

PARTICLES AND PROPAGATORS IN LORENTZ-VIOLATING SUPERGRAVITY

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We obtain the propagators for spin 1/2 fermions and sfermions in Lorentz-violating supergravity.

Any violation of Lorentz invariance must be extremely small for ordinary matter under ordinary conditions¹. However, in Lorentz-violating supergravity² there is Lorentz violation for both Standard Model particles at very high energy³ and their supersymmetric partners at even relatively low energy⁴. Here we obtain the propagators for the fermions and sfermions of this theory, with the prefix “s” standing for “supersymmetric partner” rather than “scalar” in the present context, since these particles are spin 1/2 rather than spin zero bosons⁴.

In the present paper we also extend our previous work by considering left-handed as well as right-handed sfermion fields. Initially all fermion fields are right-handed, but one can transform half of them to left-handed fields³, obtaining, e.g., the full 4-component field ψ for the electron. The same transformation can be employed for the spin 1/2 sfermions of the present theory, with only one change: In the fourth step leading up to (11) in Ref. 3, bosonic fields commute rather than anticommute, so the final Lagrangian for massless particles has the form

$$\mathcal{L}_L = \pm \frac{1}{2} \left(\bar{m}^{-1} \eta^{\mu\nu} \partial_\mu \psi_L^\dagger \partial_\nu \psi_L + \psi_L^\dagger i \bar{\sigma}^\mu \partial_\mu \psi_L \right) + h.c. \quad (1)$$

where the upper sign holds for fermions and the lower for bosons. Here ψ_L is a 2-component left-handed spinor, with $\bar{\sigma}^k = -\sigma^k$ as usual, and $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The total Lagrangian has the following form, with left- and right-handed fields combined in a 4-component spinor ψ (in the Weyl representation, and coupled by a Dirac mass m in the case of fermions):

$$\mathcal{L}_\psi = \bar{m}^{-1} \psi^\dagger \tilde{\gamma}_2 \psi + \left(\frac{1}{2} i \psi^\dagger \tilde{\gamma}_1 \psi + h.c. \right) + \mathcal{L}'_\psi \quad (2)$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are diagonal 4×4 matrices (with elements ± 1) inserted to cover all the sign possibilities.

We treat both Standard Model fermions and bosonic sfermions together, with the following conventions: (1) The canonical momenta conjugate to ψ and ψ^\dagger are respectively called π^\dagger and $\tilde{\pi}$. (2) In defining these momenta, the derivative is taken from the right. The momenta are then

$$\pi^\dagger = \frac{\partial \mathcal{L}_\psi}{\partial \dot{\psi}} = \bar{m}^{-1} \psi^\dagger \tilde{\gamma}_2 + \frac{1}{2} i \psi^\dagger \tilde{\gamma}_1, \quad \tilde{\pi} = \frac{\partial \mathcal{L}_\psi}{\partial \dot{\psi}^\dagger} = \mp \bar{m}^{-1} \tilde{\gamma}_2 \psi \pm \frac{1}{2} i \tilde{\gamma}_1 \psi. \quad (3)$$

The equation of motion has the form

$$-\tilde{\gamma}_2 \bar{m}^{-1} \frac{\partial^2}{\partial t^2} \psi + i \tilde{\gamma}_1 \frac{\partial}{\partial t} \psi - H' \psi = 0 \quad (4)$$

and we quantize by requiring that

$$\left[\psi_\alpha(\vec{x}, x^0), \pi_\beta^\dagger(\vec{x}', x^0) \right]_\pm = i \delta(\vec{x} - \vec{x}') \delta_{\alpha\beta} \quad (5)$$

$$\left[\psi_\alpha^\dagger(\vec{x}, x^0), \tilde{\pi}_\beta(\vec{x}', x^0) \right]_\pm = i \delta(\vec{x} - \vec{x}') \delta_{\alpha\beta}. \quad (6)$$

