

A STATISTICAL SUPERFIELD AND ITS OBSERVABLE CONSEQUENCES

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Abstract

A new kind of fundamental superfield is proposed, with an Ising-like Euclidean action. Near the Planck energy it undergoes its first stage of symmetry-breaking, and the ordered phase is assumed to support specific kinds of topological defects. This picture leads to a low-energy Lagrangian which is similar to that of standard physics, but there are interesting and observable differences. For example, the cosmological constant vanishes, fermions have an extra coupling to gravity, the gravitational interaction of W-bosons is modified, and Higgs bosons have an unconventional equation of motion.

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1 Introduction

The terms “superfield” and “supersymmetry” are ordinarily used in a context which presupposes local Lorentz invariance.¹⁻³ It is far from clear, however, that Lorentz invariance is still valid near the Planck scale, fifteen orders of magnitude above the energies where it has been tested. (A century ago, all accepted theories presupposed Galilean invariance.) In this paper the above terms will be used in a broader sense, to mean any field which has both commuting and Grassmann parts and any symmetry which relates these parts. At the same time, the presumption of Lorentz invariance at arbitrarily high energies will be replaced by a less stringent requirement: Lorentz invariance, and the other principles of standard physics, need only emerge at the relatively low energies where they have been tested. One is then free to consider any description which is mathematically consistent and also consistent with experiment and observation.

It appears, however, that a fundamental theory still needs four central ingredients: a space, a field, an action, and a pattern of symmetry-breaking. The specific versions assumed here are as follows:

(1) The space (or base manifold) is R^D ; i.e., it is D -dimensional, flat, and initially Euclidean.

(2) The classical field Ψ at each point x is an N -dimensional supersymmetric vector.

(3) The Euclidean action S has the same basic form as the Hamiltonian for spins on a lattice.⁴

(4) Below the Planck energy $< \Psi >$ becomes nonzero. This order parameter can then support topological defects, analogous to those in condensed matter physics.⁵⁻¹¹ Three such defects are postulated in Section 7. The first two of these cause the original symmetry group G_0 to break down locally to a reduced symmetry group $U(1) \times SU(2) \times G$. They also produce a local “filamentary” geometry with 4 extended dimensions and $(D-4)$ that are compact.

After a reasonable series of approximations, we will find that this relatively simple picture leads to a low-energy Lagrangian (9.4) which closely resembles the conventional Lagrangian of particle physics^{12,13} and general relativity.^{14,15} There are some interesting and observable differences, however, and these are discussed at the end of the paper.

2 The superfield and its Euclidean action

First consider an analogy from ordinary statistical mechanics: classical spins on a lattice in D dimensions. Each spin $s(x)$ is an N -dimensional vector whose components are real numbers. In the continuum limit, one can obtain a Ginzburg-Landau Hamiltonian

$$H = \int d^D x \left[\frac{1}{2m} \partial^M s^\dagger \partial_M s - \mu s^\dagger s + \frac{1}{2} b (s^\dagger s)^2 \right] \quad (2.1)$$

where $\partial_M = \partial/\partial x^M$. (See, e.g., (3.5.17) of Ref. 4. Summation is implied over repeated indices, and inner products involving vectors are also implied.) The physical properties of this statistical system are determined by H , via the partition function.^{4,16–19} At low temperature the order parameter $\langle s \rangle$ can become nonzero, making the system ferromagnetic.

The starting point of the present theory is very similar: a classical field $\Psi(x)$ having the form

$$\Psi = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \quad (2.2)$$

where each z consists of an ordinary complex number z_b and a complex Grassmann number z_f :

$$z = \begin{pmatrix} z_b \\ z_f \end{pmatrix}. \quad (2.3)$$

(Anticommuting Grassmann numbers are required in any classical treatment which includes fermions.^{3,4,12,18–26}) The Euclidean action is postulated to have the Ising-like form

$$S = \beta H \quad (2.4)$$

$$H = -J \sum_{ij} \Psi_i^\dagger \Psi_j + \frac{1}{2} r \sum_{ij} (\Psi_i^\dagger \Psi_j)^2 \quad (2.5)$$

where the summation is over nearest-neighbor lattice sites. The first term represents a tendency for the field to be aligned at neighboring points, and the second ensures that S has a lower bound.

Since

$$\Psi_i^\dagger \Psi_j + \Psi_j^\dagger \Psi_i = -(\Psi_i - \Psi_j)^\dagger (\Psi_i - \Psi_j) + \Psi_i^\dagger \Psi_i + \Psi_j^\dagger \Psi_j \quad (2.6)$$

the continuum version of (2.4) is

$$S = \int d^D x \left[\frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b (\Psi^\dagger \Psi)^2 \right] \quad (2.7)$$

where $\Psi(x) = (a^{-D}\beta)^{1/2}\Psi_j$, $(2m)^{-1} = a^2J$, $\mu = 2DJ$, $b = 2Da^D\beta^{-1}r$, and a is the lattice spacing. We will find below, in (7.28) and (7.34), that m , μ , and b can be related to the Planck energy m_P , defined by

$$m_P^{-1} = \ell_P = (16\pi G)^{1/2} \quad (2.8)$$

where G is the gravitational constant. (Units with $\hbar = c = 1$ are used, so mass and energy are equivalent to inverse length.) The definition (2.8) implies that $\ell_P \sim 10^{-32}$ cm and $m_P \sim 10^{15}$ TeV.

In the continuum treatment represented by (2.7), the partition function becomes a Euclidean path integral.^{4,19,21,24,25,27,28} It initially has the form

$$Z = N_1 \int \mathcal{D}(Re\Psi)\mathcal{D}(Im\Psi)e^{-S} \quad (2.9)$$

but can be rewritten in the equivalent form

$$Z = N_2 \int \mathcal{D}\Psi\mathcal{D}\Psi^\dagger e^{-S} \quad (2.10)$$

where N_1 and N_2 are constants. In (2.10), and in the following, the functions Ψ and Ψ^\dagger are taken to vary independently.²⁹

S can be interpreted as the Euclidean action for interacting Bose and Fermi fields Ψ_b and Ψ_f :

$$S = S_b^{(0)} + S_f^{(0)} + S_{int} \quad (2.11)$$

with

$$S_b^{(0)} = \int d^Dx \left(\frac{1}{2m} \partial^M \Psi_b^\dagger \partial_M \Psi_b - \mu \Psi_b^\dagger \Psi_b \right) \quad (2.12)$$

$$S_f^{(0)} = \int d^Dx \left(\frac{1}{2m} \partial^M \Psi_f^\dagger \partial_M \Psi_f - \mu \Psi_f^\dagger \Psi_f \right) \quad (2.13)$$

$$S_{int} = \int d^Dx \frac{1}{2} b \left(\Psi_b^\dagger \Psi_b + \Psi_f^\dagger \Psi_f \right)^2. \quad (2.14)$$

Notice that S is supersymmetric in an unconventional way: Ψ_b and Ψ_f have the same number of components and the same form. There is no contradiction with the spin-and-statistics theorem²³ because this theorem is based on Lorentz invariance, a symmetry that will emerge only at low energies, and after a Wick rotation to Lorentzian time.

Although Lorentz transformations can be defined only at a later stage, S is already invariant under general coordinate transformations. To make this explicit, we should replace d^Dx by the invariant volume element $d^Dx h$, where

$$h = (\det h_{MN})^{1/2} \quad (2.15)$$

and h_{MN} is the metric tensor for flat Euclidean space, initially given by

$$h_{MN} = \delta_{MN}. \quad (2.16)$$

Ψ_b and Ψ_f are both taken to transform as scalars. This is consistent with the usual convention in general relativity, according to which a spinor transforms as a scalar under general coordinate transformations.¹⁵ After the Lagrangian of (9.4) has been obtained, we will additionally have Lorentz transformations,^{14,15} with the usual behavior for spinors and the usual connection between spin and statistics.

3 The order parameter

S has the same form as the grand-canonical Hamiltonian for a conventional superfluid.⁵ This Ginzburg-Landau form indicates that $\langle \Psi_b \rangle$ will be nonzero at low temperature, so it is natural to write

$$\Psi_b = \Psi_s + \Phi_b \quad (3.1)$$

as in Ref. 5. The classical equations of motion for the order parameter Ψ_s , the bosonic excitations Φ_b , and the fermionic excitations Ψ_f follow from

$$\delta S = 0 \quad (3.2)$$

with Ψ_b , Ψ_b^\dagger , Ψ_f , and Ψ_f^\dagger all varied independently.

