

## 603: Electromagnetic Theory I

### CONTENTS

- Maxwell's Equations: Introduction; units; boundary conditions.
- Electrostatics: Uniqueness theorem; Green's theorem; gauge potentials; energy
- Boundary value problems in electrostatics: Method of images; separation of variables in Cartesian, spherical polar and cylindrical polar coordinates.
- Dielectric media
- Multipole expansion
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# 1 Introduction

## 1.1 Maxwell's equations

The equations now known as Maxwell's equations were obtained over an extended period, principally during the early nineteenth century. Here, we shall take as our starting point the set of four differential equations as they were presented by Maxwell in about 1861. It was Maxwell who completed the process of constructing the equations, thereby achieving the first *unification* of fundamental theories in physics. Prior to Maxwell, there were two essentially independent theories, one describing electricity and the other describing magnetism, and it was he who brought about the synthesis that unified them into a single theory of *electromagnetism*. It was only later, after Einstein developed the theory of *Special Relativity* in 1905, that the magnitude of Maxwell's achievement really became clear. Especially, a quite remarkable feature of Maxwell's 1861 equations is that they are already completely compatible with special relativity, with no need for modification of any kind.<sup>1</sup> Aside from changes in notation and units, Maxwell's equations have remained otherwise unaltered since 1861.

Let us begin by considering Maxwell's equations in free space, by which is meant that the space outside of any conducting surfaces is assumed to be a vacuum. Using the SI system of units, Maxwell's equations are:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}' &= \frac{\rho'}{\epsilon_0}, & \vec{\nabla} \times \vec{B}' - \mu_0 \epsilon_0 \frac{\partial \vec{E}'}{\partial t} &= \mu_0 \vec{J}', \\ \vec{\nabla} \cdot \vec{B}' &= 0, & \vec{\nabla} \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t} &= 0.\end{aligned}\tag{1.1}$$

Observe that I have written these equations with a “prime” on the electric field  $\vec{E}$  the magnetic field  $\vec{B}$ , the electric charge density  $\rho$  and the electric current density  $\vec{J}$ . This is to indicate that these quantities are all expressed in the SI system of units. (The SI system typically maximises the number of “redundant” dimensionful constants, and so one might say that it is **S**uper **I**nconvenient.) The remaining quantities appearing in (1.1) are the constants  $\epsilon_0$  and  $\mu_0$ , which are, respectively, the permittivity of free space and the permeability of free space. They have the values

$$\epsilon_0 \approx 8.85419 \times 10^{-12} \text{ Farads/metre}, \quad \mu_0 = 4\pi \times 10^{-7} \text{ Henries/metre} \tag{1.2}$$

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<sup>1</sup>This contrasts with the case of Newtonian mechanics, which is not compatible with special relativity and therefore did not survive as a “fundamental” theory after 1905.

## 1.2 Gaussian units

SI units have their virtues for some purposes, but they can also be quite inconvenient in practice. This seems to be especially true in electromagnetism, and for this reason it is often more convenient to stick with an earlier system, known as *Gaussian* units. In this system, Maxwell's equations in free space take the form

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 4\pi\rho, & \vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= \frac{4\pi}{c} \vec{J}, \\ \vec{\nabla} \cdot \vec{B} &= 0, & \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0,\end{aligned}\tag{1.3}$$

where  $c$  is the speed of light. Observe that here, we are writing the equations using the unprimed quantities  $\vec{E}$ ,  $\vec{B}$ ,  $\rho$  and  $\vec{J}$ , and it will probably therefore come as no surprise that it is this Gaussian system of units that I prefer to use. It should already be evident upon comparing (1.1) and (1.3) that Gaussian system is somewhat simpler, in that one needs only one “fundamental constant” (i.e. the speed of light) rather than two (the permittivity and the permeability of free space).<sup>2</sup> The introduction of the  $4\pi$  factors in (1.3) may perhaps seem tiresome, but the advantage of doing so will become apparent in due course. (There can be no escape from having  $4\pi$  factors somewhere, because of the fact that a unit sphere has area  $4\pi$ .) In particular, as will be seen, by using Gaussian units we avoid the need for the  $4\pi$  denominators that arise in every expression for potentials or field strengths in terms of charges when SI units are used.

