

# ANALYSIS OF THE STEADY-STATE RESPONSE OF LINEAR NETWORKS TO SINUSOIDAL VOLTAGE SOURCES

## Development and Application of the Phasor Transform and its Inverse

### I. INTRODUCTION

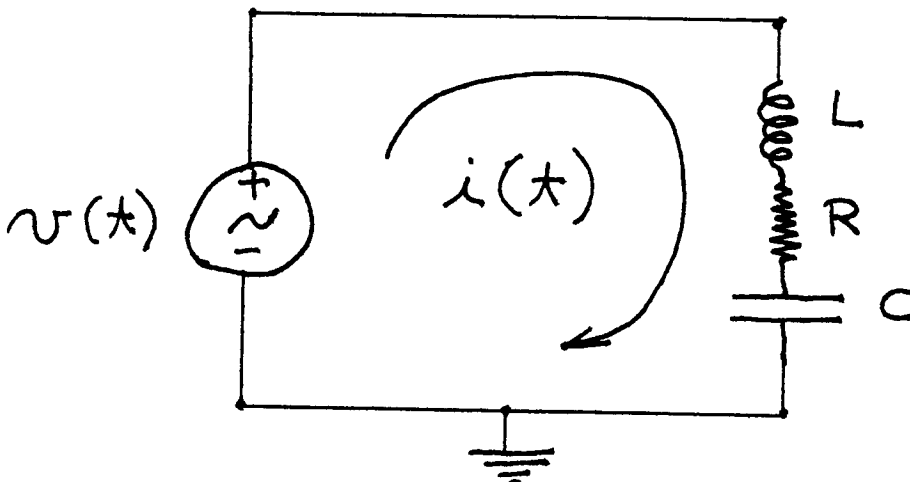
In the following presentation we will consider in detail a particular circuit which allows us to generalize on the problem of determining the *steady-state* current which flows in each branch of any linear network when a sinusoidal voltage source is applied to the network. A *linear network* is defined as one which contains resistors ( $R$ ), capacitors ( $C$ ), and inductors ( $L$ ) which are independent of the current through them and the voltage across them. (Also permissible are fixed (DC) voltage and current sources and linear controlled sources, but we will not deal with these kinds of sources immediately.)

A central idea which emerges in this presentation is that application of a sinusoidal voltage source to a linear network causes currents in the network which have the same frequency as that of the source. Appearance of any frequencies other than that of the voltage source is a *certain indicator* that the circuit is *non-linear*. The fact that the response frequency is identical to the source frequency in linear networks will lead us to an extremely practical technique for analyzing such networks: the use of the *phasor transform* (often abbreviated as *phasor*) and the *inverse phasor transform*.

### II. THE BASIC CIRCUIT AND PRELIMINARY MATHEMATICAL DESCRIPTION

#### A. The Basic Circuit

We choose to concentrate on the circuit shown below. This circuit represents the completely general linear network problem because any such network (ignoring fixed and controlled sources) can be analyzed in branches which contain *at most* the three idealized components shown in series in Figure 1.



(Fig. 1.)

A single sinusoidal voltage source  $v(t)$  is shown. This source is represented mathematically as

$$v(t) = V \cos(\omega t + \phi).$$

(Eqn. 1.)

Notice that  $v(t)$  has been expressed in the most general form for a sinusoid.

It is conventional to use the terms *AC voltage* and *AC current* when the voltages and currents vary with time in *any* manner. However, in all of the work that we will do here to develop the phasor formalism, we restrict our attention to AC voltages and currents whose time dependence is purely *sinusoidal*. A particularly useful property of all sinusoids is that they are uniquely characterized by three parameters: frequency, amplitude, and phase constant. For example, the voltage source given in Equation 1 is completely specified by its frequency,  $\omega$ , its amplitude,  $V$ , and its phase constant,  $f$ .

## B. The Differential Equation for the Voltage Drive and Current Response in the Basic Circuit

1. If we apply Kirchhoff's voltage law to the circuit in Figure 1 we immediately find

$$L \frac{di}{dt} + iR + \frac{q}{C} = V \cos(\omega t + f), \quad (\text{Eqn. 2.})$$

in which  $q(t)$  is the charge on the capacitor at any time  $t$ ,  $i(t) = dq/dt$ , and where we have used the expression given in Equation 1 for  $v(t)$ .