We will consider fermionic excitations in the next section and bosonic excitations in Section 8. For the moment, however, let us focus on the order parameter. After integration by parts (with boundary terms assumed to vanish) (2.11) becomes

$$S_0 = \int d^D x h \Psi_s^\dagger \left(T + \frac{1}{2} V - \mu \right) \Psi_s \quad (3.3)$$

in the absence of excitations, where

$$T = - \frac{1}{2m} \partial^M \partial_M \quad (3.4)$$

$$V(x) = b n_s \quad (3.5)$$

$$n_s = \Psi_s^\dagger \Psi_s \quad (3.6)$$

Then (3.2) gives

$$(T + V - \mu) \Psi_s = 0 \quad (3.7)$$

and

$$\Psi_s^\dagger (T + V - \mu) = 0 \quad (3.8)$$

with the operator acting to the left in this last equation.

For an ordinary superfluid like ^4He , one writes $\Psi_s = n_s^{1/2} \exp(i\theta)$. The appropriate generalization is

$$\Psi_s = n_s^{1/2} U \eta \quad (3.9)$$

$$\Psi_s^\dagger = \eta^\dagger U^\dagger n_s^{1/2} \quad (3.10)$$

with

$$U^\dagger U = 1 \quad (3.11)$$

$$\eta^\dagger \eta = 1. \quad (3.12)$$

$U(x)$ and $U^\dagger(x)$ are matrices, and η and η^\dagger are constant vectors. (Recall that Ψ and Ψ^\dagger vary independently, so the quantities in (3.10) are not necessarily the Hermitian conjugates of those in (3.9).) For ${}^4\text{He}$, the superfluid velocity is defined by $m\vec{v} = \vec{\nabla}\theta$. The generalization is

$$mv_M = -iU^{-1}\partial_M U. \quad (3.13)$$

Notice that (3.11) gives $\partial_M U^\dagger U = -U^\dagger \partial_M U$ with $U^\dagger = U^{-1}$, or

$$mv_M = i\partial_M U^\dagger U. \quad (3.14)$$

When (3.9) – (3.14) are used in (3.7), the result is

$$\eta^\dagger n_s^{1/2} \left[\left(\frac{1}{2} m v^M v_M + V - \frac{1}{2m} \partial^M \partial_M - \mu \right) - i \left(\frac{1}{2} \partial^M v_M + v^M \partial_M \right) \right] n_s^{1/2} \eta = 0. \quad (3.15)$$

The Schrödinger-like equations (3.7) and (3.8) also lead to the equation of continuity

$$\partial^M j_M = 0 \quad (3.16)$$

with

$$j_M = \frac{1}{2im} \left[\Psi_s^\dagger (\partial_M \Psi_s) - (\partial_M \Psi_s^\dagger) \Psi_s \right] \quad (3.17)$$

$$= \eta^\dagger n_s v_M \eta. \quad (3.18)$$

Substitution of (3.18) into (3.16) gives

$$\eta^\dagger (\partial^M v_M + v^M \partial_M) n_s \eta = 0 \quad (3.19)$$

reducing (3.15) to

$$\frac{1}{2} m \bar{v}^2 + V + P = \mu \quad (3.20)$$

where

$$\bar{v}^2 = \eta^\dagger v^M v_M \eta \quad (3.21)$$

$$P = -\frac{1}{2m} n_s^{-1/2} \partial^M \partial_M n_s^{1/2}. \quad (3.22)$$

Eq. (3.20) is a quantum version of Bernoulli's equation, with part of the kinetic energy playing the role of pressure.

In the next section we will consider an order parameter with the symmetry group $U(1) \times SU(2)$. For this case, the “superfluid velocity” v_M can be written in terms of the identity matrix σ_0 and the Pauli matrices σ_a :

$$v_M = v_M^\alpha \sigma_\alpha, \quad \alpha = 0, 1, 2, 3. \quad (3.23)$$

It is reasonable to assume that the order parameter has the symmetry

$$\eta^\dagger \sigma_a \eta = 0 \quad (3.24)$$

which means that the system is not spin-polarized when $v_M = 0$. Then cross terms involving $\sigma_a \sigma_0$ vanish, and the relation

$$\sigma_a \sigma_b + \sigma_b \sigma_a = 2\delta_{ab} \quad (3.25)$$

further reduces (3.20) to

$$\frac{1}{2} m v_\alpha^M v_M^\alpha + V + P = \mu. \quad (3.26)$$

4 Fermionic excitations, U(1) x SU(2) order parameter

When fermionic excitations are included, (2.11) becomes $S = S_0 + S_f$, with

$$S_f = \int d^D x h \Psi_f^\dagger (T + V - \mu) \Psi_f. \quad (4.1)$$

The term involving $(\Psi_f^\dagger \Psi_f)^2$ is neglected in comparison to the one containing V because fermions cannot form a condensate.

According to (3.2), Ψ_f obeys the same equation of motion as Ψ_s , and will share its rapid oscillations in regions where $\mu - V$ is large. In order to eliminate these oscillations, it is convenient to write

$$\Psi_f = U \psi_f = n_s^{1/2} U \tilde{\psi}_f. \quad (4.2)$$

For simplicity, n_s will initially be regarded as slowly varying. Then we will find that low-energy, long-wavelength excitations ψ_f also correspond to low values of the action (4.1), and that it is consistent to identify ψ_f with the fermionic fields observed in nature.

First consider the case $N=2$, $D=4$, with symmetry group U(1) x SU(2) for the order parameter. The coordinates are

$$x^\mu, \quad \mu = 0, 1, 2, 3. \quad (4.3)$$

Substitution of (4.2) into (4.1) gives

$$S_f = \int d^4 x h \psi_f^\dagger \left[\left(\frac{1}{2} m v^\mu v_\mu + V - \frac{1}{2m} \partial^\mu \partial_\mu - \mu \right) - i \left(\frac{1}{2} \partial^\mu v_\mu + v^\mu \partial_\mu \right) \right] \psi_f. \quad (4.4)$$

For low-energy (long-wavelength) excitations, $\partial^\mu \partial_\mu \psi_f$ can be neglected in comparison with $m v^\mu \partial_\mu \psi_f$. If n_s is slowly varying, P can also be neglected. Then the Bernoulli equation (3.26), together with (3.23) and (3.25), implies that

$$\frac{1}{2} m v^\mu v_\mu + V - \mu = m v_0^\mu v_\mu^a \sigma_a. \quad (4.5)$$

In Section 7, a cosmological picture will be presented in which v_μ^a is real but v_μ^0 is imaginary,

$$\frac{1}{2} m v_0^\mu v_\mu^0 < 0, \quad (4.6)$$

and the basic texture is given by

$$v_k^0 = v_0^a = 0 \text{ for } k, a = 1, 2, 3. \quad (4.7)$$

Then (4.4) becomes

$$S_f = \int d^4 x h \psi_f^\dagger \left(-\frac{1}{2} i \partial^\mu v_\mu - i v^\mu \partial_\mu \right) \psi_f \quad (4.8)$$

or

$$S_f = \int d^4x h \frac{1}{2} [\psi_f^\dagger v^\mu (-i\partial_\mu \psi_f) + (-i\partial_\mu \psi_f)^\dagger v^\mu \psi_f] \quad (4.9)$$

after integration by parts. There is no reason why the texture of (4.7) must be perfectly rigid, however, so we should permit small deformations v_k^0 and v_0^a . When second-order terms are neglected, (4.9) is changed to

$$S_f = \int d^4x \bar{\mathcal{L}}_f \quad (4.10)$$

$$\bar{\mathcal{L}}_f = -\frac{1}{2} i h \psi_f^\dagger v_\alpha^\mu \sigma^\alpha \tilde{\nabla}_\mu \psi_f + \text{conj.} \quad (4.11)$$

where

$$\tilde{\nabla}_\mu = \partial_\mu + \Gamma_\mu + i a_\mu + i b_\mu \quad (4.12)$$

with

$$a_0 = 0, \quad a_k = m v_k^0 \quad (k = 1, 2, 3) \quad (4.13)$$

$$b_k = 0, \quad b_0 = m v_0^a \sigma_a \quad (a = 1, 2, 3). \quad (4.14)$$

Here “conj.” represents a second term like that in (4.9). After the transformation to Lorentzian time in Section 9, it can be regarded as the true Hermitian conjugate of the first term, represented by “h.c.” The spin connection Γ_μ is initially zero, but must be added to (4.11) to compensate for local transformations of ψ_f when frame rotations are permitted.¹⁵

Suppose that we now define an effective vierbein e_μ^α and an effective metric tensor $g_{\mu\nu}$ by

$$e_\alpha^\mu = v_\alpha^\mu, \quad \mu, \alpha = 0, 1, 2, 3 \quad (4.15)$$

$$e_\mu^\alpha e_\beta^\mu = \delta_\beta^\alpha \quad (4.16)$$

$$g_{\mu\nu} = \eta_{\alpha\beta} e_\mu^\alpha e_\nu^\beta. \quad (4.17)$$

The Minkowski metric tensor $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is needed because of (4.6) and the requirement that a Euclidean metric tensor $g_{\mu\nu}$ should have signature (++++). $\bar{\mathcal{L}}_f$ then has nearly the same form as the standard Euclidean Lagrangian for massless spin 1/2 fermions in the Weyl representation.^{13,19,25,30–32} There are two differences: the extra couplings a_μ and b_μ , and the factor of h rather than

$$g = (\det g_{\mu\nu})^{1/2}. \quad (4.18)$$

These features will be discussed near the end of the paper, but suppose that we momentarily disregard them. The behavior of massless fermions will then be the same as in a curved spacetime with metric tensor $g_{\mu\nu}$. In the present theory, the geometry of spacetime is defined by the texture of the order parameter, with the “superfluid velocity” v_α^μ becoming the vierbein e_α^μ . The origin of spacetime curvature will be discussed in Section 7, and the transformation to Lorentzian time in Section 9.

