How to Integrate Planck's Function

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ABSTRACT

In gory detail we show how to integrate Planck's function B_ν. The hard part involves showing how a particular "undoable" integral is related to Euler's Gamma Function and the Riemann Zeta Function. The corresponding infinite sum can be calculated exactly by working out the Fourier series for $\mathrm{f}(x) = x^2$ and applying Parseval's equation. This gives the Stefan-Boltzmann Law, $E = \sigma T^4$, and allows us to determine the numerical value of the Stefan-Boltzmann constant σ. The amount of radiation emitted by a black body, and it being proportional to the fourth power of the temperature constitute a fundamental building block of astrophysics.

1. Introduction

A solid, liquid, or dense gas gives off a continuous spectrum (also known as a black body spectrum). The peak of the spectral energy distribution depends on the temperature of the black body. A rock at room temperature gives off infrared radiation that peaks at a wavelength of about 10 microns. If we heated a solid with a very high melting point to 6000 K, it would be a black body source that produces mostly visible light, but it also produces significant amounts of ultraviolet and infrared light. The Sun's photosphere has just such a temperature, but because the photosphere is cooler than the deeper layers that produce continuous radiation, light is absorbed and scattered by the atoms in the photosphere. Which photons are absorbed? It depends on the composition and ionization state of the atoms. An absorption spectrum results.

The spectral radiance of a black body at absolute temperature T is given by Planck's function (Carroll & Ostlie 2007, p. 73)[2]:

$$B_\nu = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} , \tag{1}$$

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[2] See also the *Wikipedia* article on "Planck's law".

where ν is the frequency of light, h is Planck's constant, c is the velocity of light, and k is Boltzmann's constant. The units of B_ν are ergs/s/cm^2/sr/Hz.

A similar form of the spectral radiance, but as a function of wavlength λ, is:

$$B_\lambda \;=\; \frac{2hc^2/\lambda^5}{e^{\frac{hc}{\lambda kT}}-1} \; . \tag{2}$$

Here the units are ergs/s/cm^3/sr.

We can multiply Eqs. 1 and 2 by $4\pi/c$ to give the spectral energy density u_ν or u_λ, which is measured in terms of energy per unit volume per spectral unit.

In Fig. 1 we see an example of the spectral energy density of a black body of temperature 6000 K, both in frequency form and wavelength form. While each function clearly has a maximum, and for an individual photon $\nu = c/\lambda$, it is *not* the case that $\nu_{max} = c/\lambda_{max}$.

From radiative transfer theory we note that the amount of radiation emitted by a black body at a particular frequency will be

$$F_\nu \;=\; \int_0^{2\pi} d\phi \; \int_{-1}^{+1} I \; \mu \; d\mu \; , \tag{3}$$

where $\mu = cos\ \theta$. $I = B_\nu$ for μ ranging from 0 to +1. $I = 0$ for μ ranging from -1 to 0. This means that we only need to consider photons emitted in the 2π steradians between the source and the observer. Photons emitted in the other direction are not received by the observer. We then have

$$F_\nu \;=\; 2\;\pi\; B_\nu \; \left(\frac{\mu^2}{2}\right)\Big|_0^1 \;=\; \pi\; B_\nu \; . \tag{4}$$

Integrating B_ν over all frequencies and multiplying by π gives the total amount of energy given off by the black body each second. This is known as the Stefan-Boltzmann Law:

$$E \;= \pi \int_0^\infty B_\nu \; d\nu \;=\; \sigma T^4 \; , \tag{5}$$

where σ is the Stefan-Boltzmann constant. In this writer's opinion, if you want to call yourself an astrophysicist, at least once you should go through the full derivation of integrating Eq. 5, which is the purpose of this note. This is non-trivial, as it is rather long winded and involves at least one unusual trick from advanced calculus.

2. A Simple Substitution

Let $x = (\frac{h\nu}{kT})$. Then $dx = (\frac{h}{kT})d\nu$. In other words, $\nu = (\frac{kT}{h})x$ and $d\nu = (\frac{kT}{h})dx$. Eq. 5 becomes

$$E \;=\; \pi \left(\frac{2h}{c^2}\right)\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}\ . \tag{6}$$

Using numerical integration and a simple Python program, we can show that the integral on the right hand side of Eq. 6 is approximately equal to 6.493939, and below we show that the integral is *exactly* equal to $\pi^4/15$. This gives us $E = \sigma T^4$, with

$$\sigma \;=\; \frac{2\pi^5 k^4}{15c^2h^3} = 5.6704 \times 10^{-5} erg/cm^2/s/K^4\ . \tag{7}$$