We now differentiate Equation 2 with respect to time to give the equation a simpler form involving only  $i(t)$  as the dependent variable:

$$L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = -\omega V \sin(\omega t + f). \quad (\text{Eqn. 3.})$$

2. Special comments on the nature of the solution which we are seeking:

(a) We are interested in the *steady-state solution*. This will always be identified as the so-called *particular solution* (in the language of linear ordinary differential equations) which satisfies Equation 3 with no regard whatever for *initial conditions*.

(b) For the sake of perspective it is useful to recall that the *complete or general solution* of Equation 3 is the sum of two solutions to two different equations: the *general solution* of the *homogeneous equation* and the *steady-state solution* of the *inhomogeneous equation*. The solution of the homogeneous equation takes into account the initial current in the inductor and the initial voltage across the capacitor, and this solution gives what is frequently called the *natural response* to the initial conditions, completely free of any driving voltage. This solution always gives a *transient response* to the initially stored energies in the linear components. The steady-state solution describes the response of the system to the applied *forcing function*, the applied voltage  $v(t)$ .

## III. THE STEADY-STATE SOLUTION FOR THE CURRENT RESPONSE

### A. Complex Notation for the Current Response

We now launch into complex expressions for currents—and eventually voltages—because it is in the complex domain that the *phasor* is invented and used. There is no systematic *a priori* justification for this abrupt introduction of complex quantities. It just turns out that the descriptive mathematics of simple harmonic physical systems is convenient and compact when complex notation is carefully employed.

Following the conventional method of solution of linear, inhomogeneous, ordinary differential equations, we guess a form for the particular solution of Equation 3:

$$i(t) = I \cos(\omega t + \phi). \quad (\text{Eqn. 4})$$

Notice that this trial solution is identical in form to  $v(t)$  as given in Equation 1. Also note that the frequency  $\omega$  in our trial solution is the same as the frequency of  $v(t)$ , and that  $\phi$  is a phase constant—not equal to  $f$ , in general—which is necessary in order to give  $i(t)$  the completely general sinusoid form.

We will soon substitute Equation 4 into the differential equation (Equation 3), but we will do so with  $i(t)$  expressed in a special form using (as promised above) complex notation:

$$i(t) = \text{Re} \left[ I e^{j(\omega t + \phi)} \right], \quad (\text{Eqn. 5})$$

in which  $j \equiv \sqrt{-1}$  and  $\text{Re}[\ ]$  is an operator which performs the action “take the real part.”

**B. Phasor Solution of the Differential Equation (Equation 3)  
Using Complex Expressions for  $i(t)$  and  $v(t)$**

We will now try to find a particular solution of Equation 3 by substituting  $i(t)$  in the form given by Equation 5. This substitution gives the following result:

$$L \frac{d^2}{dt^2} \left\{ \text{Re} \left[ I e^{j(\omega t + \phi)} \right] \right\} + R \frac{d}{dt} \left\{ \text{Re} \left[ I e^{j(\omega t + \phi)} \right] \right\} + \frac{1}{C} \text{Re} \left[ I e^{j(\omega t + \phi)} \right] = -\omega V \sin(\omega t + f). \quad (\text{Eqn. 6})$$

In order to write the right-hand side of Equation 6 in the complex notation established on the left-hand side, we make use of the identity

$$\sin(\omega t + f) = -\cos(\omega t + f + \pi/2). \quad (\text{Eqn. 7})$$

Applying this identity to the right-hand side of Equation 6 and expressing the result in complex notation, we find:

$$L \frac{d^2}{dt^2} \left\{ \text{Re} \left[ I e^{j(\omega t + \phi)} \right] \right\} + R \frac{d}{dt} \left\{ \text{Re} \left[ I e^{j(\omega t + \phi)} \right] \right\} + \frac{1}{C} \text{Re} \left[ I e^{j(\omega t + \phi)} \right] = \omega \text{Re} \left[ V e^{j(\omega t + f + \pi/2)} \right]. \quad (\text{Eqn. 8})$$