The retarded Green's function is defined by

$$i G_{\alpha\beta}^R(x, x') = \theta(t - t') \left\langle 0 \left[\psi_\alpha(x), \psi_\beta^\dagger(x') \right]_\pm \middle| 0 \right\rangle \quad (7)$$

and we can show that it is in fact a Green's function by using

$$\begin{aligned} \frac{\partial^2}{\partial t^2} (\theta(t - t') f(t)) &= \frac{\partial \delta(t - t')}{\partial t} f(t) + 2\delta(t - t') \frac{f(t)}{\partial t} + \theta(t - t') \frac{\partial^2 f(t)}{\partial t^2} \\ &= \delta(t - t') \frac{\partial f(t)}{\partial t} + \theta(t - t') \frac{\partial^2 f(t)}{\partial t^2} \end{aligned} \quad (8)$$

to obtain

$$\begin{aligned} & i \left(-\tilde{\gamma}_2 \bar{m}^{-1} \frac{\partial^2}{\partial t^2} + i \tilde{\gamma}_1 \frac{\partial}{\partial t} - H' \right) G_{\alpha\beta}^R(x, x') \\ &= \delta(t - t') \left(-\tilde{\gamma}_2 \bar{m}^{-1} \frac{\partial}{\partial t} + i \tilde{\gamma}_1 \right) \left\langle 0 \left[\psi_\alpha(x), \psi_\beta^\dagger(x') \right]_\pm \middle| 0 \right\rangle \\ & \quad + \theta(t - t') \left(-\tilde{\gamma}_2 \bar{m}^{-1} \frac{\partial^2}{\partial t^2} + i \tilde{\gamma}_1 \frac{\partial}{\partial t} - H' \right) \left\langle 0 \left[\psi_\alpha(x), \psi_\beta^\dagger(x') \right]_\pm \middle| 0 \right\rangle \\ &= 2\delta(t - t') \left\langle 0 \left[\psi_\beta^\dagger(x'), \tilde{\pi}_\alpha(x) \right]_\pm \middle| 0 \right\rangle + O(\bar{m}^{-1}) + 0 \\ &= 2i\delta^{(4)}(x - x') \delta_{\alpha\beta} + O(\bar{m}^{-1}) \end{aligned} \quad (9)$$

where $O(\bar{m}^{-1})$ represents a term which will become negligibly small at energies low compared to \bar{m} , after extremely high energy terms have been

discarded from the representation of G^R . (See the discussion below (30).) The causal Green's function is defined by

$$iG_{\alpha\beta}(x, x') = \left\langle 0 \left| T \left(\psi_\alpha(x) \psi_\beta^\dagger(x') \right) \right| 0 \right\rangle \quad (10)$$

$$= iG_{\alpha\beta}^R(x, x') \mp \left\langle 0 \left| \psi_\beta^\dagger(x') \psi_\alpha(x) \right| 0 \right\rangle \quad (11)$$

so it satisfies the equation

$$\begin{aligned} & \left(-\tilde{\gamma}_2 \bar{m}^{-1} \frac{\partial^2}{\partial t^2} + i\tilde{\gamma}_1 \frac{\partial}{\partial t} - H' \right) G(x, x') \\ &= \left(-\tilde{\gamma}_2 \bar{m}^{-1} \frac{\partial^2}{\partial t^2} + i\tilde{\gamma}_1 \frac{\partial}{\partial t} - H' \right) G^R(x, x') + 0 \\ &= 2\delta^{(4)}(x - x') + O(\bar{m}^{-1}) \end{aligned} \quad (12)$$

where a 4×4 identity matrix implicitly multiplies $\delta^{(4)}(x - x')$ and the factor of 2 will be explained below.