5 Fermionic excitations, U(1) x SU(2) x G order parameter

The treatment of the preceding section contains only one fermion species and no forces other than gravity. Let us now move to a more realistic description, with $N > 2$ and $D > 4$, which is similar to standard higher-dimensional theories.^{3,33-35} The ordered phase described by Ψ_s is assumed to locally have a “filamentary” geometry, with 4 extended dimensions and d that are compact. To be more precise, it occupies only a very small volume

$$V_B = \int d^d x \quad (5.1)$$

in an internal space x_B with coordinates

$$x^m, \quad m = 4, 5, \dots, 3 + d \quad (5.2)$$

but a very large volume in the 4-dimensional external spacetime x_A with coordinates x^μ . In the simplest picture, the ordered phase is a d -dimensional ball of condensate in internal space, which can be described by one radial coordinate and $(d-1)$ angular coordinates. To avoid confusion, however, we will retain the original rectangular coordinates x^m in the discussion below, so that $\det h_{mn} = 1$ and $h = (\det h_{MN})^{1/2} = (\det h_{\mu\nu})^{1/2}$.

It is also assumed that the order parameter locally has the form of a product:

$$\Psi_s = \Psi_A \Psi_B \quad (5.3)$$

where Ψ_A has the symmetry group U(1) x SU(2) and Ψ_B has an unspecified symmetry group G with generators σ_c . Then (3.23) must be generalized to

$$v_M = v_M^A \sigma_A = v_M^\alpha \sigma_\alpha + v_M^c \sigma_c, \quad c \geq 4. \quad (5.4)$$

Both the filamentary geometry and the form of the order parameter originate from two topological defects discussed in Section 7, associated respectively with the symmetry groups G and U(1) x SU(2).

In generalizing the definition of the effective vierbein e_μ^α , it will be convenient to choose

$$e_M^c = v_M^c \quad (5.5)$$

while retaining (4.15) and (4.16):

$$e_\alpha^M = v_\alpha^M \quad (5.6)$$

$$e_M^\alpha e_\beta^M = \delta_\beta^\alpha. \quad (5.7)$$

The effective metric tensor is then

$$g_{MN} = \eta_{AB} e_M^A e_N^B = \eta_{\alpha\beta} e_M^\alpha e_N^\beta + e_M^c e_N^c. \quad (5.8)$$

Standard Kaluza-Klein theory^{33,34} begins with an unperturbed metric tensor having the form $g_{\mu\nu} = g_{\mu\nu}(x_A)$, $g_{mn} = g_{mn}(x_B)$, $g_{\mu m} = g_{m\mu} = 0$. The present theory similarly begins with an unperturbed order parameter having the form

$$\Psi_s = \Psi_A(x_A) \Psi_B(x_B) \quad (5.9)$$

which implies the texture

$$v_\alpha^\mu = v_\alpha^\mu(x_A) \quad (5.10)$$

$$v_\mu^c = 0 \quad (5.11)$$

$$v_m^c = v_m^c(x_B) \quad (5.12)$$

$$v_\alpha^m = 0 \quad (5.13)$$

and the effective geometry $g_{\mu\nu} = g_{\mu\nu}(x_A)$, $g_{mn} = g_{mn}(x_B)$, $g_{\mu m} = g_{m\mu} = 0$.

The form (5.9) requires that $n_A = \Psi_A^\dagger \Psi_A$ and $\mu_A = \frac{1}{2} m v_\alpha^\mu v_\mu^\alpha$ be regarded as constant in treating the rapid variations of the internal order parameter Ψ_B . Then (3.7) gives

$$\left(-\frac{1}{2m} \partial^m \partial_m + V \right) \Psi_B = \mu_B \Psi_B \quad (5.14)$$

where $V(x_B) = b n_A n_B(x_B)$, $n_B = \Psi_B^\dagger \Psi_B$, and $\mu_A = \mu - \mu_B$. The internal versions of (3.9) and (3.13) are

$$\Psi_B = n_B^{1/2} U_B \eta_B \quad (5.15)$$

and

$$m v_m = -i U_B^{-1} \partial_m U_B \quad (5.16)$$

with

$$v_m = v_m^c \sigma_c. \quad (5.17)$$

Now let us turn to the fermion field ψ_f of (4.2). It can be expanded in a complete set of states $\psi_r^B(x_B)$ with coefficients $\psi_r(x_A)$:^{3,33-35}

$$\psi_f(x_A, x_B) = \sum_r \psi_r(x_A) \psi_r^B(x_B). \quad (5.18)$$

The boson-fermion symmetry suggests that we should choose each term in (5.18) to have the same form as (5.3). We can also write

$$\Psi_r^B = U_B \psi_r^B = n_B^{1/2} U_B \tilde{\psi}_r^B \quad (5.19)$$

as in (5.15), and choose the Ψ_r^B to be eigenfunctions of the operator in (5.14):

$$\left(-\frac{1}{2m}\partial^m\partial_m + V - \mu_B\right)\Psi_r^B = \varepsilon_r\Psi_r^B. \quad (5.20)$$

We will find, as usual, that only the solutions with $\varepsilon_r = 0$ can be retained in the low-energy limit.^{3,33–35} The above choices and ideas are similar to those of other higher-dimensional theories, and it will be seen that they lead to consistent results.

The internal space x_B has an effective geometry determined by the effective metric tensor g_{mn} . One can then define Killing vectors K_i^n , or

$$K_i = K_i^n \partial_n. \quad (5.21)$$

They have an algebra^{14,36}

$$K_i K_j - K_j K_i = -c_{ij}^k K_k \quad (5.22)$$

and satisfy Killing's equation^{15,37}

$$K_i^p \partial_p g_{mn} + g_{pn} \partial_m K_i^p + g_{mp} \partial_n K_i^p = 0. \quad (5.23)$$

For a scalar function F which is invariant under the symmetry operation specified by K_i , the corresponding equation is

$$K_i F = 0. \quad (5.24)$$

We now need two assumptions: First, the condensate density n_B is assumed to have the same symmetry as the geometry defined by g_{mn} :

$$K_i n_B(x_B) = 0. \quad (5.25)$$

This will be the case if the velocity v_m results from an instanton with spherical symmetry which is “frozen into” internal space, as in the examples of Section 7. Second, the physically significant zero modes of (5.20) are assumed to share this symmetry:

$$K_i \tilde{\psi}_r^B = 0, \quad \varepsilon_r = 0. \quad (5.26)$$

This assumption is reasonable because $\tilde{\psi}_r^B$, defined in (5.19), satisfies the same equation as the constant vector η_B of (5.15). It is also plausible that zero modes should reflect the symmetry of the space in which they are defined. A more detailed discussion of these modes is given in Section 8.

In the simple picture mentioned below (5.2) and (5.25), the K_i are associated with rotations in d dimensions, and thus with the symmetry group $\text{SO}(d)$.

Since the Ψ_r^B serve as basis functions, let

$$\langle r|Q|s \rangle = \int d^d x \Psi_r^{B\dagger} Q \Psi_s^B \quad (5.27)$$

where Q is any operator. In the next section we will need these functions to be orthogonal,

$$\langle r|s \rangle = \delta_{rs} \quad (5.28)$$

and we will also need the result

$$K_i \Psi_s^B = U_B (imK_i^n v_n^c \sigma_c) \psi_s^B \quad (5.29)$$

which follows from (5.16), (5.25), and (5.26). This implies the relation

$$\int d^d x \psi_r^{B\dagger} \sigma_i \psi_s^B = \langle r | (-iK_i) | s \rangle \quad (5.30)$$

where

$$\sigma_i = mK_i^n v_n^c \sigma_c \quad (5.31)$$

is a matrix associated with the i^{th} internal symmetry direction.