Since the operation “take the real part” commutes with the differentiation operator and since the sum of the real parts of any set of complex expressions is equal to the real part of the sum of the same set of complex expressions, we can write:

$$\text{Re} \left\{ L \frac{d^2}{dt^2} \left[ I e^{j(\omega t + \phi)} \right] + R \frac{d}{dt} \left[ I e^{j(\omega t + \phi)} \right] + \frac{1}{C} \left[ I e^{j(\omega t + \phi)} \right] - \omega \left[ V e^{j(\omega t + f + \pi/2)} \right] \right\} = 0. \quad (\text{Eqn. 9})$$

Now carrying out the differentiation and factoring out  $e^{j\omega t}$ , we find:

$$\text{Re} \left\{ \left[ -\omega^2 L I e^{j\phi} + j\omega R I e^{j\phi} + \frac{1}{C} I e^{j\phi} - \omega V e^{j(f + \pi/2)} \right] e^{j\omega t} \right\} = 0. \quad (\text{Eqn. 10})$$

The expression shown in square brackets immediately above is clearly complex. It follows that it can

be written in cartesian form as

$$a + jb,$$

in which  $a$  and  $b$  are real expressions and are constants in this case since the only time dependence in Equation 10 is in the factor  $e^{j\omega t}$ . If we also expand  $e^{j\omega t}$  to its cartesian form, Equation 10 becomes:

$$\operatorname{Re}\{(a + jb)(\cos \omega t + j \sin \omega t)\} = 0. \quad (\text{Eqn. 11.})$$

Carrying out the multiplication and taking the real part, we get

$$a \cos \omega t - b \sin \omega t = 0. \quad (\text{Eqn. 12.})$$

This equation must hold *for all times*. In particular, at  $t = 0$ , we find

$$a = 0.$$

Similarly, at  $t = p/2\omega$  we find

$$b = 0.$$

It follows that in Equation 10, the expression in square brackets is identically zero:

$$-\omega^2 L I e^{jq} + j\omega R I e^{jq} + \frac{1}{C} I e^{jq} - \omega V e^{j(\omega t + p/2)} = 0. \quad (\text{Eqn. 13.})$$

Factoring out the quantity  $I e^{jq}$  from the first three terms and dividing both sides of the equation by  $\omega$  we get

$$\left(-\omega L + jR + \frac{1}{\omega C}\right) I e^{jq} - V e^{j(\omega t + p/2)} = 0$$

We now continue to simplify the basic expression in Equation 13 taking into account that  $e^{jp/2} = j$  and using  $1/j = -j$ :

$$\left(j\omega L + R + \frac{1}{j\omega C}\right) I e^{jq} - V e^{j\omega t} = 0. \quad (\text{Eqn. 14.})$$

We now define the phasor current  $\hat{I}$  and the phasor voltage  $\hat{V}$  as follows:

$$\begin{aligned} \hat{I} &\equiv I e^{jq} \\ \hat{V} &\equiv V e^{j\omega t}. \end{aligned} \quad (\text{Eqn. 15.})$$

These expressions are obtained from the real current  $i(t)$  and voltage  $v(t)$  by the following set of steps:

(1) Replace the sinusoidal factor in the real expression by a complex exponential expression whose real part is equal to the original sinusoidal function:

$$\text{Example: } V \cos(\omega t + f) \Rightarrow V e^{j(\omega t + f)} \text{ in Equation 1.}$$

(2) Divide the resulting complex exponential expression by  $e^{j\omega t}$ :

$$\text{Example: } \hat{V} = \frac{V e^{j(\omega t + f)}}{e^{j\omega t}} = V e^{jf}.$$

These steps constitute the phasor transformation of real voltages and currents into the corresponding phasors. The inverse phasor transformation converts the phasor representation back

to the corresponding time-domain (real) representation, as follows:

- (1.) Multiply the phasor expression by  $e^{j\omega t}$  :

$$\text{Example: } \hat{V}e^{j\omega t} = Ve^{j(\omega t + \phi)} .$$

- (2.) Take the real part of the resulting expression:

$$\text{Example: } \operatorname{Re}\left[Ve^{j(\omega t + \phi)}\right] = V \cos(\omega t + \phi) \equiv v(t) .$$