In order to ensure that we can, whenever desired, revert to SI units, it is useful to work out explicitly the relation between the Gaussian quantities (denoted without primes, as in (1.3)) and the SI quantities (denoted with primes, as in (1.1)). In order to do this, we first need to understand a very important property of the Maxwell equations, namely that they imply the existence of *electromagnetic waves* that propagate at the speed of light.

Consider the Maxwell equations (1.1) in a completely empty region of space, where there is no charge density  $\rho'$  and no current density  $\vec{J}'$ . Taking the curl of the last equation, and using the vector identity (we shall assume here we are using Cartesian coordinates)  $\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \nabla^2 \vec{V}$ , we obtain

$$-\nabla^2 \vec{E}' + \vec{\nabla}(\vec{\nabla} \cdot \vec{E}') + \vec{\nabla} \times \frac{\partial \vec{B}'}{\partial t} = 0,\tag{1.4}$$

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<sup>2</sup>Actually, one can do even better by changing the units in which one measures length from the *metre* to the *light second*, or alternatively, changing the unit of time to the *light metre* (the time light takes to travel 1 metre). In either of these systems of units, the speed of light becomes equal to 1.

and hence, since the partial derivative  $\partial/\partial t$  commutes with the spatial partial derivatives in  $\vec{\nabla}$ ,

$$-\nabla^2 \vec{E}' + \vec{\nabla}(\vec{\nabla} \cdot \vec{E}') + \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}') = 0. \quad (1.5)$$

Using the first equation in (1.1) (with  $\rho' = 0$ ) and the second equation (with  $\vec{J}' = 0$ ) then gives

$$\nabla^2 \vec{E}' - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}' = 0. \quad (1.6)$$

Analogous manipulations show that  $\vec{B}'$  satisfies an identical equation. We see, therefore, that the electric and magnetic fields satisfy an equation for waves that propagate at the speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 2.99792 \times 10^8 \text{ metres/second}. \quad (1.7)$$

This is precisely the speed of light *in vacuo*, and these wave solutions describe the propagation of radio waves, light, etc.

With this preliminary, we are nearly ready to establish the relation between the SI units used in (1.1), and the Gaussian units used in (1.3). The procedure for doing this is to introduce constant factors  $\alpha, \beta, \gamma$  and  $\delta$  that relate the primed to the unprimed quantities,

$$\vec{E}' = \alpha \vec{E}, \quad \vec{B}' = \beta \vec{B}, \quad \rho' = \gamma \rho, \quad \vec{J}' = \delta \vec{J}, \quad (1.8)$$

and to fix the values of these constants by demanding that plugging (1.8) into (1.1) should give (1.3). It is essential, in doing so, that we have the relation (1.7) between  $c, \mu_0$  and  $\epsilon_0$ . Elementary algebra then gives

$$\alpha = \frac{\gamma}{4\pi\epsilon_0}, \quad \beta = \frac{\gamma}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}}, \quad \delta = \gamma. \quad (1.9)$$

Notice that the fact that the constants  $\gamma$  and  $\delta$ , the rescalings of  $\rho$  and  $\vec{J}$ , turned out to be equal is a not a coincidence. These rescalings reflect the fact that the unit of *charge* is being rescaled in the transformation from SI to Gaussian units. Now the dimensions of charge density and current density are

$$[\rho] = (\text{Charge}) L^{-3}, \quad [\vec{J}] = (\text{Charge}) L^{-2} T^{-1}, \quad (1.10)$$

so they must both scale the same way under a rescaling of the charge.

