SUPERSYMMETRY METHODS, in statistical physics –

Methods which transform problems in statistical physics to problems in supersymmetric field theory, or vice-versa. These techniques usually involve the replacement of random variables (in a functional integral or differential equation) by Grassmann variables, with a resulting effective action that exhibits some form of supersymmetry. One can then exploit both the techniques of standard field theory and the supersymmetry itself, so that the original problem is simplified or made more tractable. In some cases the transformation also provides deeper insights. The original variables may represent disorder or other random perturbations of the system. Let us consider several examples [1-4].

Parisi and Sourlas demonstrated that the supersymmetry hidden in a stochastic differential equation can have physical implications [1] (specifically, for a spin system in a random magnetic field), and that particular stochastic differential equations lead to Wess-Zumino models. They also considered general stochastic differential equations of the form

$$U_x\left[\phi\right] \equiv \frac{\delta U}{\delta \phi\left(x\right)} = \eta\left(x\right)$$

where η is a random variable. An example is

$$U\left[\phi\right] = \int d^D x\,\left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + V\left(\phi\right)\right)$$

which describes a system in a random magnetic field when $V\left(\phi\right) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}g\phi^4$, a dilute system of branched polymers when $V\left(\phi\right) = \frac{1}{2}m^2\phi^2 + \frac{1}{3}\lambda\phi^3$, etc. For the case with a ϕ^4 term, the differential equation is

$$-\Delta\phi + m^2\phi + g\phi^3 = \eta.$$

For simplicity in the notation, let us now consider a one-dimensional system. As η varies randomly, so does the solution $\phi_\eta\left(x\right)$. The expectation value of any function $F\left[\phi\right]$ is

$$\begin{aligned}\left\langle F\right\rangle &= \int D\eta\,\exp\left(-\frac{1}{2}\int dx\,\eta^2\left(x\right)\right)F\left[\phi_\eta\left(x\right)\right]\\ &= \int D\eta\,D\phi\,\exp\left(-\frac{1}{2}\int dx\,\eta^2\left(x\right)\right)F\left[\phi\right]\\ &\qquad\times\delta\left(U_x - \eta\left(x\right)\right)\det\left[U_{xy}\right]\end{aligned}$$

where $U_{xy} = \delta^2 U/\delta\phi\left(x\right)\delta\phi\left(y\right)$ and standard manipulations have been performed within the functional integral [1].

The central trick which yields the supersymmetric reformulation of stochastic theories is the following. Since Grassmann variables have the useful property [4]

$$\begin{aligned}\det\left[U_{xy}\right] &= \int D\psi\,D\overline{\psi}\\ &\qquad\times\exp\left(-\int dx\,dy\,\overline{\psi}\left(x\right)U_{xy}\psi\left(y\right)\right)\end{aligned}$$

the stochastic variable $\eta\left(x\right)$ can be replaced by the anticommuting variable $\psi\left(x\right)$:

$$\left\langle F\right\rangle = \int D\phi\,D\,\psi\,D\,\overline{\psi}\,F\left[\phi\right]e^{-S}$$

$$S = \frac{1}{2}\int dx\,U_x^2 + \int dx\,dy\,\overline{\psi}\left(x\right)U_{xy}\psi\left(y\right).$$

The effective action S is then invariant under the supersymmetry transformations

$$\begin{aligned}&\delta\phi\left(x\right) = \overline{\varepsilon}\psi\left(x\right) + \overline{\psi}\left(x\right)\varepsilon\\ &\delta\psi\left(x\right) = -\varepsilon U_x\ ,\ \delta\overline{\psi}\left(x\right) = -\overline{\varepsilon}U_x\end{aligned}$$

where ε and $\overline{\varepsilon}$ are anticommuting parameters. One can also introduce an auxiliary field A, to obtain supersymmetry transformations that are linear in the fields.

Parisi and Sourlas showed that the stochastic equation

$$\partial_\mu\gamma_\mu\phi = \eta$$

corresponds to the effective action

$$S_0 = \int d^D x\left(\frac{1}{2}\sum_i\left(\partial_\mu\phi_i\right)^2 + \overline{\psi}\,\partial_\mu\gamma_\mu\psi\right)$$

which is the $N = 2$ free Wess-Zumino model. A slightly more complicated differential equation in two dimensions gives

$$\begin{aligned}S = \int d^2x\,\Bigg(&\frac{1}{2}\sum_{i=1}^{2}\left(\partial_\mu\phi_i\right)^2 + \frac{1}{2}g^2\left(\phi_1^2+\phi_2^2\right)^2\\ &+\overline{\psi}\left(\partial\!\!\!/ + g\left(\phi_1 + i\gamma_5\phi_2\right)\right)\psi\Bigg)\end{aligned}$$

which is the $N = 2$ Wess-Zumino model with interaction.