Even before we carry out the difficult integral we can easily see that the energy radiated by a black body varies with the fourth power of the temperature. Let us consider a blue main sequence star of photospheric temperature 30000 K and a red supergiant star of photospheric temperature 3000 K. And say the diameter of the red star is 100 times the diameter of the blue star. The luminosity (L) of a star of radius R is equal to the area of the star times the intensity of light emitted per unit area:

$$L \;=\; 4\pi R^2 \sigma T^4\ . \tag{8}$$

The blue star radiates most of its energy in the ultraviolet, while the red star radiates most of its energy in the infrared. The red star has a surface area 10^4 times that of the blue star, but each square meter of the blue star radiates 10^4 times as much energy (integrated over all frequencies) compared to each square meter of the red star. So the bolometric luminosties of the two stars are equal.

3. Euler's Gamma Function

We define Euler's Gamma Function (Hogg & Craig 1970, pp. 99-101) as

$$\Gamma(s) \;=\; \int_0^\infty e^{-x} x^{s-1} dx\ . \tag{9}$$

So,

$$\Gamma(1) \;=\; \int_0^\infty e^{-x} dx \;= -e^{-x}|_0^\infty \;=\; 1\;.$$

$$\Gamma(2) \;=\; \int_0^\infty e^{-x}\; x\; dx\;.$$

Using integration by parts, let $u = x$ and $dv = e^{-x}dx$ so that $du = dx$ and $v = -e^{-x}$. Then $uv \;-\; \int vdu$ equals

$$-xe^{-x}|_0^\infty + \int_0^\infty e^{-x}dx \;=\; -0+0-e^{-x}|_0^\infty \;= 1\;.$$

Furthermore,

$$\Gamma(3) \;=\; \int_0^\infty e^{-x}\; x^2\; dx\;.$$

Using integration by parts, let $u = x^2$ and $dv = e^{-x}dx$ so that $du = 2xdx$ and $v = -e^{-x}$. Then $uv \;-\; \int vdu$ equals

$$-x^2e^{-x}|_0^\infty + \int_0^\infty e^{-x}2x\; dx \;=\; -0+0+2\cdot 1 \;=\; 2\;.$$

Let us do one more for good measure.

$$\Gamma(4) \;=\; \int_0^\infty e^{-x}\; x^3\; dx\;.$$

Using integration by parts, let $u = x^3$ and $dv = e^{-x}dx$ so that $du = 3x^2dx$ and $v = -e^{-x}$. Then $uv \;-\; \int vdu$ equals

$$-x^3e^{-x}|_0^\infty + \int_0^\infty e^{-x}3x^2\; dx \;=\; -0+0+3\cdot 2\cdot 1 \;=\; 6\;.$$

Thus, each subsequent integral is $s-1$ times an integral we have already done. In summary, for positive integer s

$$\Gamma(s) \; = \; (s-1)! \tag{10}$$

Soon we will need the following (Whittaker & Watson 1935, p. 243):

$$\int_0^\infty e^{-bx} x^{s-1} dx \; = \frac{\Gamma(s)}{b^s} \; . \tag{11}$$

Let $z = bx$, and let us require that $b > 0$. Then $dx = dz/b$ and the integral becomes

$$\int_0^\infty e^{-z} \left(\frac{z}{b}\right)^{s-1} \frac{dz}{b} \; = \; \frac{\int_0^\infty e^{-z} z^{s-1} dz}{b^s} \; = \; \frac{\Gamma(s)}{b^s} \; . \quad (Q.E.D.) \tag{12}$$

4. The Riemann Zeta Function

Following Whittaker & Watson (1935, pp. 265-6), we define the Riemann Zeta Function as follows:

$$\zeta(s) \; = \; \sum_{n=1}^{\infty} \frac{1}{n^s} \; . \tag{13}$$

We can also define a more general Zeta Function thusly:

$$\zeta(s,a) \; = \; \sum_{n=0}^{\infty} \frac{1}{(a+n)^s} \; , \tag{14}$$

where a is a constant. For simplicity we shall suppose that $0 < a \leq 1$.

Since

$$\sum_{n=0}^{\infty} \frac{1}{(1+n)^s} \; = \; \sum_{n=1}^{\infty} \frac{1}{n^s} \; , \tag{15}$$

it is evident that $\zeta(s,1) = \zeta(s)$.

