (48)$$

since $a_0 = m^{-1} \sim m_P^{-1} = \ell_P$ according to (78).

With the definition (47), positive and negative coordinates correspond to the same occupancies. There are two relevant facts, however, which make this definition physically acceptable: First, two points whose coordinates differ by a minus sign are typically separated by cosmologically large distances. Second, and more importantly, the fields ϕ_k need not return to their original values when they are evolved, according to their equation of motion, from points with positive coordinates to points with negative coordinates. I. e., the field configurations described by the two sets of points can be regarded as distinct, and in this sense the points themselves are distinct. The different branches of the field configuration are analogous to the branches of a multivalued function like $z^{1/2}$, which are taken to correspond to distinct Riemann sheets.

At extremely small distances, spacetime is discrete in the present theory, with a finite interval between two adjacent points X^M and $X^M + \delta X^M$: $\delta X^M = a_0$. As in Section 2, the X^M are divided into 4 external coordinates X^μ and $(D - 4)$ internal coordinates X^m . In the internal space it is natural to have variations on a length scale that is comparable to ℓ_P . In external spacetime, on the other hand, we wish to consider fields which vary much more slowly, and it is convenient to average over a more physically meaningful length scale. Let us therefore consider a D -dimensional rectangular box centered on a point \bar{X} , with X^M ranging from $\bar{X}^M - a^M/2$ to $\bar{X}^M + a^M/2$. For the $(D - 4)$ coordinates of internal space, a^m is taken to be the original fundamental length a_0 . For the 4 coordinates of external spacetime, a^μ

is taken to be an arbitrary length a , and we will find that the final form of the action is independent of this parameter.

In this coarse-grained picture, the density of whits in the i th state is

$$\rho_i(\bar{X}) = N_i/\Delta V \quad , \quad i = 1, 2, \dots, M_w \quad (49)$$

where

$$N_i = \sum_X n_i(X) \quad , \quad \Delta V = \prod_M a^M = a^4 a_0^{D-4} \quad (50)$$

and the values of X are those lying within the box centered on \bar{X} . Let

$$\phi_k^2 = \rho_k \quad , \quad k = 1, 2, \dots, m_w. \quad (51)$$

The initial bosonic fields ϕ_k are then real (but defined only up to a phase factor ± 1).

Let $\bar{S}(\bar{X})$ be the entropy of the single box at \bar{X} for a given set of densities ρ_i , as defined by $\bar{S}(\bar{X}) = \log W(\bar{X})$ (in units with $k_B = \hbar = c = 1$). Here $W(\bar{X})$ is the total number of microstates in this box at fixed ρ_i or N_i : $W(\bar{X}) = \mathcal{N}(\bar{X})!/\prod_i N_i(\bar{X})!$, with

$$\mathcal{N}(\bar{X}) = \sum_i N_i(\bar{X}). \quad (52)$$

The total number of available microstates for all points \bar{X} is $W = \Pi_{\bar{X}} W(\bar{X})$, so the total entropy is

$$\bar{S} = \sum_{\bar{X}} \bar{S}(\bar{X}) \quad (53)$$

$$\bar{S}(\bar{X}) = \log \Gamma(\mathcal{N}(\bar{X}) + 1) - \sum_i \log \Gamma(N_i(\bar{X}) + 1). \quad (54)$$

It follows that

$$\frac{\partial \bar{S}}{\partial N_i(\bar{X})} = \psi(\mathcal{N}(\bar{X}) + 1) - \psi(N_i(\bar{X}) + 1) \quad (55)$$

$$\frac{\partial^2 \bar{S}}{\partial N_{i'}(\bar{X}) \partial N_i(\bar{X})} = \psi^{(1)}(\mathcal{N}(\bar{X}) + 1) - \psi^{(1)}(N_i(\bar{X}) + 1) \delta_{i'i} \quad (56)$$

where $\psi(z) = d \log \Gamma(z)/dz$ and $\psi^{(n)}(z) = d^{n+1} \log \Gamma(z)/dz^{n+1}$ are the digamma and polygamma functions, with the asymptotic expansions [12]

$$\psi(z) = \log z - \frac{1}{2z} - \sum_{l=1}^{\infty} \frac{B_{2l}}{2l z^{2l}} \quad (57)$$

$$\psi^{(n)}(z) = (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{l=1}^{\infty} B_{2l} \frac{(2l+n-1)!}{(2l)! z^{n+2l}} \right] \quad (58)$$

as $z \rightarrow \infty$. For $a \gg \ell_P$, we have $\mathcal{N}(\bar{X}) \gg \gg \bar{n}_\mu = (\bar{X}^\mu/a_0)^2 \gg \gg 1$, so it is an extremely good approximation to write

$$\frac{\partial \bar{S}}{\partial N_k(\bar{X})} = \log \mathcal{N}(\bar{X}) - \psi(N_k(\bar{X}) + 1) \quad (59)$$

$$\frac{\partial^2 \bar{S}}{\partial N_{k'}(\bar{X}) \partial N_k(\bar{X})} = -\psi^{(1)}(N_k(\bar{X}) + 1) \delta_{k'k}. \quad (60)$$