As a different kind of example, consider

$$\langle F\rangle = \left\langle \frac{\int D\phi\, D\phi^\dagger\, e^{iS_L\left[\phi,\phi^\dagger\right]} F\left[\phi,\phi^\dagger\right]}{\int D\phi\, D\phi^\dagger\, e^{iS_L\left[\phi,\phi^\dagger\right]}} \right\rangle$$

where

$$S_L\left[\phi,\phi^\dagger\right] = \int d^D x\, \left(-\partial^\mu \phi^\dagger \partial_\mu \phi - \mu^2 \phi^\dagger \phi + u\phi^\dagger \phi\right)$$

and $u(x)$ is a random variable satisfying

$$\langle u\,\rangle = 0\ ,\ \langle u(x)\, u(y)\rangle = b\,\delta(x-y)\ ,$$

which is again assumed to have a Gaussian distribution. $F\left[\phi,\phi^\dagger\right]$ is some physical quantity and $\langle\cdots\rangle$ represents an average over the perturbing potential u. (F may involve a product of the fields at different points, in which case $\langle F\rangle$ is a Green's function or correlation function.) The presence of the denominator makes it difficult to perform the average. However, since

$$\int D\,\phi\, D\,\phi^\dagger\, e^{iS_L\left[\phi,\phi^\dagger\right]} = (\det\ A)^{-1}$$

$$\int D\,\psi\, D\,\psi^\dagger\, e^{iS_L\left[\psi,\psi^\dagger\right]} = \det\ A$$

where A represents the operator of iS_L, it follows that

$$\langle F\rangle = \left\langle \int D\,\phi\, D\,\phi^\dagger\, D\,\psi\, D\,\psi^\dagger \times F\, e^{iS_L\left[\phi,\phi^\dagger\right]} e^{iS_L\left[\psi,\psi^\dagger\right]} \right\rangle = \left\langle \int D\Psi\ D\,\Psi^\dagger\ F\, e^{iS_L\left[\Psi,\Psi^\dagger\right]} \right\rangle$$

where Ψ is a statistical superfield as defined elsewhere in this volume,

$$\Psi = \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

and

$$S_L\left[\Psi,\Psi^\dagger\right] = \int d^D x\, \left(-\partial^\mu \Psi^\dagger \partial_\mu \Psi - \mu^2 \Psi^\dagger \Psi + u\Psi^\dagger \Psi\right).$$

For a Gaussian random variable v whose mean is zero, the result

$$\left\langle e^{iv}\right\rangle = e^{-\frac{1}{2}\left\langle v^2\right\rangle}$$

implies that

$$\left\langle exp\left(i\int d^D x\, u\, \Psi^\dagger \Psi\right)\right\rangle = exp\left(-\frac{1}{2} b \int d^D x\, \left(\Psi^\dagger(x)\, \Psi(x)\right)^2\right).$$

It follows that

$$\langle F\rangle = \int D\,\Psi\, D\,\Psi^\dagger\, F\, e^{-S}$$

with

$$S = \int d^D x \Big[-i\left(\partial^\mu \Psi^\dagger \partial_\mu \Psi + \mu^2 \Psi^\dagger \Psi\right) + \frac{1}{2} b \left(\Psi^\dagger \Psi\right)^2 \Big].$$

Again, the stochastic variable u has been replaced by the anticommuting variable ψ.

Efetov has used somewhat similar ideas to treat a wide variety of problems in condensed matter physics [4]. The applications involve rather complicated products of Green's functions, but he has developed a repertoire involving integrals over supervectors and supermatrices, etc.

Let us summarize the principal motivations behind the use of supersymmetry in statistical physics:

1) It is an alternative to the replica trick and other techniques for treating disordered systems. As Efetov has emphasized, the use of anticommuting variables in problems that originally contain only commuting variables is analogous to the use of complex numbers in evaluating integrals that initially involve only real numbers.

2) It establishes connections between stochastic systems and supersymmetry which are very intriguing, and which may possibly turn out to have fundamental significance.

BIBLIOGRAPHY

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