6 Gauge fields

In conventional Kaluza-Klein theories,^{33,34} the metric tensor is perturbed by letting

$$g_{\mu m}(x_A, x_B) = A_\mu^i(x_A) K_i^n(x_B) g_{nm}. \quad (6.1)$$

In the present theory, this corresponds to letting

$$v_\mu^c = A_\mu^i K_i^n v_n^c \quad (6.2)$$

since $g_{\mu m} = v_\mu^c v_m^c$ (with v_m^α still zero) and $g_{nm} = v_n^c v_m^c$. It is also equivalent to writing

$$m v_\mu^c \sigma_c = A_\mu^i \sigma_i. \quad (6.3)$$

We now need to determine how the gauge fields A_μ^i are coupled to the fermion fields ψ_r . When (4.2) and (5.18) are substituted into (4.1), and (5.20) is used (with $\varepsilon_r = 0$), the result is

$$S_f = \sum_{rs} \int d^D x \, h \psi_r^{B\dagger} \psi_r^\dagger U^\dagger \left(-\frac{1}{2m} \partial^\mu \partial_\mu - \frac{1}{2} m v_\alpha^\mu v_\mu^\alpha \right) U \psi_s \psi_s^B. \quad (6.4)$$

Let us focus on the term involving $\partial^\mu \partial_\mu$, and a particular r and s , which becomes

$$S_{rs} = \frac{1}{2m} \int d^D x \, h \psi_r^{B\dagger} [\partial^\mu (U \psi_r)]^\dagger \partial_\mu (U \psi_s) \psi_s^B \quad (6.5)$$

after integration by parts. Since (3.13) and (5.4) give

$$\partial_\mu U = imU (v_\mu^\alpha \sigma_\alpha + v_\mu^c \sigma_c) \quad (6.6)$$

we need to consider

$$\partial_\mu (U \psi_s) = U (imv_\mu^\alpha \sigma_\alpha + \partial_\mu) \psi_s + U imv_\mu^c \sigma_c \psi_s. \quad (6.7)$$

This expression is multiplied by its conjugate (with $s \rightarrow r$). The product of the first term with its conjugate was already treated in Section 4. The product of the second term with its conjugate is second order in v_μ^c , and can consequently be neglected. The extra contribution to (6.5) thus involves the cross terms $(im\psi_r)^\dagger v_c^\mu \sigma^c (imv_\mu^\alpha \sigma_\alpha + \partial_\mu) \psi_s + (\text{conj. with } r \leftrightarrow s)$. For low-energy excitations, however, $\partial_\mu \psi_s$ can be neglected in comparison with $mv_\mu \psi_s$. We are left with

$$m^2 \psi_r^\dagger v_\mu^c \sigma_c v_\mu^\alpha \sigma^\alpha \psi_s \quad (6.8)$$

plus its conjugate. The additional term in (6.5) is then

$$\Delta S_{rs} = \frac{1}{2} \int d^4 x \, h \psi_r^\dagger v_\alpha^\mu \sigma^\alpha \left[\int d^d x \, \psi_r^{B\dagger} m v_\mu^c \sigma_c \psi_s^B \right] \psi_s + \text{conj.} \quad (6.9)$$

When (6.3) and (5.30) are employed, the factor in square brackets reduces to a remarkably nice form:

$$\Delta S_{rs}^\mu = \int d^d x \psi_r^{B\dagger} m v_\mu^c \sigma_c \psi_s^B \quad (6.10)$$

$$= A_\mu^i \int d^d x \psi_r^{B\dagger} \sigma_i \psi_s^B \quad (6.11)$$

$$= A_\mu^i \langle r | (-iK_i) | s \rangle. \quad (6.12)$$

Then (6.9) can be rewritten as

$$\Delta S_{rs} = \frac{1}{2} \int d^4 x h \psi_r^\dagger v_\alpha^\mu \sigma^\alpha A_\mu^i t_i^{rs} \psi_s + conj. \quad (6.13)$$

where

$$t_i^{rs} = \langle r | (-iK_i) | s \rangle. \quad (6.14)$$

Let t_i be the matrix with elements t_i^{rs} . Since it corresponds to the operator $-iK_i$, it has the same algebra:

$$t_i t_j - t_j t_i = i c_{ij}^k t_k. \quad (6.15)$$

This is exactly what is needed for (6.13) to represent a proper gauge interaction.

To simplify notation, let ψ be the vector with components ψ_r . Then the extra contribution to (6.4) is

$$\Delta S = \frac{1}{2} \int d^4 x h \psi^\dagger e_\alpha^\mu \sigma^\alpha A_\mu^i t_i \psi + conj. \quad (6.16)$$

After (5.28) is used, the other terms in (6.4) can be treated just as in Section 4. The Lagrangian density corresponding to (6.16) can then be added to (4.11), giving

$$\bar{\mathcal{L}}_f = -\frac{1}{2} i h \psi^\dagger e_\alpha^\mu \sigma^\alpha \tilde{D}_\mu \psi + conj. \quad (6.17)$$

where

$$\tilde{D}_\mu = \tilde{\nabla}_\mu + i A_\mu^i t_i. \quad (6.18)$$

The present theory thus yields the correct form for initially massless fermions coupled to both gravity and gauge fields.

7 Instantons

In an ordinary superfluid, the definition $m\vec{v} = \vec{\nabla}\theta$ implies that

$$\vec{\nabla} \times \vec{v} = 0. \quad (7.1)$$

For the condensate of the present theory, the definition $mv_M = -iU^{-1}\partial_M U$, together with the condition $\partial_M\partial_N U - \partial_N\partial_M U = 0$, immediately gives the generalization

$$G_{MN} = 0 \quad (7.2)$$

where

$$G_{MN} = \partial_M v_N - \partial_N v_M + im[v_M, v_N]. \quad (7.3)$$

If (7.2) were to hold everywhere, the present theory would be untenable, since there is no such constraint on the vielbein and metric tensor in standard physics. It seems to be a general principle, however, that constraints like (7.2) can be relieved by topological defects, with important physical consequences. Let us consider a few examples.

(a) U(1) vortices. Since (7.1) states that \vec{v} is irrotational, it was originally a mystery how superfluid ^4He could exhibit its observed rotation. Feynman provided an answer by postulating the existence of vortices,⁵ which were later seen experimentally. Integration over an area A containing a vortex gives

$$\begin{aligned} \int_A \vec{\nabla} \times \vec{v} \cdot d\vec{S} &= \int_C \vec{v} \cdot d\vec{\ell} \\ &= 2\pi n/m \end{aligned} \quad (7.4)$$

where n is an integer. The singularity at the center of a vortex thus relieves the constraint (7.1), in the sense that the integrated value of $\vec{\nabla} \times \vec{v}$ in (7.4) is nonzero. This has the important effect of allowing the superfluid to rotate.

(b) SU(2) instantons in four dimensions. The velocity field around an $n=1$ vortex is given by⁵

$$m\vec{v}(\vec{r}) = r^{-1}\hat{\phi} \quad (7.5)$$

where $\vec{r} = (r, \phi)$ in the xy plane. For an $n=1$ BPST instanton the corresponding result is^{12,32}

$$m\vec{v}(x) = \frac{\vec{\sigma}x_0 + \vec{\sigma} \times \vec{x}}{\rho^2}, \quad mv_0(x) = \frac{\vec{\sigma} \cdot \vec{x}}{\rho^2} \quad (7.6)$$

where

$$\rho^2 = x_0^2 + \vec{x}^2. \quad (7.7)$$

(The replacement $A_\mu/i \rightarrow mv_\mu$ has been made in the usual expressions, and the instanton size λ has been set equal to zero.) Even though (7.2) is satisfied at all points except $\rho=0$, the integrated value of $G^{\mu\nu}G_{\mu\nu}$ is nonzero:

$$\int d^4x \operatorname{tr}(G^{\mu\nu}G_{\mu\nu}) = 16\pi^2/m^2. \quad (7.8)$$

(c) Gravitational instantons in four dimensions. The Eguchi-Hanson instanton has a metric which can be written in the form²⁸

$$ds^2 = (1 - a^4r^{-4})^{-1} dr^2 + (1 - a^4r^{-4})(r^2/4)(d\psi + \cos\theta d\phi)^2 + (r^2/4)(d\theta^2 + \sin^2\theta d\phi^2). \quad (7.9)$$

As $r/a \rightarrow \infty$ this becomes the metric of flat Euclidean space, but the Euler number

$$\chi = (128\pi^2)^{-1} \int d^4x g\varepsilon^{\tau\omega}R_{\tau\omega\alpha\beta}\varepsilon^{\alpha\beta}R^{\mu\nu\rho\sigma} + \text{boundary terms} \quad (7.10)$$

and the signature

$$\tau = (96\pi^2)^{-1} \int d^4x gR_{\mu\nu\rho\sigma}\varepsilon^{\rho\sigma\tau\omega}R^{\mu\nu}{}_{\tau\omega} + \text{boundary terms} \quad (7.11)$$

are nonzero: $\chi=2$ and $\tau=1$. Gibbons and Hawking obtained a generalization with the form²⁸

$$ds^2 = u^{-1}(d\tau + \vec{\omega} \cdot d\vec{x})^2 + u d\vec{x} \cdot d\vec{x} \quad (7.12)$$

where

$$u = u_0 + \sum_{i=1}^s q_i |\vec{x} - \vec{x}_i|^{-1} \quad (7.13)$$

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} u. \quad (7.14)$$

When $u_0 = 0$ and all the q_i are equal, these solutions are also asymptotically locally Euclidean, but with $\chi=s$ and $\tau=s-1$.