Observe that the value of the constant  $\gamma$  has not yet been determined. This means that we can choose any value for  $\gamma$ , and we may use this freedom in order to make some other equation as nice as possible by removing superfluous constant factors. Consider Coulomb's law, giving the force between two electric charges separated by a distance  $R$ . We again need

to distinguish between the charges  $q'_1$  and  $q'_2$  expressed in SI units, and the charges  $q_1$  and  $q_2$  expressed in Gaussian units. Since we have the relation  $\rho' = \gamma\rho$  between charge *densities* in the two systems, and since the unit of *volume* is the same in the two systems, it follows that the charges will also be related by the same factor of  $\gamma$ :

$$q' = \gamma q. \quad (1.11)$$

Now, in the SI system the force between the two charges is given by

$$F = \frac{q'_1 q'_2}{4\pi\epsilon_0 R^2} = \frac{\gamma^2 q_1 q_2}{4\pi\epsilon_0 R^2}. \quad (1.12)$$

Clearly, since we are free to choose  $\gamma$  to be whatever we like, the most convenient choice is to take

$$\gamma = \sqrt{4\pi\epsilon_0}, \quad (1.13)$$

so that the force between charges  $q_1$  and  $q_2$  is simply

$$F = \frac{q_1 q_2}{R^2}. \quad (1.14)$$

This is precisely the choice made in the Gaussian system of units.

Going back to (1.9), and combining it with the additional relation (1.13), we see that the four constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  are now uniquely determined in terms of  $\mu_0$  and  $\epsilon_0$ . Thus we arrive at the following “dictionary” for relating the SI (primed) quantities to the Gaussian (unprimed) quantities:

$$\begin{aligned} \vec{E}' &= \frac{1}{\sqrt{4\pi\epsilon_0}} \vec{E}, & \vec{B}' &= \sqrt{\frac{\mu_0}{4\pi}} \vec{B}, \\ \rho' &= \sqrt{4\pi\epsilon_0} \rho, & \vec{J}' &= \sqrt{4\pi\epsilon_0} \vec{J}, & q' &= \sqrt{4\pi\epsilon_0} q. \end{aligned} \quad (1.15)$$

With these relations established, we can happily proceed by using the more convenient Gaussian units in this course, and anyone who wishes to re-express things in SI units can do so using the *SI-Gauss Dictionary* (1.15).

### 1.3 Macroscopic media

In principle, every problem in classical electromagnetism can be viewed as a problem formulated in free space, together with a number of electric point charges carried by electrons, protons, etc. In practice, however, it is often the case that the number of individual point charges is so large that it would not be convenient to consider them all separately, and instead, it is preferable to make a “macroscopic approximation.” One obvious example is

the notion of a conductor: It would be very clumsy and unwieldy to treat every electrostatics problem involving a conducting surface as a problem involving  $10^{23}$  or so positive and negative point charges that are bound together in such a way as to make what we conventionally think of as a sheet of metal. Instead, we can typically just forget about the microscopic explanation of why the protons, neutrons and electrons have formed themselves into a metal, and instead simply abstract from this the macroscopic notion of a surface on which the electrostatic potential is constant.

Another example where a macroscopic viewpoint is very useful is when one considers materials (such as glass) that exhibit a dielectric permittivity, or else materials that exhibit a magnetic permeability. One certainly can give a microscopic understanding of why these materials behave as they do, but it is convenient not to have to delve into these details every time we want to work out the effect of a slab of glass in an electrostatics problem.

In order to give a macroscopic formulation of Maxwell's theory in the presence of media, we now interpret  $\vec{E}$  and  $\vec{B}$  as *averaged* values of the electric and magnetic fields, where the averaging is performed over the distance scale of order the interatomic spacing in the medium. The point here is that we don't want to get involved in looking at the (enormous) microscopic variations in the fields that occur on the atomic length scale as one moves around close to individual electrons and protons. Having performed this averaging, the meanings of  $\vec{E}$  and  $\vec{B}$  are the same as they are in free space. For example,  $\vec{E}$  still measures the potential difference between neighbouring points divided by their spatial separation.