Now, using Eq. 11 and $b = a + n$,

$$\int_0^\infty x^{s-1} e^{-(a+n)x} dx \;=\; \frac{\Gamma(s)}{(a+n)^s} \; . \tag{16}$$

Whittaker & Watson (1935) have a more formal proof, involving taking the upper limit of the sum in the next equation as $n \rightarrow \infty$, but it boils down to

$$\sum_{n=0}^{\infty} \frac{\Gamma(s)}{(a+n)^s} \;=\; \sum_{n=0}^{\infty} \int_0^\infty x^{s-1} e^{-(a+n)x} dx \tag{17}$$

On the right hand side we have an infinite sum of integrals, which can be expressed as follows:

$$\zeta(s,a)\,\Gamma(s) \;=\; \int_0^\infty x^{s-1} e^{-ax} \left[1 \;+\; e^{-x} \;+\; e^{-2x} \;+\; ...\right] dx \tag{18}$$

The reader should recall (or you may consult Hodgman 1952, p. 271) that for $0 < q < 1$ the infinite sum of

$$1 \;+\; q \;+\; q^2 + \;...\; \longrightarrow \; \frac{1}{1-q} \; . \tag{19}$$

Letting $q = e^{-x}$, this leaves us with

$$\zeta(s,a)\,\Gamma(s) \;=\; \int_0^\infty \frac{x^{s-1} e^{-ax}}{(1-e^{-x})} dx \; . \tag{20}$$

For $a = 1$

$$\zeta(s,1)\,\Gamma(s) \;=\; \zeta(s)\,\Gamma(s) \;=\; \int_0^\infty \frac{x^{s-1} dx}{e^x - 1} \; . \tag{21}$$

Since $\Gamma(4) = 3! = 6$, this means that for $s = 4$

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} \;=\; 6 \sum_{n=1}^{\infty} \frac{1}{n^4} \; , \tag{22}$$

and we are halfway to finding the exact answer to Eq. 6.

5. An Application of Fourier Series

Following Kaplan (1952, p. 391), let f(x) be a function that is piecewise very smooth in the interval $-\pi \leq x \leq \pi$. Then the Fourier series of f(x) is:

$$\frac{a_0}{2} \ + \ \sum_{n=1}^{\infty}(a_n \ cos \ nx \ + \ b_n \ sin \ nx) \ , \tag{23}$$

with

$$a_0 \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \ dx \ , \tag{24}$$

$$a_n \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \ cos \ nx \ dx \ , \tag{25}$$

and

$$b_n \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \ sin \ nx \ dx \ . \tag{26}$$

Here we wish to determine the Fourier series for f(x) = x^2. The coefficients b_n are all zero because sine is an odd function (f(x) = $-$f($-x$)), while x^2 is an even function (f(x) = f($-x$)), and the integral of this even function times this odd function over the interval of $-\pi$ to π is zero.

We have

$$a_0 \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \ dx \ = \ \frac{x^3}{3\pi}\Big|_{-\pi}^{\pi} \ = \frac{2\pi^2}{3} \ . \tag{27}$$

Integration by parts allows us to show that

$$\int x^2 \ cos \ nx \ dx \ = \ \frac{x^2}{n} \ sin \ nx \ + \ \frac{2x}{n^2} \ cos \ nx \ - \ \frac{2}{n^3} \ sin \ nx \ + \ C. \tag{28}$$

Evaluating this result from $x = -\pi$ to $+\pi$ for $n = 1$ gives $a_1 = -4$. For $n = 2$, $a_2 = +1$. And in general $a_n = \frac{4}{n^2}(-1)^n$. In the context of Eq. 23 we find

$$f(x) \; = \; x^2 \; = \; \frac{\pi^2}{3} \; + \; 4 \sum_{n=1}^{\infty} \frac{(-1)^n \; cos \; nx}{n^2} \; . \tag{29}$$

In Fig. 2 we show the sum of the first 10 terms of the Fourier series given in Eq. 29. Except close to $x \; = \; \pm\pi$ radians the agreement using only 10 cosine terms is quite good.