(d) Multidimensional instantons. The above ideas are known to generalize to larger symmetry groups and higher dimensions.³⁸⁻⁴⁰ For example, the Kaluza-Klein monopole in 5 dimensions is given by⁴¹

$$ds^2 = dt^2 + u^{-1}(d\tau + \vec{\omega} \cdot d\vec{x})^2 + u d\vec{x} \cdot d\vec{x} \quad (7.15)$$

where the fifth coordinate τ is periodic and $u_0 = 1$, $s = 1$ in (7.13).

(e) Other topological defects in field theory^{12,28,42-48} which play an important role in grand-unified theories, higher-dimensional theories, and cosmological models.

(f) Other topological defects in condensed-matter physics,⁶⁻¹¹ which have a pervasive influence on the properties of superfluid ³He and ⁴He, type I and type II superconductors, liquid crystals, crystalline solids, magnetic materials, one-dimensional organic systems, and two-dimensional phase transitions.

Given the ubiquity of topological defects, it is not unreasonable to assume that they exist in an ordered phase of the kind proposed here. The vortices postulated by Feynman

relaxed the constraint (7.1), permitting the integrated vorticity to be nonzero. The topological defects postulated here will similarly relax the constraint (7.2), permitting integrated curvature scalars like (7.8) to be nonzero.

Three distinct kinds of defects are needed:

(1) An internal instanton, associated with the symmetry group G , which accounts for the internal velocity field v_m^c . This topological defect is analogous to the monopoles, instantons, etc. which are postulated in other higher-dimensional theories. Since G is left unspecified in the present paper, so is the detailed nature of this instanton. The simplest toy models are the following:

(a) $d \rightarrow 2$ and $G \rightarrow U(1)$. Then the condensate is bounded by a circle of radius r_B in internal space, and $V_B = \pi r_B^2$. The internal instanton is a vortex, with $v_\phi = (mr)^{-1}$. There is only one Killing vector

$$K = \partial_\phi \tag{7.16}$$

and the gauge group is $U(1)$.

(b) $d \rightarrow 4$ and $G \rightarrow SU(2)$. In this case the condensate is enclosed by a 3-sphere of radius r_B . The internal instanton has the form (7.6). There are now 6 Killing vectors K_i , associated with the 6 rotational degrees of freedom, and the gauge group is $SO(4)$.

A more realistic model is provided by $d \rightarrow 10$. Then the condensate lies within a 9-sphere of radius r_B , and the internal instanton is a hypothetical extension of (7.6) to a larger symmetry group which is contained in G . There are 45 Killing vectors and the gauge group is $SO(10)$, perhaps the most appealing possibility for grand unification.⁴⁹

A finite internal volume V_B is required for this instanton to have finite action: If $mv_m^c \propto r^{-1}$, as in (7.5) and (7.6), then the kinetic energy contribution

$$\int d^d x n_B \cdot \frac{1}{2} m v_c^m v_m^c \tag{7.17}$$

will diverge unless $n_B \rightarrow 0$ for $r > r_B$. Since the natural length scale in (5.14) is the correlation length⁵

$$\xi = (2m\mu)^{-1/2} \tag{7.18}$$

it is plausible that

$$r_B \sim \xi, \quad V_B \sim \xi^d. \tag{7.19}$$

Notice that the internal velocity v_m has no radial component. The metric tensor g_{mn} is then defined only along the tangential directions, with $v_m^c \propto (mr)^{-1}$ and $g_{mn} \propto (mr)^{-2}$ within a $(d-1)$ -sphere of radius r . Let V'_B be the effective volume of this sphere:

$$V'_B = \int d^{d'} x g_{d'} \quad , \quad d' = d - 1 \tag{7.20}$$

where

$$g_{d'} = (\det g_{mn})^{1/2} \tag{7.21}$$

and

$$m, n = 4, 5, \dots, 3 + d'.$$

We can similarly define

$$g_{D'} = (\det g_{MN})^{1/2}, \quad (7.22)$$

where $D' = D-1$ and the coordinates are restricted to those describing the manifold $R^4 \times S^{d'}$:

$$M, N = 0, 1, \dots, D' - 1.$$

Since $g_{d'} \propto (mr)^{-d'}$ and $d^{d'}x = r^{d'} d\Omega$, where $d\Omega$ is a solid angle, (7.20) implies that V'_B is independent of r and

$$V'_B \sim m^{-d'}. \quad (7.23)$$

(2) A cosmological instanton, with an $SU(2)$ velocity field like that of (7.6). If we choose $\vec{x} = 0$ at our position in the universe, then the 3-vector $v_k^a \sigma_a$ has the form

$$\vec{v} \propto \vec{\sigma}/mx^0 \quad (7.24)$$

and $v_0^a = 0$. The singularity at $x^0 = 0$ is interpreted as the big bang. Recall that there is also a $U(1)$ velocity field v_μ^0 , which need not be real, and that Ψ and Ψ^\dagger vary independently. These features can be exploited in minimizing the action (3.3), by requiring the $U(1)$ kinetic energy $\frac{1}{2}mv_0^\mu v_\mu^0$ to be negative. Symmetry indicates that v_μ^0 is radial, or along the x^0 direction at our position in the universe, giving the texture (4.7). Ψ_s then varies as $\exp(-\omega x^0)$ within the present Euclidean picture, with $\Psi_s^\dagger \propto \exp(+\omega x^0)$ to keep n_s constant. In Section 9 we will transform to a Lorentzian picture by performing a Wick rotation $x^0 \rightarrow ix^0$. The above dependences are then changed to $\exp(-i\omega x^0)$ and $\exp(+i\omega x^0)$, with the condensate density still constant.

The continuity equation (3.16) appears to impose a constraint on the velocity field v_μ^0 , but this constraint may also be relieved by topological defects: There can be monopole-like defects which act as sources or sinks for the current $j_\mu^0 = \eta^\dagger n_s v_\mu^0 \eta$, with $\partial^\mu j_\mu^0 = 0$ everywhere except at the singularities themselves (where $n_s \rightarrow 0$). These defects are physically allowed because v_μ^0 is not a true superfluid velocity; it instead specifies a field configuration, analogous to the configuration of spins on a lattice.

(3) Planck-scale instantons which are dilutely distributed throughout external spacetime, and which give rise to a twisting of the field e_μ^A . Just as an ordinary gravitational instanton is embedded in a surrounding metric $g_{\mu\nu}$, or vierbein e_μ^α , the instantons postulated here are embedded in a more general field e_μ^A which includes both the gravitational field (for $A = \alpha \leq 3$) and the gauge fields (for $A = c \geq 4$).

The effective vielbein e_M^A and metric tensor g_{MN} of (5.8) are defined on a manifold \mathcal{M} of dimension D' . (Recall that $D' = D - 1$ and $\mathcal{M} = R^4 \times S^{D'-4}$ in the models above (7.23).) We can then define a Riemannian curvature scalar ${}^{(D')}R$ and a scalar density $g_{D'} = \det g_{MN}$, with the coordinates M and N restricted to this manifold.

Let S_{in} be the action of one instanton, and R_{in} be its contribution to the quantity

$$- \int d^{D'}x g_{D'} {}^{(D')}R. \quad (7.25)$$

(It is assumed that each instanton has a core singularity which makes R_{in} nonzero. It is also assumed that instantons of the same kind have the same values of S_{in} and R_{in} .) A comparison of (7.29) with (7.30) shows that R_{in} must be positive.

Although it costs an action S_{in} to form an instanton, the action of the matter fields can be lowered by the resulting change in curvature. We will find below that minimization of the total action with respect to $g^{\mu\nu}$ and A_μ^i leads to the Einstein and Maxwell field equations.