Let ψ_n and ψ_m respectively represent the positive-frequency and negative-frequency solutions to (4). With $b_m^\dagger = a_m$, the field can be represented as in (4.17), (4.45), and (4.55) of Ref. 4:

$$\psi = \sum_n a_n \psi_n + \sum_m b_m^\dagger \psi_m \quad (13)$$

with

$$\psi_n(x) = A_\lambda(p') u_\lambda(p') \exp(-i\varepsilon_\lambda(\vec{p}')t) \exp(i\vec{p}' \cdot \vec{x}) \quad (14)$$

$$\psi_m(x) = A_\kappa(p') v_\kappa(p') \exp(+i\varepsilon_\kappa(\vec{p}')t) \exp(-i\vec{p}' \cdot \vec{x}) \quad (15)$$

so that $n \leftrightarrow \vec{p}', \lambda$ and $m \leftrightarrow -\vec{p}', \kappa$. Here u and v are 4-component spinors, and the normalization is the same as in (4.35) of Ref. 4:

$$A_\lambda^*(p') A_\lambda(p') = (1 + 2\varepsilon_\lambda(\vec{p}')/\bar{m})^{-1} V^{-1} \quad (16)$$

$$A_\kappa^*(p') A_\kappa(p') = (1 - 2\varepsilon_\kappa(\vec{p}')/\bar{m})^{-1} V^{-1}. \quad (17)$$

To obtain the Green's function we need

$$\langle 0 | \psi(x) \psi^\dagger(x') | 0 \rangle = \left\langle 0 \left| \sum_{nn'} \psi_n(x) \psi_{n'}^\dagger(x') \left(\delta_{nn'} \mp a_n^\dagger a_n \right) \right| 0 \right\rangle \quad (18)$$

$$= \sum_n \psi_n(x) \psi_n^\dagger(x') \quad (19)$$

$$\langle 0 | \psi^\dagger(x') \psi(x) | 0 \rangle = \sum_m \psi_m(x) \psi_m^\dagger(x'). \quad (20)$$

4

The Fourier transform of the causal Green's function

$$G(\omega, \vec{p}) = \int dt \exp(i\omega(t-t')) \int d^3x \exp(-i\vec{p} \cdot (\vec{x} - \vec{x}')) G(x, x')$$

can be found by using

$$\theta(t-t') = \int \frac{d\omega'}{2\pi i} \frac{\exp(i\omega'(t-t'))}{\omega' - i\epsilon} \quad (21)$$

$$\int d^4x \exp(i(\vec{p}' - \vec{p}) \cdot (\vec{x} - \vec{x}')) = V \delta_{\vec{p}\vec{p}'} \quad (22)$$

in each of the two terms:

$$\begin{aligned} & \int dt \exp(i\omega(t-t')) \int d^3x \exp(-i\vec{p} \cdot (\vec{x} - \vec{x}')) \theta(t-t') \sum_n \psi_n(x) \psi_n^\dagger(x') \\ &= \int \frac{d\omega'}{2\pi i} \frac{2\pi}{\omega' - i\epsilon} \delta(\omega + \omega' - \varepsilon_\lambda(\vec{p})) \sum_{\vec{p}'\lambda} A_\lambda^*(p') A_\lambda(p') u_\lambda(p') u_\lambda^\dagger(p') V \delta_{\vec{p}\vec{p}'} \\ &= i \sum_\lambda \frac{u_\lambda(\vec{p}, p^0) u_\lambda^\dagger(\vec{p}, p^0)}{\omega - \varepsilon_\lambda(\vec{p}) + i\epsilon} \frac{1}{(1 + 2\varepsilon_\lambda(\vec{p})/\bar{m})} \quad , \quad p^0 = \varepsilon(\vec{p}) \end{aligned} \quad (23)$$

and

$$\begin{aligned} & \int dt \exp(i\omega(t-t')) \int d^3x \exp(-i\vec{p} \cdot (\vec{x} - \vec{x}')) \theta(t'-t) \sum_m \psi_m(x) \psi_m^\dagger(x') \\ &= \int \frac{d\omega'}{2\pi i} \frac{2\pi}{\omega' - i\epsilon} \delta(\omega - \omega' + \varepsilon_\kappa(\vec{p})) \sum_{\vec{p}'\kappa} A_\kappa^*(p') A_\kappa(p') v_\kappa(p') v_\kappa^\dagger(p') V \delta_{\vec{p}, -\vec{p}'} \\ &= -i \sum_\kappa \frac{v_\kappa(-\vec{p}, -p^0) v_\kappa^\dagger(-\vec{p}, -p^0)}{\omega + \varepsilon_\kappa(\vec{p}) - i\epsilon} \frac{1}{(1 - 2\varepsilon_\kappa(\vec{p})/\bar{m})} \quad , \quad -p^0 = \varepsilon(\vec{p}) \end{aligned} \quad (24)$$