Here is an interesting intermediate result. For all positive n, $(-1)^n \; cos \; n\pi \; = \; +1$. If $x = \pi$, Eq. 29 gives

$$\pi^2 \; = \; \frac{\pi^2}{3} \; + \; 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \; , \tag{30}$$

or

$$\zeta(2) \; = \; \sum_{n=1}^{\infty} \frac{1}{n^2} \; = \; \frac{\pi^2}{6} \; . \tag{31}$$

Finally, using the Fourier series for f(x) = x^2 (Eq. 29), we can determine $\zeta(4)$ using Parseval's equation (Kaplan 1952, pp. 412-3):

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \; dx \; = \; \frac{a_0^2}{2} \; + \; \sum_{n=1}^{\infty} (a_n^2 \; + \; b_n^2) \; . \tag{32}$$

The left hand side is trivial:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \; dx \; = \; \frac{x^5}{5\pi} \Big|_{-\pi}^{\pi} \; = \; \frac{2\pi^4}{5} \; .$$

This gives us

$$\frac{2\pi^4}{5} \; = \; \frac{2\pi^4}{9} \; + \; 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \; .$$

After a little rearrangement, we have

$$\zeta(4) \; = \; \sum_{n=1}^{\infty} \frac{1}{n^4} \; = \; \frac{\pi^4}{90} \; , \tag{33}$$

which resolves Eqs. 6 and 22.

We thank Eddie Baron for useful discussions regarding fundamentals of radiative transfer discussed in §1.

A. More on the Riemann Zeta Function

If one wishes to calculate $\zeta(6)$ in a similar fashion, one needs the Fourier series

$$f(x) \;=\; x^3 \;=\; 2\sum_{n=1}^{\infty}\left[(-1)^{n+1}\left(\frac{\pi^2}{n} \;-\; \frac{6}{n^3}\right)\right]\; sin\; nx\;. \tag{A1}$$

Then one needs Parseval's equation, $\zeta(2)$, and $\zeta(4)$, which we have already determined. Using Eqs. 32 and A1 we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^6\, dx \;=\; \sum_{n=1}^{\infty} 4\left(\frac{\pi^2}{n} \;-\; \frac{6}{n^3}\right)^2$$

After dividing both sides by 4, the left hand side is trivial:

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} x^6\, dx \;=\; \frac{x^7}{28\pi}\bigg|_{-\pi}^{\pi} \;=\; \frac{\pi^6}{14}\;.$$

The right hand side becomes

$$\sum_{n=1}^{\infty}\left(\frac{\pi^4}{n^2} \;-\; \frac{12\pi^2}{n^4} \;+\; \frac{36}{n^6}\right) \;=\; \pi^4\left(\frac{\pi^2}{6}\right) \;-\; 12\pi^2\left(\frac{\pi^4}{90}\right) \;+\; 36\sum_{n=1}^{\infty}\frac{1}{n^6}\;.$$

Combining all π^6 terms on the left hand side gives

$$\pi^6\left(\frac{1}{14} \;-\; \frac{1}{6} \;+\; \frac{2}{15}\right) \;=\; 36\sum_{n=1}^{\infty}\frac{1}{n^6}\;.$$

We are almost done.

$$\pi^6\left(\frac{90 \;-\; 210 \;+\; 168}{1260}\right) \;=\; \pi^6\left(\frac{48}{1260}\right) \;=\; 36\sum_{n=1}^{\infty}\frac{1}{n^6}\;.$$

Finally, since $\frac{3}{4}$ of 1260 is equal to 945, we have

$$\zeta(6) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^6} \;=\; \frac{\pi^6}{945}\;. \tag{A2}$$

The *Wikipedia* article whose URL is given below[3] gives a regression formula for values of the Riemann Zeta Function for an even, positive argument:

$$\zeta(2n) \;=\; \left(\frac{1}{n\,+\,\frac{1}{2}}\right) \sum_{k=1}^{n-1} \zeta(2k)\,\zeta(2n-2k)\;,\; n > 1\;. \tag{A3}$$

For $n = 2$,

$$\zeta(4) \;=\; \left(\frac{1}{2\,+\,\frac{1}{2}}\right) \Big[\zeta(2)\,\zeta(4-2)\,\Big]$$

$$=\; \frac{2}{5}\left(\frac{\pi^2}{6}\right)\left(\frac{\pi^2}{6}\right) \;=\; \frac{\pi^4}{90}\;,$$

which is the same as the result given in Eq. 33.

For $n = 3$,

$$\zeta(6) \;=\; \left(\frac{1}{3\,+\,\frac{1}{2}}\right) \Big[\zeta(2)\,\zeta(6-2) \;+\; \zeta(4)\,\zeta(6-4)\Big]$$

$$=\; \left(\frac{1}{3\,+\,\frac{1}{2}}\right) \left[2\left(\frac{\pi^2}{6}\right)\left(\frac{\pi^4}{90}\right)\right]$$

$$=\; \frac{\pi^6}{945}\;,$$

as given above in Eq. A2. Values of $\zeta(8)$ through $\zeta\;(34)$ are found in the article whose URL is given in footnote 3.