We could express \bar{S} as a Taylor series expansion about the bare vacuum with $\phi_k(\bar{X}) = 0$ for all k and \bar{X} :

$$\bar{S} = S_{bare} + \sum_{\bar{X},k} \sum_n b_n(\bar{X}) N_k(\bar{X})^n \quad (61)$$

$$b_1(\bar{X}) = \log \mathcal{N}_{bare}(\bar{X}) - \psi(1) \quad (62)$$

$$b_{n+1} = -\psi^{(n)}(1)/n! \quad , \quad n = 1, 2, \dots \quad (63)$$

with

$$\psi(1) = -\gamma \quad , \quad \gamma = \text{Euler's constant} \quad (64)$$

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad (65)$$

where $\mathcal{N}_{bare}(\bar{X})$ is the value of $\mathcal{N}(\bar{X})$ when $N_k(\bar{X}) = 0$ for all the observable states k and $\zeta(z)$ is the Riemann zeta function. This is not physically appropriate, however, because bosonic fields exhibit extremely large zero-point fluctuations in the physical vacuum [13]. (These are analogous to the zero-point oscillations $\langle x^2 \rangle$ of a harmonic oscillator, but with a very large number of modes extending up to a Planck-scale cutoff.) In fact, it is consistent with both standard physics and the treatment of this paper to assume that

$$\langle \phi_k^2 \rangle_{vac} = \langle \rho_k \rangle_{vac} = \langle N_k \rangle_{vac} / \Delta V \sim \ell_P^{-D}. \quad (66)$$

Since there is no initial distinction between the states ϕ_k , it is reasonable to perform a Taylor series expansion about the same value N_{vac} for each k , where

$$N_{vac} \sim \ell_P^{-D} \Delta V \sim (a/\ell_P)^4 \gg \gg 1 \quad (67)$$

if, e.g., $a^{-1} \sim 10^{10}$ TeV (with $\ell_P^{-1} = m_P \sim 10^{15}$ TeV). It is then an extremely good approximation to use the asymptotic formulas above and write

$$\bar{S} = S_{vac} + \sum_{\bar{X},k} a_1 \Delta N_k(\bar{X}) + \sum_{\bar{X},k} a_2 [\Delta N_k(\bar{X})]^2 \quad (68)$$

$$\Delta N_k(\bar{X}) = N_k(\bar{X}) - N_{vac} \quad (69)$$

$$a_1 = \log \mathcal{N}_{vac} - \log N_{vac} \quad , \quad a_2 = -1/(2N_{vac}) \quad (70)$$

where $\mathcal{N}_{vac}(\bar{X})$ is the value of $\mathcal{N}(\bar{X})$ when $N_k(\bar{X}) = N_{vac}$ for all k , and the neglected terms are of order $[\Delta N_k(\bar{X})/N_{vac}]^n \Delta N_k(\bar{X})$, $n \geq 2$.

It is not conventional or convenient to deal with ΔN_k and $(\Delta N_k)^2$, so let us instead write \bar{S} in terms of the fields ϕ_k and their derivatives $\partial\phi_k/\partial x^M$ via the following procedure: First, we can switch from the original points \bar{X} , which are defined to be the centers of the boxes, to a new set of points \tilde{X} , which will be defined to be the corners of the boxes. It is easy to see that

$$\bar{S} = S_{vac} + \sum_{\tilde{X},k} a_1 \langle \Delta N_k(\bar{X}) \rangle + \sum_{\tilde{X},k} a_2 \langle [\Delta N_k(\bar{X})]^2 \rangle \quad (71)$$

where $\langle \dots \rangle$ in the present context indicates an average over the 2^D boxes labeled by \bar{X} which have the common corner \tilde{X} . Second, we can write $\Delta N_k = \Delta\rho_k \Delta V = (\langle \Delta\rho_k \rangle + \delta\rho_k) \Delta V$, with $\langle \delta\rho_k \rangle = 0$:

$$\bar{S} = S_{vac} + \sum_{\tilde{X},k} a_1 \langle (\langle \Delta\rho_k \rangle + \delta\rho_k) \Delta V \rangle + \sum_{\tilde{X},k} a_2 \langle (\langle \Delta\rho_k \rangle + \delta\rho_k)^2 \rangle (\Delta V)^2 \quad (72)$$

$$= S_{vac} + \sum_{\tilde{X},k} a_1 \langle \Delta\rho_k \rangle \Delta V + \sum_{\tilde{X},k} a_2 [\langle \Delta\rho_k \rangle^2 + \langle (\delta\rho_k)^2 \rangle] (\Delta V)^2. \quad (73)$$

Each of the 2^D points \bar{X} surrounding \tilde{X} is displaced by $\pm a/2$ along the x^μ axes and $\pm a_0/2$ along the x^m axes. The last term above can therefore be rewritten

$$\langle (\delta\rho_k)^2 \rangle = \sum_{\mu} \left(\frac{\partial\rho_k}{\partial X^\mu} \right)^2 \left(\frac{a}{2} \right)^2 + \sum_m \left(\frac{\partial\rho_k}{\partial X^m} \right)^2 \left(\frac{a_0}{2} \right)^2 \quad (74)$$

$$= \sum_{\mu} \rho_k \left(\frac{\partial\phi_k}{\partial X^\mu} \right)^2 a^2 + \sum_m \rho_k \left(\frac{\partial\phi_k}{\partial X^m} \right)^2 a_0^2 \quad (75)$$

where the neglected terms involve higher derivatives and higher powers of a and a_0 . Since $\rho_k = \rho_{vac} + \Delta\rho_k$, with $\Delta\rho_k \ll \rho_{vac} = N_{vac}/\Delta V$ for normal fields, it is an extremely good approximation to replace ρ_k by ρ_{vac} in the above expression, and to neglect the term involving $a_2 (\Delta V)^2 (\Delta\rho_k)^2 = -(\Delta N_k)^2/2N_{vac}$, so that we have

$$\bar{S} = S'_{vac} + \sum_{\tilde{X},k} \Delta V \left\{ \mu \bar{\phi}_k^2 - \frac{1}{2m} \left[\sum_{\mu} \left(\frac{\partial\bar{\phi}_k}{\partial X^\mu} \right)^2 \left(\frac{a}{a_0} \right)^2 + \sum_m \left(\frac{\partial\bar{\phi}_k}{\partial X^m} \right)^2 \right] \right\} \quad (76)$$

where

$$m = a_0^{-1} \quad , \quad \mu = m (\log \mathcal{N}_{vac} - \log N_{vac}) \quad , \quad \bar{\phi}_k = \phi_k/m \quad (77)$$

and $S'_{vac} = S_{vac} - \sum_{\tilde{X},k} N_{vac} (\log \mathcal{N}_{vac} - \log N_{vac})$. Recall that

$$m \sim m_P = \ell_P^{-1}. \quad (78)$$

The philosophy behind the above treatment is simple: We essentially wish to replace $\langle f^2 \rangle$ by $(\partial f / \partial x)^2$, and this can be accomplished because

$$\langle f^2 \rangle - \langle f \rangle^2 = \langle (\delta f)^2 \rangle \approx \langle (\partial f / \partial x)^2 (\delta x)^2 \rangle = (\partial f / \partial x)^2 (a/2)^2. \quad (79)$$

The form of (76) also has a simple interpretation: The entropy \bar{S} increases with the number of whits, but decreases when the whits are not uniformly distributed.

In the continuum limit,

$$\sum_{\tilde{X}} \Delta V = \sum_{\tilde{X}} a^4 a_0^{D-4} \rightarrow \int d^D X = \int_a^\infty d^4 X \int_{a_0}^\infty d^{D-4} X \quad (80)$$

(76) becomes

$$\begin{aligned} \bar{S} &= S'_{vac} + \int_a^\infty d^4 X \int_{a_0}^\infty d^{D-4} X \sum_k \quad (81) \\ &\quad \times \left\{ \mu \bar{\phi}_k^2 - \frac{1}{2m} \left[\sum_\mu \left(\frac{\partial \bar{\phi}_k}{\partial X^\mu} \right)^2 \left(\frac{a}{a_0} \right)^2 + \sum_m \left(\frac{\partial \bar{\phi}_k}{\partial X^m} \right)^2 \right] \right\} \\ &= S'_{vac} + \int_{a_0}^\infty d^D x \sum_k \left[\mu \Phi_k^2 - \frac{1}{2m} \sum_M \left(\frac{\partial \Phi_k}{\partial x^M} \right)^2 \right] \quad (82) \end{aligned}$$

where

$$x^m = X^m, \quad x^\mu = (a_0/a) X^\mu, \quad \Phi_k = (a_0/a)^2 \bar{\phi}_k. \quad (83)$$