We must also introduce two new quantities, called  $\vec{D}$  and  $\vec{H}$ , which are related to  $\vec{E}$  and  $\vec{B}$  respectively. The standard names for all four fields are:

$\vec{E}$ :	Electric field
$\vec{D}$ :	Electric displacement
$\vec{B}$ :	Magnetic induction
$\vec{H}$ :	Magnetic field

In free space, we have

$$\vec{D} = \vec{E}, \quad \vec{H} = \vec{B}. \quad (1.16)$$

In a medium, on the other hand,  $\vec{D}$  represents a “back-reacted” version of  $\vec{E}$ , which takes into account the fact that the positive and negative charges in the medium are displaced because of the presence of the externally-applied  $\vec{E}$  field, and thus they feed back into the system. To leading order, the system of positive and negative charges in the medium (which is neutral on balance) distorts so that each atom or molecule acquires a small electric dipole



moment, leading to a dipole moment density, or *polarisation*,  $\vec{P}$ , and

$$\vec{D} = \vec{E} + 4\pi\vec{P}. \quad (1.17)$$

In a similar way, if the medium has magnetic properties there will be a similar relation

$$\vec{H} = \vec{B} - 4\pi\vec{M}, \quad (1.18)$$

where  $\vec{M}$  is a magnetic dipole density, or *magnetisation*, term.

The effect of all this is that the Maxwell equations are modified in the presence of the medium. Instead of the free-space equations (1.3) we shall now have

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= 4\pi\rho, & \vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} &= \frac{4\pi}{c} \vec{J}, \\ \vec{\nabla} \cdot \vec{B} &= 0, & \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0, \end{aligned} \quad (1.19)$$

Notice that it is the first two equations, the ones that have the  $\rho$  and  $\vec{J}$  source-terms on the right-hand side, that are modified. The remaining two equations are completely unchanged from their free-space forms.

Later in the course, we shall return to a more detailed discussion of the microscopic understanding of dielectric phenomena.

A common situation is when the medium is completely homogeneous and isotropic (meaning that it is the same everywhere, and the same in all directions), and for which  $\vec{D}$  and  $\vec{H}$  are simply constant multiples of  $\vec{E}$  and  $\vec{B}$  respectively:

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}. \quad (1.20)$$

The constant  $\epsilon$  is called the *relative permittivity* of the medium, and the constant  $\mu$  is called the *relative permeability* of the medium. In free space, where (1.16) holds, we clearly have

$$\epsilon = 1, \quad \mu = 1. \quad (1.21)$$

In this course, when we consider electromagnetism in a medium, we shall commonly assume the relations (1.20).<sup>3</sup> Note that the permittivity  $\epsilon$  of the medium is also often called the *dielectric constant*.

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<sup>3</sup>Note that the adjective “relative” preceding the permittivity or permeability is really superfluous when using the Gaussian system of units, since the permittivity and permeability of free space (i.e. vacuum) are both equal to unity. In the SI system the adjective “relative” is used in order to signify that it means the factor by which the permittivity or permeability is larger (or smaller) than that in the vacuum. We shall drop the prefix “relative” from now on.

## 1.4 Boundary conditions at media interfaces

A situation that one encounters frequently when studying physical problems in electromagnetism is where there is a boundary or interface between two different materials or media. The simplest such situation in electrostatics is the case where there is a conducting surface in otherwise free space. Another example would be an interface between two materials with different dielectric constants. In fact the conductor in free space can just be viewed as a special case of the interface between two dielectric materials, with one of them (free space) having  $\epsilon = 1$  and the other (the conductor) having  $\epsilon = \infty$ .

The boundary conditions on the electric and magnetic fields at an interface between two media can be determined by performing appropriate integrals of the Maxwell equations (1.19). Let us label the media by “1” and “2,” and likewise place “1” and “2” subscripts on the various electric and magnetic fields on the two sides of the interface.

Beginning with  $\vec{D}$ , we can integrate  $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$  over a so-called “Gaussian pillbox” that straddles the interface. The pillbox is like a very short length of circular cylinder, with the ends capped off so as to form a closed volume. One should imagine that the size of the whole pillbox is very small, and in fact eventually one takes the limit where the size tends to zero. At all stages in the limiting process, the height of the box (i.e. the length of the cylinder) is very small compared with its radius. The caps of the cylinder are taken to be parallel to the interface, with the interface slicing through the box; one cap on each side.