Derbyshire (2003) has written an interesting book about the Riemann Zeta Function. It is a combination math book and biography about Berhard Riemann.

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[3]https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function (accessed on February 16, 2018).

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Fig. 1.— Examples of the spectral energy distribution of a black body of temperature 6000 K as a function of frequency (top figure) and as a function of wavlength (bottom figure). In the bottom figure we have multiplied the wavelengths in cm by 10^8 to give wavelengths in Ångströms. Since spectral radiance and spectral energy density only differ by a multiplicative factor, the *shapes* of these curves are the same no matter which set we plot.

Fig. 2.— The sum of first 10 terms of the Fourier series given in Eq. 29 for f(x) = x^2.

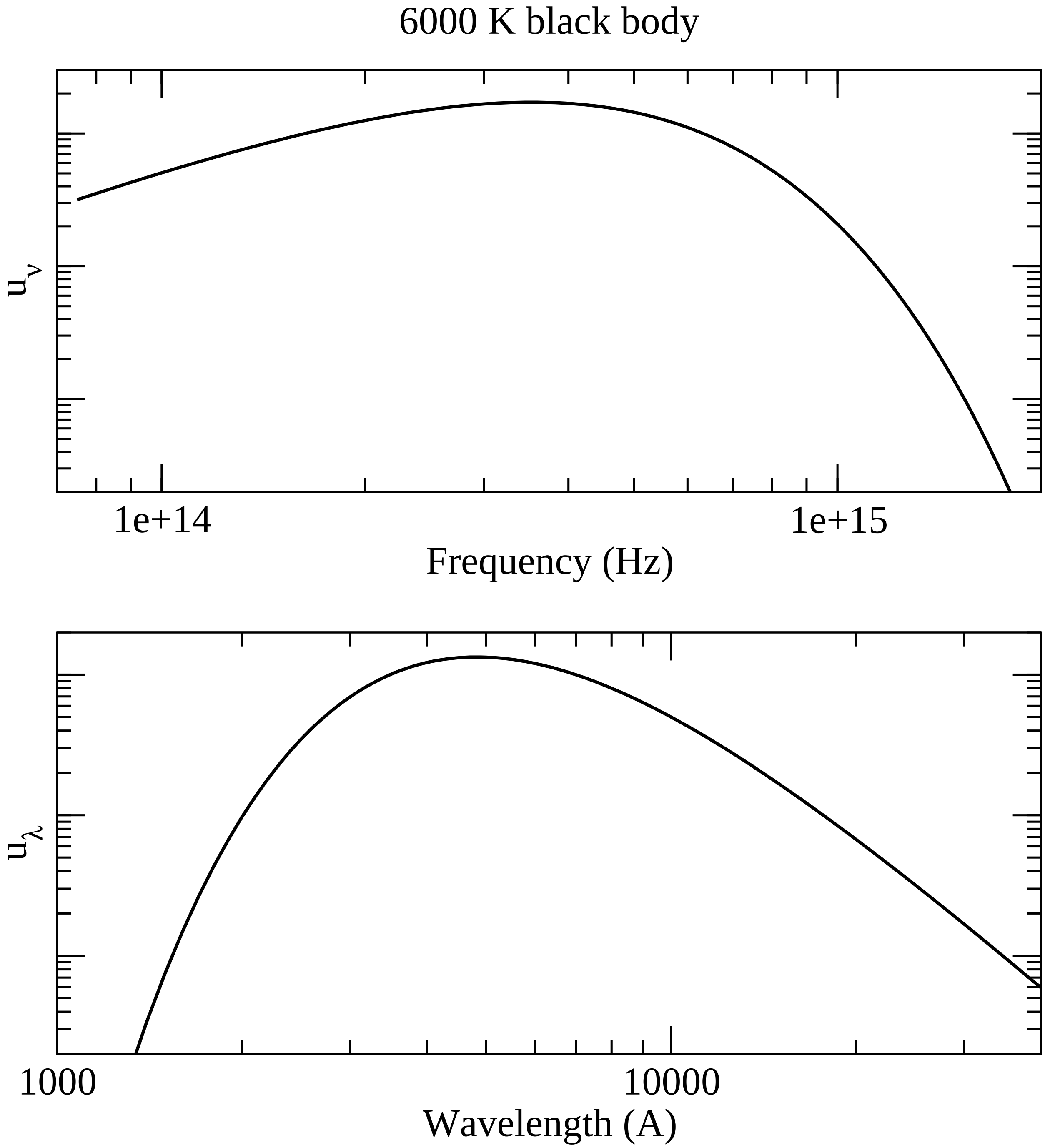
6000 K black body
u_ν
1e+14
1e+15
Frequency (Hz)
u_λ
1000
10000
Wavelength (A)

Fig. 1.