Since S_{in} is dimensionless and $(D')R$ has dimension length⁻², we can write

$$S_{in} = \ell_0^{-D'+2} R_{in}. \quad (7.26)$$

To obtain precise values of S_{in} , R_{in} , and ℓ_0 would require detailed calculations for a specific model. We can, however, obtain estimates if the instantons are assumed to have the following general properties: First, the presence of m in (3.13) suggests a ‘‘velocity core’’ of size $r_v \sim m^{-1}$, within which $v_\mu^A \sim 1$ and $\partial_\mu v_\nu^A \sim m$. (This behavior can be seen explicitly in (7.5) and (7.6), which become dimensionless if distances are scaled by m^{-1} .) Second, the presence of a singularity suggests a ‘‘density core’’ of size $r_n \sim \xi$. One then expects $R_{in} \sim m^{-4} V_B' m^2$ and $S_{in} \sim \xi^4 V_B \mu^2 / b$. (The action (3.3) becomes $\int d^D x \left(-\frac{1}{2} b n_s^2\right)$ after (3.7) is used. If Ψ_s is constant, (3.7) also implies that the density is $\bar{n}_s = \mu/b$. Then in a core region of radius r_n , whose density is depleted by a singularity at the center, the change in the action is $\Delta S = \int d^D x \left(-\frac{1}{2} b n_s^2 + \frac{1}{2} b \bar{n}_s^2\right) \sim \xi^4 V_B \mu^2 / b$.) It follows that

$$\ell_0^{D'-2} \sim \xi^{-d} V_B' b \sim (\mu/m)^{d/2} m b \quad (7.27)$$

where (7.18), (7.19), and (7.23) have been used. Finally, (7.31) relates the Planck length ℓ_P to the parameters m , μ , and b of (2.7):

$$\ell_P^2 \sim (m\mu)^{d/2} b, \quad d = D - 4. \quad (7.28)$$

Since the contributions are additive for dilutely distributed instantons, (7.26) implies that they have a net action

$$S_{D'} = -\ell_0^{-D'+2} \int d^{D'} x g_{D'} (D') R_{in} \quad (7.29)$$

where $(D')R_{in}$ represents their total contribution to the scalar curvature. This is the Euclidean Einstein-Hilbert action in D' dimensions, and the usual Kaluza-Klein reduction gives^{33,34}

$$S_{D'} = -\ell_P^{-2} \int d^4 x g^{(4)} R + \frac{1}{4} g_0^{-2} \int d^4 x g F_{\mu\nu}^i F_{\rho\sigma}^i g^{\mu\rho} g^{\nu\sigma} \quad (7.30)$$

where

$$\ell_P^{-2} = \ell_0^{-D'+2} V_B' \quad (7.31)$$

$$\ell_P^{-2} \langle g_{mn} K_i^m K_j^n \rangle = g_0^{-2} \delta_{ij} \quad (7.32)$$

and

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + c_{jk}^i A_\mu^j A_\nu^k. \quad (7.33)$$

V'_B is the internal volume of (7.20) and $\langle - \rangle$ represents an average over this volume. (If σ_i is constant in (5.31), however, then so is $g_{mn}K_i^m K_j^n$, eliminating the need for an average.) As in conventional Kaluza-Klein theories, ${}^{(4)}R$ is the curvature scalar associated with the vierbein e_μ^α , and $g = (\det e_\mu^\alpha e_\nu^\alpha)^{1/2}$.

Suppose that $v_m^c v_n^c K_i^m K_j^n$ is $\sim m^{-2} \delta_{ij}$, as in the models above (7.23). Since g_0 is ~ 1 ,^{12,13} (7.32) then implies the relationship

$$m \sim m_P. \quad (7.34)$$

The Lagrangian densities corresponding to (7.30) are

$$\bar{\mathcal{L}}_G = -\ell_P^{-2} g {}^{(4)}R \quad (7.35)$$

$$\bar{\mathcal{L}}_g = \frac{1}{4} g_0^{-2} g F_{\mu\nu}^i F_{\rho\sigma}^i g^{\mu\rho} g^{\nu\sigma}. \quad (7.36)$$

8 Bosonic excitations

When bosonic excitations Φ_b are included, (2.11) becomes

$$S = S_b + S_f \quad (8.1)$$

$$S_b = \int d^D x h \Psi_b^\dagger \left(T + \frac{1}{2} \tilde{V} - \mu \right) \Psi_b \quad (8.2)$$

$$S_f = \int d^D x h \Psi_f^\dagger \left(T + \tilde{V} - \mu \right) \Psi_f \quad (8.3)$$

where

$$\tilde{V} = b \Psi_b^\dagger \Psi_b \quad (8.4)$$

and $\Psi_b = \Psi_s + \Phi_b$. If we now require that

$$\left(T + \tilde{V} - \mu \right) \Psi_s = 0 \quad (8.5)$$

the treatment of the order parameter and fermionic excitations in the preceding sections is unchanged, except that $V \rightarrow \tilde{V}$. The bosonic action (8.2) can be written

$$S_b = S_0 + \Delta S_b + \Delta S'_b \quad (8.6)$$

with

$$\Delta S_b = \int d^D x h \Phi_b^\dagger \left(T + V - \mu + \frac{1}{2} b \Phi_b^\dagger \Phi_b \right) \Phi_b \quad (8.7)$$

$$\Delta S'_b = \int d^D x h \Phi_b^\dagger \left(T + \frac{1}{2} \tilde{V} + \frac{1}{2} V + \frac{1}{2} b \Phi_b^\dagger \Phi_b - \mu \right) \Psi_s + conj. \quad (8.8)$$

For the excitations considered below, we will find that $\Phi_b^\dagger \Psi_s = 0$, so that (8.8) is unchanged if the interaction terms in parentheses are replaced by \tilde{V} . But the equation of motion (8.5) then gives

$$\Delta S'_b = 0. \quad (8.9)$$

Let us expand the boson field Φ_b in the complete set of internal states Ψ_r^B , with coefficients Φ_r :

$$\Phi_b = \sum_r \Phi_r \Psi_r^B \quad (8.10)$$

where

$$\Psi_0^B = N_B^{-1/2} \Psi_B, \quad (8.11)$$

$N_B = \int d^d x n_B$, and the prime means $r \neq 0$. Recall that these basis functions are the solutions to (5.20), and are written in the form (5.19). In treating low-energy bosonic excitations, it is necessary to assume the orthogonality condition

$$\tilde{\psi}_r^{B\dagger} \tilde{\psi}_s^B = N_B^{-1} \delta_{rs} \quad , \varepsilon_r = \varepsilon_s = 0 \quad (8.12)$$

or equivalently

$$\Psi_r^{B\dagger} \Psi_s^B = N_B^{-1} n_B \delta_{rs} \quad , \varepsilon_r = \varepsilon_s = 0. \quad (8.13)$$

Only those functions satisfying this condition and (5.26) are considered to be physically significant in the present context. There is another set of solutions to (5.20) with $\Psi_r^B \rightarrow (\Psi_r^B)^*$; since these involve motion counter to that of the condensate, however, it is assumed that radiative corrections will break the degeneracy between these states and those of (8.13), so they are omitted from the sums of (5.18) and (8.10) at low energy. The state with $r=0$ is already occupied by the order parameter, so it is also omitted from (8.10). Then (5.9) and (8.10) – (8.13) imply that

$$\Phi_b^\dagger \Psi_s = 0 \quad (8.14)$$

and

$$\Phi_b^\dagger \Phi_b = \sum_r ' \Phi_r^\dagger \Phi_r \Psi_r^{B\dagger} \Psi_r^B. \quad (8.15)$$