The lower limit on each integral is the cutoff imposed by the size of the rectangular boxes used in the coarse-graining above: a for X^μ , a_0 for X^m , and a_0 for any x^M . The continuum limit is an extremely good approximation for slowly varying fields in external spacetime, but only a moderately good approximation within the internal space, where the order parameter varies on a length scale comparable to ℓ_P . This implies that terms involving higher derivatives $\partial^n \bar{\phi}_k / \partial (x^m)^n$ can be significant in the internal space.

Notice that the final form (82) is independent of the arbitrary length a which was used for coarse-graining in external spacetime. The fields must be rescaled as a is varied, but this is already a familiar feature in standard physics [14].

A physical configuration of all the fields $\phi_k(x)$ corresponds to a specification of all the densities $\rho_k(x)$. In the present picture, the probability of such a configuration is proportional to $W = e^{\bar{S}}$. In the Euclidean path integral, the probability is proportional to e^{-S_E} , where S_E is the Euclidean action. We conclude that

$$S_E = -\bar{S} + \text{constant}. \quad (84)$$

Choosing the constant to be zero, and employing the Einstein summation convention for all repeated indices, we obtain

$$S_E = -S'_{vac} + \int d^D x \left(\frac{1}{2m} \frac{\partial \Phi_k}{\partial x_M} \frac{\partial \Phi_k}{\partial x^M} - \mu \Phi_k \Phi_k \right). \quad (85)$$

The above result neglects interactions among the observable and unobservable fields, which will arise from the higher-order terms neglected above. Since a detailed treatment of these interactions would be quite complicated, we resort at this point to a phenomenological description: We assume that probability can flow out of and into each field, and that this effect can be modeled by a random optical potential $i\tilde{V}$ which has a Gaussian distribution, with

$$\langle \tilde{V} \rangle = 0 \quad , \quad \langle \tilde{V}(x) \tilde{V}(x') \rangle = b \delta(x - x') \quad (86)$$

where b is a constant.

If we also assume that the number of observable real fields Φ_k is even, we can group them in pairs to form complex fields $\Psi_{b,k}$. Then we finally have $S_E = S_0 + \bar{S}_E[\Psi_b, \Psi_b^\dagger]$ with

$$\bar{S}_E[\Psi_b, \Psi_b^\dagger] = \int d^D x \left(\frac{1}{2m} \partial^M \Psi_b^\dagger \partial_M \Psi_b - \mu \Psi_b^\dagger \Psi_b + i\tilde{V} \Psi_b^\dagger \Psi_b \right) \quad (87)$$

where Ψ_b is the vector with components $\Psi_{b,k}$.

4 Supersymmetric Action

If F is a physical quantity which is determined by the observable fields, its average value is given by

$$\langle F \rangle = \left\langle \frac{\int \mathcal{D}\Psi_b \mathcal{D}\Psi_b^\dagger F[\Psi_b, \Psi_b^\dagger] e^{-\bar{S}_E[\Psi_b, \Psi_b^\dagger]}}{\int \mathcal{D}\Psi'_b \mathcal{D}\Psi'^{\dagger}_b e^{-\bar{S}_E[\Psi'_b, \Psi'^{\dagger}_b]}} \right\rangle \quad (88)$$

where $\langle \dots \rangle$ represents an average over the perturbing potential $i\tilde{V}$. The presence of the denominator makes it difficult to perform this average, but there is a trick for removing the bosonic degrees of freedom Ψ'_b in the denominator and replacing them with fermionic degrees of freedom Ψ_f in the numerator [15, 16, 17, 18]: Since

$$\int \mathcal{D}\Psi'_b \mathcal{D}\Psi'^{\dagger}_b e^{-\bar{S}_E[\Psi'_b, \Psi'^{\dagger}_b]} = (\det A)^{-1} \quad (89)$$

$$\int \mathcal{D}\Psi_f \mathcal{D}\Psi_f^\dagger e^{-\bar{S}_E[\Psi_f, \Psi_f^\dagger]} = \det A \quad (90)$$