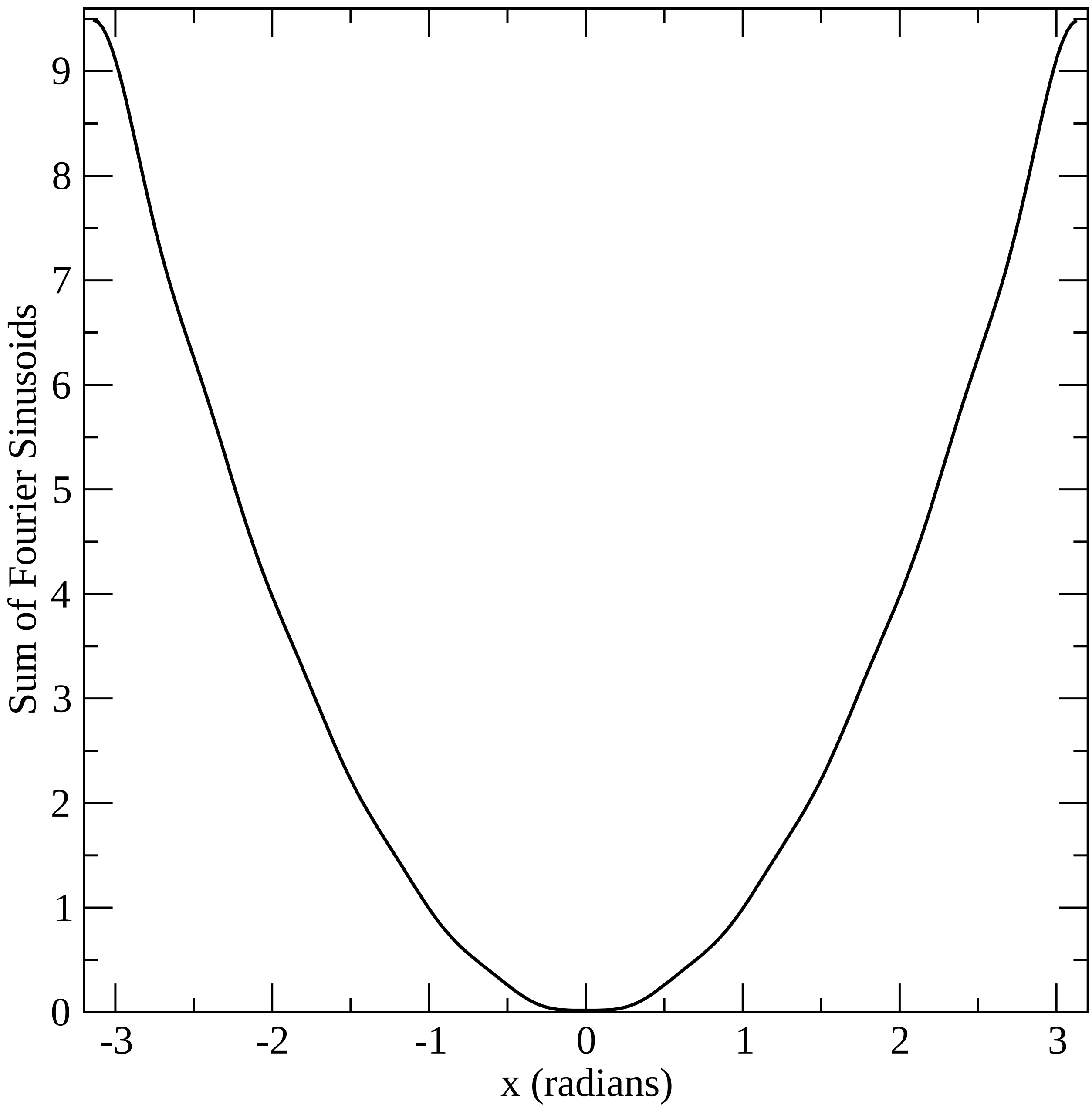

Fig. 2.