since $\varepsilon(-\vec{p}) = \varepsilon(\vec{p})$. When combined these expressions give

$$\begin{aligned} G(\omega, \vec{p}) &= \frac{1}{2} \sum_\lambda \frac{u_\lambda(p) u_\lambda^\dagger(p)}{\omega - \varepsilon_\lambda(\vec{p}) + i\epsilon} \frac{1}{1 + 2\varepsilon_\lambda(\vec{p})/\bar{m}} \\ &\quad \mp \frac{1}{2} \sum_\kappa \frac{v_\kappa(-p) v_\kappa^\dagger(-p)}{\omega + \varepsilon_\kappa(\vec{p}) - i\epsilon} \frac{1}{1 - 2\varepsilon_\kappa(\vec{p})/\bar{m}} \end{aligned} \quad (25)$$

with $u_\lambda(p) u_\lambda^\dagger(p)$ and $v_\kappa(-p) v_\kappa^\dagger(-p)$ on the energy shell, in the sense that $p^0 = \varepsilon(\vec{p})$ in the first term and $-p^0 = \varepsilon(\vec{p})$ in the second. However, $G(x-x')$ can be equally well represented by

$$G(x-x') = \int \frac{dp^0}{2\pi} \exp(-ip^0(t-t')) \sum_{\vec{p}} \exp(i\vec{p} \cdot (\vec{x} - \vec{x}')) \tilde{G}(p) \quad (26)$$

$$\begin{aligned} & \tilde{G}(p) \\ &= \frac{1}{2} \sum_{\lambda} \frac{u_{\lambda}(p) u_{\lambda}^{\dagger}(p)}{p^0 - \varepsilon_{\lambda}(\vec{p}) + i\epsilon} \frac{1}{1 + 2p^0/\bar{m}} \mp \frac{1}{2} \sum_{\kappa} \frac{v_{\kappa}(-p) v_{\kappa}^{\dagger}(-p)}{p^0 + \varepsilon_{\kappa}(\vec{p}) - i\epsilon} \frac{1}{1 - 2p^0/\bar{m}} \end{aligned} \quad (27)$$

with p^0 unrestricted, since, when the residue is evaluated at one of the poles, p^0 is forced to equal $\pm(\varepsilon(\vec{p}) - i\epsilon)$.

Let us define a modified Green's function \tilde{S}_f for a fermion by

$$\tilde{S}_f(x, x') = \langle 0 | T(\psi(x) \bar{\psi}(x')) | 0 \rangle \quad , \quad \bar{\psi} = \psi^{\dagger} \gamma^0 \quad (28)$$

$$\begin{aligned} \tilde{S}_f(p) &= i\tilde{G}(p) \gamma^0 \\ &= \frac{i}{2} \sum_{\lambda} \frac{u_{\lambda}(p) \bar{u}_{\lambda}(p)}{p^0 - \varepsilon_{\lambda}(\vec{p}) + i\epsilon} \frac{1}{1 + 2p^0/\bar{m}} - \frac{i}{2} \sum_{\kappa} \frac{v_{\kappa}(-p) \bar{v}_{\kappa}(-p)}{p^0 + \varepsilon_{\kappa}(\vec{p}) - i\epsilon} \frac{1}{1 - 2p^0/\bar{m}}. \end{aligned} \quad (29)$$