With the introduction of the phasor quantities  $\hat{I}$  and  $\hat{V}$ , we can rewrite Equation 14 as:

$$\hat{V} = \hat{I}Z , \quad (\text{Eqn. 16.})$$

in which

$$Z \equiv \left[ j\omega L + R + \frac{1}{j\omega C} \right] . \quad (\text{Eqn. 17.})$$

So we find the phasor solution of Equation 3 to be:

$$\hat{I} = \frac{\hat{V}}{Z} . \quad (\text{Eqn. 18.})$$

Equation 16 is a generalized form of Ohm's law,  $V = IR$ , and is the basis for a great deal of simplification in the analysis of AC circuits containing inductors and capacitors. The complex impedance  $Z$  clearly consists of three terms, representing the impedance of  $L$ ,  $R$ , and  $C$ , respectively:

$$\begin{aligned} Z_L &= \text{impedance of inductor } L = j\omega L \\ Z_R &= \text{impedance of resistor } R = R \\ Z_C &= \text{impedance of capacitor } C = \frac{1}{j\omega C} . \end{aligned}$$

Furthermore, these three terms appear *added together* in the expression for  $Z$  because they combine in series according to the same rule as that for resistors. In fact, impedances in *parallel* also combine according to the rule for combining resistors in parallel.

The complex impedance  $Z$  can be written in cartesian form as

$$\begin{aligned} Z &= R + j\left[\omega L - \frac{1}{\omega C}\right] \\ &= R + jX , \end{aligned} \quad (\text{Eqn. 19.})$$

in which  $X \equiv \omega L - \frac{1}{\omega C}$ .  $X$  is called the total reactance of the circuit in Figure 1. Inductors and capacitors are frequently referred to as reactive components, and each has its own reactance, as follows:

$$\begin{aligned} X_L &= \text{inductive reactance} = \omega L \\ X_C &= \text{capacitive reactance} = \frac{1}{\omega C} , \end{aligned}$$

so that the impedances and reactances of inductors and capacitors are related by:

$$\begin{aligned} Z_L &= jX_L \\ Z_C &= -jX_C . \end{aligned} \quad (\text{Eqn. 20.})$$

### C. The Time-Domain Solution of Equation 3; Application of the Inverse Phasor Transform

We now follow the steps outlined in the preceding section for converting the phasor representation to the corresponding time-domain representation. Applying this inverse phasor transform to our phasor solution  $\hat{I}$  in Equation 18, we first multiply both sides of Equation 18 by  $e^{j\omega t}$ :

$$\hat{I}e^{j\omega t} = \frac{\hat{V}e^{j\omega t}}{Z} = \frac{\hat{V}e^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}, \quad (\text{Eqn. 21.})$$

in which we have written  $Z$  in explicit cartesian form. The last step in getting  $i(t)$  is to take the real part of both sides of Equation 21. But this will be made easier if we first multiply both numerator and denominator by the complex conjugate of the present denominator:

$$\hat{I}e^{j\omega t} = \frac{\hat{V}e^{j\omega t} \left[ R - j\left(\omega L - \frac{1}{\omega C}\right) \right]}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

Now writing the complex expression in square brackets in polar form, we get

$$\hat{I}e^{j\omega t} = \frac{\hat{V}e^{j\omega t} \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} e^{ja}}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

in which  $a = \tan^{-1} \left( \frac{\frac{1}{\omega C} - \omega L}{R} \right)$ . Cancelling common factors, substituting  $V e^{jf}$  for  $\hat{V}$ , and combining the exponential factors give

$$\hat{I}e^{j\omega t} = \frac{V e^{j(\omega t + f + a)}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

Now taking the real part of both sides,

$$i(t) = \text{Re} \left[ \hat{I}e^{j\omega t} \right] = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t + f + a). \quad (\text{Eqn. 22.})$$