Since $T = (2m)^{-1} (\partial^\mu \partial_\mu + \partial^m \partial_m)$, the first term in (8.7) involves

$$\partial_\mu \Phi_b = \partial_\mu \sum_s ' \Phi_s U_B \psi_s^B \quad (8.16)$$

$$= \sum_s ' U_B \left(\psi_s^B \partial_\mu \Phi_s + i m v_\mu^c \sigma_c \psi_s^B \Phi_s \right) \quad (8.17)$$

$$= \sum_s ' \left(\Psi_s^B \partial_\mu \Phi_s + A_\mu^i K_i \Psi_s^B \Phi_s \right) \quad (8.18)$$

where (8.10), (5.19), (6.6), (6.3), (5.31), and (5.29) have been used. After integration by parts, the $\partial^\mu \partial_\mu$ term of (8.7) then has the form

$$\Delta S_1 = \int d^4 x h(2m)^{-1} \sum_{rs} ' \int d^d x \left(P_r^\mu \Psi_r^B \right)^\dagger \left(P_{\mu s} \Psi_s^B \right) \quad (8.19)$$

where

$$P_{\mu s} = \partial_\mu \Phi_s + \Phi_s A_\mu^i K_i. \quad (8.20)$$

The integral over the internal coordinates can be written

$$\langle r | P_r^{\mu\dagger} P_{\mu s} | s \rangle = \sum_t \langle r | P_r^{\mu\dagger} | t \rangle \langle t | P_{\mu s} | s \rangle \quad (8.21)$$

$$= \sum_t \left(\delta_{tr} \partial^\mu \Phi_r + i A_i^\mu t_i^{tr} \Phi_r \right)^\dagger \left(\delta_{ts} \partial_\mu \Phi_s + i A_\mu^j t_j^{ts} \Phi_s \right) \quad (8.22)$$

after (6.14) is used. Then (8.19) becomes

$$\Delta S_1 = \int d^4x h(2m)^{-1} D^\mu \Phi^\dagger D_\mu \Phi \quad (8.23)$$

where Φ is the vector with components Φ_r and

$$D_\mu = \partial_\mu + iA_\mu^i t_i. \quad (8.24)$$

Notice that the bosons of this section have not been treated in the same way as the fermions of Sections 4–6. This is because the bosons can undergo condensation at low energy. Their equation of motion is then less important than their coupling to the gauge fields A_μ^i , and it is appropriate to deal directly with the boson field Φ_b rather than writing it in the form (4.2) and neglecting terms that are second order in A_μ^i .

With $\varepsilon_r = 0$, (5.20) and (8.10) imply that

$$\left(-\frac{1}{2m} \partial^m \partial_m + V - \mu_B \right) \Phi_b = 0 \quad (8.25)$$

so the next term from (8.7) is

$$\Delta S_2 = - \int d^4x h \Phi^\dagger \mu_A \Phi. \quad (8.26)$$

Also, (8.15) gives

$$\int d^d x (\Phi_b^\dagger \Phi_b)^2 = \sum_{rs} ' \Phi_r^\dagger \Phi_r I_{rs} \Phi_s^\dagger \Phi_s \quad (8.27)$$

where

$$I_{rs} = \int d^d x \Psi_r^{B\dagger} \Psi_r^B \Psi_s^{B\dagger} \Psi_s^B. \quad (8.28)$$

For the solutions of (8.13), however, this expression is independent of r and s : $I_{rs} = I$. The last term from (8.7) is then

$$\Delta S_3 = \frac{1}{2} b I \int d^4x h (\Phi^\dagger \Phi)^2. \quad (8.29)$$

To obtain a standard form, let

$$\phi = (2m)^{-1/2} \Phi \quad (8.30)$$

$$\bar{\mu}^2 = 2m\mu_A. \quad (8.31)$$

The total Lagrangian density resulting from (8.23), (8.26), and (8.29) becomes

$$\bar{\mathcal{L}}_b = h \left[D^\mu \phi^\dagger D_\mu \phi - \bar{\mu}^2 \phi^\dagger \phi + \frac{1}{2} \bar{b} (\phi^\dagger \phi)^2 \right] \quad (8.32)$$

where

$$\bar{b} = (2m)^2 b I. \quad (8.33)$$

The prefactor in (8.32) is $h = (\det h_{\mu\nu})^{1/2}$ rather than $g = (\det g_{\mu\nu})^{1/2}$, and the first term involves

$$D^\mu \phi^\dagger D_\mu \phi = h^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi \quad (8.34)$$

rather than $g^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi$. Suppose for simplicity that $v_a^k = \lambda \delta_a^k$ and $v_0^0 = i\lambda$ (with $v_a^0 = v_0^k = 0$), since a similar scaling is implied by the cosmological model of Section 7. It follows that

$$g^{\mu\nu} = \lambda^2 \delta^{\mu\nu}, g = \lambda^{-4}. \quad (8.35)$$

Letting

$$\phi' = \lambda \phi \quad (8.36)$$

we can write

$$\bar{\mathcal{L}}_b = g \left[g^{\mu\nu} D_\mu \phi'^\dagger D_\nu \phi' - \lambda^2 \bar{\mu}^2 \phi'^\dagger \phi' + \frac{1}{2} \bar{b} (\phi'^\dagger \phi')^2 \right]. \quad (8.37)$$

We can similarly rescale (6.17):

$$\bar{\mathcal{L}}_f = -\frac{1}{2} i g \psi'^\dagger e_\alpha^\mu \sigma^\alpha \tilde{D}_\mu \psi' + \text{conj.} \quad (8.38)$$

where

$$\psi' = \lambda^2 \psi. \quad (8.39)$$

The specific scaling of the preceding paragraph is simplistic, but it suggests that the second term in (8.37) may be neglected, leaving

$$\bar{\mathcal{L}}_b = h \left[D^\mu \phi^\dagger D_\mu \phi + \frac{1}{2} \bar{b} (\phi^\dagger \phi)^2 \right]. \quad (8.40)$$

(There is another reason for neglecting this term: If μ_B is constant in (5.14), μ_A must also be constant, and it is asymptotically equal to zero in the cosmological picture of Section 7. It follows that $\bar{\mu} = 0$.) The final Lagrangian for fundamental bosons then has no mass terms or Yukawa couplings. This is consistent with the idea that radiative effects may give rise to such additional interactions at the electroweak scale. On the other hand, symmetry-breaking at a grand-unified scale is attributed to formation of the order parameter itself: The argument that led to (8.23) and (8.29) also implies that

$$S_0 = \int d^d x h \left[(2m)^{-1} D^\mu \tilde{\Phi}^\dagger D_\mu \tilde{\Phi} - \mu N_B n_A + \frac{1}{2} (2m)^{-2} \bar{b} (N_B n_A)^2 \right] \quad (8.41)$$

where $\tilde{\Phi}$ is the vector corresponding to Ψ_s , with all its components $\tilde{\Phi}_r$ equal to zero except $\tilde{\Phi}_0 = N_B^{1/2} \Psi_A$. The gauge fields A_μ^i that are coupled to Ψ_s , through the term $\tilde{\Phi}^\dagger (A_\mu^i t_i A_\mu^j t_j) \tilde{\Phi}$ in (8.41), will acquire large masses when Ψ_s becomes nonzero. According to (8.19) and (5.30), these are the fields for which

$$\int d^d x (K_i \Psi_0^B)^\dagger (K_j \Psi_0^B) = N_B^{-1} \int d^d x n_B \eta_B^\dagger \sigma_i \sigma_j \eta_B \quad (8.42)$$

is nonzero, where σ_i is defined in (5.31). For example, if the σ_i were proportional to the SU(3) Gell-Mann matrices λ_i ,¹² and if η_B^\dagger were (0,0,1), then the gauge fields corresponding to $i = 4,5,6,7,8$ would acquire masses at the grand-unified scale, and those corresponding to $i = 1,2,3$ would not, leaving an unbroken SU(2) gauge group at lower energy. The true internal symmetry group G should, of course, leave an unbroken gauge group SU(3) \times SU(2) \times U(1).

Notice that the Bernoulli equation (3.26) is unchanged when $mv_\mu^c \sigma_c = A_\mu^i \sigma_i$ is introduced at low energy. For those A_μ^i which do not couple to η_B , $\eta^\dagger \sigma_i \sigma_j \eta$ vanishes in (3.21). But those which do couple have large masses, so they do not appear at low energy.

The scaling above (8.35) is also relevant to the extra fields of (4.11) – (4.14): If $v_\alpha^\mu \sim \lambda$, (4.16) shows that $e_\mu^\alpha \sim \lambda^{-1}$, or $v_\mu^\alpha \sim \lambda^2 e_\mu^\alpha$, giving

$$a_k \sim \lambda^2 m e_k^0, \quad b_0 \sim \lambda^2 m e_0^a \sigma_a. \quad (8.43)$$

There is then an extra coupling to gravity for spin-polarized fermions which involves a mass $\lambda^2 m$.