In the remainder of this paper we limit attention to energies that are low compared to \bar{m} . In this case, and for massless right-handed fermions, it can be seen in (4.18)-(4.21) of Ref. 4 that there is only one value of λ , and it corresponds to the normal branch with $\varepsilon_{\lambda}(\vec{p}) = |\vec{p}|$. On the other hand, there are three values of κ : One corresponds to the normal branch with $\varepsilon_{\kappa}(\vec{p}) = |\vec{p}|$, and two to extremely high energy branches with $\varepsilon_{\kappa}(\vec{p}) = \bar{m} \pm |\vec{p}|$. When a Dirac mass is introduced, these last two branches are hardly perturbed, and they will still give extremely large denominators in the expression above for \tilde{S}_f . They can then be neglected in calculations at normal energies, and at the same time $2p^0/\bar{m}$ can be neglected. With the high energy branches omitted, we have the relevant low energy propagator

$$S_f(p) = \frac{i}{2} \sum_{\lambda=1,2} \frac{u_{\lambda}(p) \bar{u}_{\lambda}(p)}{p^0 - \varepsilon_{\lambda}(\vec{p}) + i\epsilon} - \frac{i}{2} \sum_{\kappa=1,2} \frac{v_{\kappa}(-p) \bar{v}_{\kappa}(-p)}{p^0 + \varepsilon_{\kappa}(\vec{p}) - i\epsilon} \quad (31)$$

where $\varepsilon(\vec{p}) = (\vec{p}^2 + m^2)^{1/2}$. In the sums, λ and κ are now each limited to the 2 usual values for a 4-component Dirac spinor, rather than the total of 8 values that one would have if the 4 extremely high energy solutions were retained. We should note, however, that the high-energy solutions give a contribution in the equation of motion (9) for the Green's function that is equal to that of the low-energy solutions, because the derivatives bring down large energies which cancel those in the denominator. This accounts for the factor of 2 in (9) and (12), and at normal energies we obtain

$$(\not{p} - m) iS_f(x, x') = \delta^{(4)}(x - x') \quad (32)$$

where $\not{p} = -\gamma^\mu \partial_\mu$ with our metric tensor $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The usual Dirac spinors u_λ^D and v_κ^D have the completeness relation

$$\sum_{\lambda=1,2} u_\lambda^D(p) \bar{u}_\lambda^D(p) = \not{p} + m \quad , \quad \sum_{\kappa=1,2} v_\kappa^D(p) \bar{v}_\kappa^D(p) = \not{p} - m \quad (33)$$

and they are normalized to $2p^0$, whereas our u_λ and v_κ are normalized to unity. We then have

$$S_f(p) = \frac{i}{2} \frac{1}{2p^0} \frac{\not{p} + m}{p^0 - \varepsilon(\vec{p}) + i\epsilon} - \frac{i}{2} \frac{1}{2p^0} \frac{-\not{p} - m}{p^0 + \varepsilon(\vec{p}) - i\epsilon} = \frac{i}{\not{p} - m + i\epsilon} \quad (34)$$

and the standard expression for the Feynman propagator is regained.

For sfermions, however, one must make a distinction even at low energy between the mathematical Green's function $\tilde{G}(p)$, in which negative-norm solutions have been included, and the physical propagator $S_b(p)$, which can contain only positive-norm solutions. It is a fundamental requirement of quantum mechanics that physical operators are allowed to connect only states in a positive-norm Hilbert space. The physical field operator ψ_{phys} , and other physical operators, should therefore contain only creation and destruction operators for positive-norm states. To obtain the physical propagator, one can repeat the above treatment with only positive-norm solutions retained. In the present theory, it fortunately turns out that one still has a complete set of functions ψ_n and ψ_m , as is required to provide a proper representation of the original classical field and satisfy the quantization condition (5). On the other hand, one finds that the restriction to positive-norm solutions permits only one value each for λ and κ in (25) or (27) at low energy, corresponding to sfermions and anti-sfermions which are right-handed before a mass is introduced. The physical implications of this, and the issue of sfermion masses, will be discussed elsewhere.

References

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