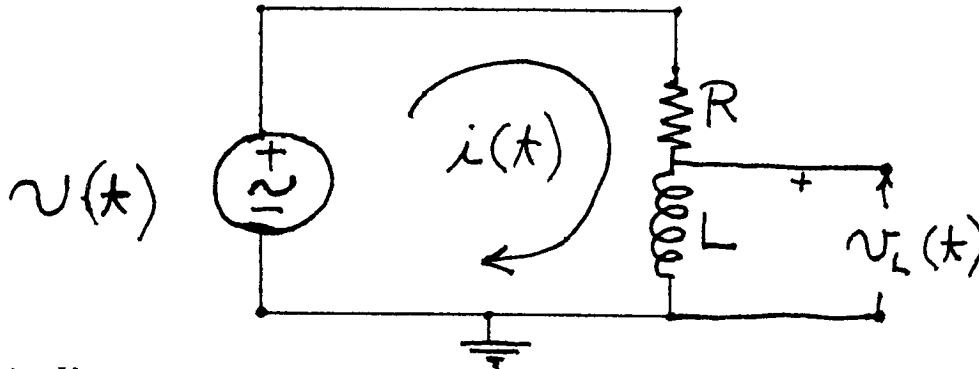
We have now performed the inverse phasor transform on the phasor representation of  $i(t)$  to recover  $i(t)$  itself. In actual exercises involving AC circuit analysis, one frequently leaves the solution in the phasor form, since this form contains all the information needed for interpretation and for recovering the time-domain functions when they are preferred.

It is important to realize that Equation 22 relates the current and the voltage for any conceivable branch of a linear network not involving fixed or controlled sources, since any such branch can be considered to be the series combination of components  $R$ ,  $L$ , and  $C$ . If either  $R$  or  $L$  does not appear in a given branch, we simply let  $R = 0$  or  $L = 0$ , respectively, in Equation 22. And if no capacitance appears we must let  $C = \infty$  in Equation 22.

#### IV. PRACTICAL USE OF PHASORS IN STEADY-STATE AC CIRCUIT PROBLEMS

It would be extremely cumbersome, inconvenient, and time consuming to repeat *all* of the analysis presented in Section III (beginning with the differential equation) each time you face the problem of analyzing an AC circuit. Fortunately, you do not have to and, in fact, you *should* not. All of the techniques that we are familiar with for analyzing purely resistive networks—voltage dividers, Thevenin and Norton equivalents, etc.—can be applied to networks containing inductors and capacitors in addition to resistors by direct use of the phasor formalism. To demonstrate the direct use of phasors, we consider the following example:

*Example:* Consider the circuit drawn below.



Given that  $v(t) = V \cos \omega t$ , find the voltage  $v_L(t)$  across the inductor  $L$ .

*Solution:*

- (1.) Let  $\hat{I}$  denote the phasor current and  $\hat{V}$  denote the phasor source voltage. Then

$$\hat{I} = \frac{\hat{V}}{Z} = \frac{\hat{V}}{R + Z_L}.$$

- (2.) Let  $\hat{V}_L$  be the phasor voltage across the inductor. Then

$$\hat{V}_L = \hat{I} Z_L = \hat{V} \frac{Z_L}{R + Z_L}.$$

Notice that this equation is in every way similar to the equation which results from the application of the voltage divider rule for resistors!

- (3.) To find the time-domain expression  $v_L(t)$  we take the inverse phasor transform of the result immediately above:

$$v_L(t) = \text{Re} \left[ \hat{V}_L e^{j\omega t} \right] = \text{Re} \left[ \frac{\hat{V} \omega L e^{j(\omega t + \pi/2)}}{R + j\omega L} \right],$$

in which we have used  $Z_L = j\omega L$  and the identity  $j = e^{j\pi/2}$ . Recognizing that  $\hat{V} = V$  and writing

**$R + j\omega L$  in polar form, we find**

$$v_L(t) = \operatorname{Re} \left\{ \frac{V \omega L e^{j(\omega t + \phi/2 - a)}}{\sqrt{R^2 + \omega^2 L^2}} \right\},$$

**in which**  $a = \tan^{-1} \left( \frac{\omega L}{R} \right)$ .

**Now taking the real part explicitly, we find the final result**

$$v_L(t) = \frac{V \omega L \cos(\omega t + \phi/2 - a)}{\sqrt{R^2 + \omega^2 L^2}}.$$