9 Observable consequences

The low-energy Lagrangian

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}_f + \bar{\mathcal{L}}_b + \bar{\mathcal{L}}_g + \bar{\mathcal{L}}_G \quad (9.1)$$

still corresponds to Euclidean spacetime. We now need to perform a Wick rotation^{4,12,24–28}

$$x^0 \rightarrow ix^0 \quad (9.2)$$

to obtain the Lorentzian action

$$S_L = iS = \int d^4x \mathcal{L} \quad (9.3)$$

where

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_b + \mathcal{L}_g + \mathcal{L}_G \quad (9.4)$$

$$\mathcal{L}_f = \frac{1}{2}if\psi^\dagger e_\alpha^\mu \sigma^\alpha \tilde{D}_\mu \psi + h.c. \quad (9.5)$$

$$\mathcal{L}_b = -f \left[D^\mu \phi^\dagger D_\mu \phi + \frac{1}{2}\bar{b} (\phi^\dagger \phi)^2 \right] \quad (9.6)$$

$$\mathcal{L}_g = -\frac{1}{4}g_0^{-2}eF_{\mu\nu}^i F_{\rho\sigma}^i g^{\mu\rho} g^{\nu\sigma} \quad (9.7)$$

$$\mathcal{L}_G = \ell_P^{-2}e {}^{(4)}R \quad (9.8)$$

$$e = |\det e_\mu^\alpha| = (-\det g_{\mu\nu})^{1/2} \quad (9.9)$$

$$f = (-\det h_{\mu\nu})^{1/2} \quad (9.10)$$

and

$$D_\mu = \partial_\mu + iA_\mu^i t_i \quad (9.11)$$

A_0^i, e_0^α , etc. are now real-valued Lorentzian fields (see, e.g., p. 329 of Ref. 28), and the metric tensors $h_{\mu\nu}$ and $g_{\mu\nu}$ have Lorentzian signature $(-+++)$.

\mathcal{L} contains four terms, corresponding respectively to spin 1/2 fermions, scalar bosons, gauge fields, and the gravitational field. It has the same form as the Lagrangian postulated in standard fundamental physics, except for several differences that it is now appropriate to discuss.

For the sake of generality, suppose that radiative effects give rise to additional interaction terms and an effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L} - fu(\phi) + \mathcal{L}_{int} \quad (9.12)$$

$$\mathcal{L}_{int} = -\frac{1}{2}f\gamma\psi^\dagger\phi\psi + h.c. \quad (9.13)$$

$$= -\frac{1}{2}f \sum_{rps} \gamma_{rps} \psi_r^\dagger \phi_p \psi_s + h.c. \quad (9.14)$$

where $u(\phi)$ contains terms of the form $\pm \mu_p^2 \phi_p^\dagger \phi_p$. The complete matter field Lagrangian is then

$$\mathcal{L}_m = \mathcal{L}_f + \mathcal{L}_B + \mathcal{L}_{int} \quad (9.15)$$

with $\mathcal{L}_B = \mathcal{L}_b - fu(\phi)$. Since the fermions and fundamental bosons described by \mathcal{L}_m are defined on an initially flat spacetime with metric tensor $h_{\mu\nu}$, this Lagrangian does not contain a conventional factor $e = (-\det g_{\mu\nu})^{1/2}$. Instead it contains the nondynamical factor $f = (-\det h_{\mu\nu})^{1/2}$:

$$\mathcal{L}_m = f \tilde{\mathcal{L}}_m . \quad (9.16)$$

The variational principle (3.2) also holds for the Lorentzian action S_L :

$$\delta S_L = 0. \quad (9.17)$$

In addition, it holds for variations in $g^{\mu\nu}$, or $e_\alpha^\mu = v_\alpha^\mu$, since these are equivalent to variations in Ψ_s or Ψ_b . The Einstein field equations are given as usual by $\delta S_L / \delta g^{\mu\nu} = 0$. With the present action

$$S_m = \int d^4x f \tilde{\mathcal{L}}_m \quad (9.18)$$

they are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} {}^{(4)}R = -\ell_P^2 e^{-1} f \frac{\delta \tilde{\mathcal{L}}_m}{\delta g^{\mu\nu}} - \ell_P^2 e^{-1} \frac{\delta \mathcal{L}_g}{\delta g^{\mu\nu}} \quad (9.19)$$

since $\delta e / \delta g^{\mu\nu} = -\frac{1}{2}g_{\mu\nu} e$ and $\delta {}^{(4)}R / \delta g^{\mu\nu}$ is effectively $R_{\mu\nu}$.^{14,15} With the conventional matter field action

$$S'_m = \int d^4x e \tilde{\mathcal{L}}'_m \quad (9.20)$$

they are instead

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} {}^{(4)}R = -\ell_P^2 \left(\frac{\delta \tilde{\mathcal{L}}'_m}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu} \tilde{\mathcal{L}}'_m \right) - \ell_P^2 e^{-1} \frac{\delta \mathcal{L}_g}{\delta g^{\mu\nu}}. \quad (9.21)$$

Let us now consider the consequences of this modification.

(i) The cosmological constant. In conventional physics, the vacuum has a Lagrangian density $e \tilde{\mathcal{L}}_0$ due to Higgs fields.⁵⁰ If $\delta \tilde{\mathcal{L}}_0 / \delta g^{\mu\nu}$ is neglected, this density gives a contribution $\frac{1}{2} \ell_P^2 g_{\mu\nu} \tilde{\mathcal{L}}_0$ in the field equations (9.21), so it corresponds to an effective cosmological constant Λ .^{14,15}

$$\Lambda = -\frac{1}{2} \ell_P^2 \tilde{\mathcal{L}}_0 . \quad (9.22)$$

This prediction of conventional physics is in error by at least 50 orders of magnitude.⁵⁰ In the present theory, however, the Lagrangian density is $f \tilde{\mathcal{L}}_0$, and there is no contribution involving $\tilde{\mathcal{L}}_0$ directly in the field equations (9.19):

$$\Lambda = 0. \quad (9.23)$$

There may be a much weaker term involving $\delta \tilde{\mathcal{L}}_0 / \delta g^{\mu\nu}$, but this appears to be consistent with observation. There is also a more poorly defined contribution due to vacuum fluctuations which is not considered here.

(ii) Ordinary matter as a gravitational source. Since \mathcal{L}_B does not contribute in the field equations (9.19), we are left with

$$\begin{aligned} \mathcal{L}_F &= f \tilde{\mathcal{L}}_F = \mathcal{L}_f + \mathcal{L}_{int} \\ &= \frac{1}{2} f \psi^\dagger \left(i e_\alpha^\mu \sigma^\alpha \tilde{D}_\mu - \gamma \phi \right) \psi + h.c. \end{aligned} \quad (9.24)$$

The variational principle (9.17), for arbitrary $\delta\psi^\dagger$, then gives the Dirac equation for initially massless fermions coupled to gauge fields and scalar bosons:

$$\left(i e_\alpha^\mu \sigma^\alpha \tilde{D}_\mu - \gamma \phi \right) \psi = 0. \quad (9.25)$$

But this makes $\mathcal{L}_F = 0$. The same reasoning applies to the corresponding action \mathcal{L}'_F in conventional physics, so the conventional field equations (9.21) reduce to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} {}^{(4)}R = -\ell_P^2 \frac{\delta \tilde{\mathcal{L}}'_F}{\delta g^{\mu\nu}} - \ell_P^2 e^{-1} \frac{\delta \mathcal{L}_g}{\delta g^{\mu\nu}}. \quad (9.26)$$

In \mathcal{L}'_F , the fermion field ψ' has a normalization

$$\int d^4x e \psi'^\dagger \psi' = N_f \quad (9.27)$$

where N_f is the total number of fermions. In \mathcal{L}_F , on the other hand, ψ has a normalization

$$\int d^4x f \psi^\dagger \psi = N_f. \quad (9.28)$$

When this difference is taken into account, the conventional field equations (9.26) and the present field equations (9.19) make nearly the same predictions. It appears that both are in agreement with the classic and more recent tests of general relativity.^{14,15,51,52}

As mentioned below (8.43), however, there is an extra coupling of fermions to gravity through the fields a_k and b_0 , which might be observable.

(iii) Massless vector bosons. For photons and gluons the only coupling to gravity is through the Lagrangian \mathcal{L}_g of (9.7), and this is the same in the present theory as in conventional physics.

(iv) Massive vector bosons. For the W and Z particles there is an additional term resulting from (9.6). In the present theory it is

$$-f h^{\mu\nu} \left(A_\mu^i t_i \phi \right)^\dagger \left(A_\nu^j t_j \phi \right).$$

whereas in conventional physics it would have the form

$$-eg^{\mu\nu} (A_\mu^i t_i \phi')^\dagger (A_\nu^j t_j \phi') .$$

There is thus a difference in the coupling of these particles to gravity. The resulting violation of the equivalence principle will be small, because virtual W-bosons have large masses, but it is potentially observable.

In addition to the above gravitational effects, the present theory predicts unconventional behavior of propagators at high energy: For $p^\mu > mv^\mu$, the approximation below (4.4) will fail, and fermion propagators should begin to go as p^{-2} rather than \not{p}^{-1} . Also, the equation of motion for scalar bosons involves $h^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi$ rather than $g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi$. Since the model scaling above (8.35) is not quantitatively correct, there will be a violation of Lorentz invariance which should lead to observable effects for Higgs bosons.

Finally, the present theory provides a new cosmological picture, with implications for the Hubble constant⁵³ and other large-scale properties of the universe.

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