where A represents the operator of (87), it follows that

$$\langle F \rangle = \left\langle \int \mathcal{D}\Psi_b \mathcal{D}\Psi_b^\dagger \mathcal{D}\Psi_f \mathcal{D}\Psi_f^\dagger F e^{-\bar{S}_E[\Psi_b, \Psi_b^\dagger]} e^{-\bar{S}_E[\Psi_f, \Psi_f^\dagger]} \right\rangle \quad (91)$$

$$= \left\langle \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger F e^{-\bar{S}_E[\Psi, \Psi^\dagger]} \right\rangle \quad (92)$$

where Ψ_b and Ψ_f have been combined into Ψ ,

$$\Psi = \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix}, \quad (93)$$

and

$$\bar{S}_E [\Psi, \Psi^\dagger] = \int d^D x \left[\partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + i\tilde{V} \Psi^\dagger \Psi \right]. \quad (94)$$

(In (93), Ψ_f consists of Grassmann variables $\Psi_{f,k}$, just as Ψ_b consists of ordinary variables $\Psi_{b,k}$.) For a Gaussian random variable v whose mean is zero, the result

$$\langle e^{-iv} \rangle = e^{-\frac{1}{2}\langle v^2 \rangle} \quad (95)$$

implies that

$$\left\langle e^{-\int d^D x i\tilde{V} \Psi^\dagger \Psi} \right\rangle = e^{-\frac{1}{2} \int d^D x d^D x' \Psi^\dagger(x) \Psi(x) \langle \tilde{V}(x) \tilde{V}(x') \rangle \Psi^\dagger(x') \Psi(x')} \quad (96)$$

$$= e^{-\frac{1}{2} b \int d^D x [\Psi^\dagger(x) \Psi(x)]^2}. \quad (97)$$

It follows that

$$\langle F \rangle = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger F e^{-S} \quad (98)$$

with

$$S = \int d^D x \left[\frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b (\Psi^\dagger \Psi)^2 \right]. \quad (99)$$

A special case is

$$Z = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger e^{-S} \quad (100)$$

but according to (88)

$$Z = 1. \quad (101)$$

To make the expression for $\langle F \rangle$ independent of how the measure is defined in the path integral, we can rewrite (98) as

$$\langle F \rangle = \frac{1}{Z} \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger F e^{-S}. \quad (102)$$

Notice that the fermionic variables Ψ_f represent true degrees of freedom, and that they originate from the bosonic variables Ψ'_b . The coupling between the fields Ψ_b and Ψ_f (or Ψ'_b) is due to the random perturbing potential $i\tilde{V}$.

5 Conclusion

The goal of this paper, and of the program represented by Refs. 2 and 4, is ambitious: It is to start with a simple and convincing microscopic picture, to show that this picture reproduces standard physics in the appropriate regime (of known particles at energies far below the Planck scale), and to propose extensions of standard physics that are experimentally testable.

A truly fundamental theory should aspire to explaining the origins of

- Lorentz invariance
- bosonic fields

- fermionic fields
- supersymmetry
- gauge fields and their symmetry
- gravity
- quantum mechanics
- spacetime.

In the present paper, Lorentz invariance emerges for fermions at energies that are far below the Planck scale, in part because the second-order term in (43) can be neglected at low energy. Bosonic fields, fermionic fields, and supersymmetry emerge via the arguments of Sections 3 and 4, immediately above. Gauge fields and gravity, and well as gaugino and gravitino fields, emerge as collective modes of the vacuum, described by the supermatrix \mathcal{U} of (20). Quantum mechanics emerges through the chain of reasoning which begins in Section 3: Starting with a statistical picture, one obtains an entropy \bar{S} which is then essentially interpreted as the negative of the Euclidean action, according to (84). After a transformation to Lorentzian time, and a change from path-integral quantization to canonical quantization, one obtains the usual formulation of quantum field theory. Finally, spacetime emerges according to the definition (47) of initial spacetime coordinates. Essentially, each coordinate x^M corresponds to a different state labeled by M , and the occupancy of this state measures the position along the x^M axis.

There are many obvious philosophical issues involved in, e.g., the transition from Euclidean to Lorentzian time, and the identification of occupancies with spacetime coordinates. It appears, nevertheless, that these issues can be resolved and that the picture of the above paragraph makes physical sense.

Most importantly, there are predictions that should be experimentally testable within the next few years. In particular, the simplest form of the present theory predicts that each particle and its superpartner have the same spin, so that bosonic sfermions have spin 1/2 and fermionic gauginos have spin 1.

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