The *divergence theorem* states that for any vector field  $\vec{v}$  we have

$$\int_V \vec{\nabla} \cdot \vec{v} dV = \int_S \vec{v} \cdot d\vec{S}, \quad (1.22)$$

where  $S$  is a closed surface enclosing a volume  $V$ . Integrating  $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$  over the pillbox and using (1.22), we therefore find

$$\int_V \vec{\nabla} \cdot \vec{D} dV = \int_S \vec{D} \cdot d\vec{S} = 4\pi \int_V \rho dV = 4\pi q, \quad (1.23)$$

where  $q$  is the charge inside the pillbox. Because the height of the pillbox is taken to be very small compared to its diameter, we can neglect the contributions to the  $\vec{D}$  integral coming from the sides. Since the pillbox itself will eventually be taken to have infinitesimal size we can think of the interface where the pillbox is placed as being planar. Let  $\vec{n}$  be the unit normal vector pointing from medium 1 into medium 2. If the cross-sectional area of the pillbox is  $\Delta A$ , then (1.23) gives

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) \Delta A = 4\pi\sigma \Delta A, \quad (1.24)$$

where  $\sigma$  is the surface charge density<sup>4</sup> at the interface. Thus we have

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 4\pi\sigma \quad (1.25)$$

at the interface.

By the same token, the integration of the Maxwell equation  $\vec{\nabla} \cdot \vec{B} = 0$  over the same pillbox gives

$$\vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0 \quad (1.26)$$

at the interface. The zero on the right-hand side reflects the fact that there are no magnetic charges.

Further boundary conditions follow by appropriately integrating the remaining two Maxwell equations across the interface. This time, we consider a rectangular loop formed by two parallel line elements, one on each side of the interface, joined into a loop by adding connecting lines at the two ends. In the discussion that follows we first take the lengths of the two end connecting lines to zero, meaning that the area of the loop will go to zero. Eventually, we also take the lengths of the long sides of the loop to zero also. We now make use of Stokes' theorem, which states that for any vector field  $\vec{v}$  we have

$$\int_{\Sigma} (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{\ell}, \quad (1.27)$$

where  $\Sigma$  denotes an (open) surface whose boundary is the closed loop  $C$ .

Suppose again the unit normal from medium 1 to medium 2 is  $\vec{n}$  at the chosen point on the interface that we are considering. Let  $\vec{m}$  be a unit vector that is perpendicular to the rectangular loop, and therefore it is tangent to the interface at the selected point. Integrating the Maxwell equation  $\vec{\nabla} \times \vec{E} = -(1/c)\partial\vec{B}/\partial t$  over the area  $\Sigma$  of the loop and applying (1.27) gives

$$\int_{\Sigma} (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{1}{c} \frac{\partial}{\partial t} \int_{\Sigma} \vec{B} \cdot d\vec{S}. \quad (1.28)$$

Since  $\vec{B}$  will be assumed to be finite, as also is  $\partial\vec{B}/\partial t$ , it follows that the right-hand side goes to zero as we send the lengths of the two end lines of the loop to zero, since the area of the rectangular loop will go to zero also. If the length of each of the two long sides of the loop is  $\Delta\ell$ , then it follows that

$$0 = \oint \vec{E} \cdot d\vec{\ell} = (\vec{m} \times \vec{n}) \cdot (\vec{E}_2 - \vec{E}_1) \Delta\ell, \quad (1.29)$$

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<sup>4</sup>The surface charge resides entirely in the surface of the interface; it is a charge per unit area. In the idealisation where the interface is a sharp boundary between the two media, it is an infinitesimally thin sheet of charge.

(since the line integrations along the long sides of the rectangle are in the directions of the unit vectors  $\pm \vec{m} \times \vec{n}$ ). Using the vector identity  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$  it follows that

$$\vec{m} \cdot [\vec{n} \times (\vec{E}_2 - \vec{E}_1)] = 0. \quad (1.30)$$

Now, we could have chosen a rectangular loop with any orientation for this discussion. That is to say,  $\vec{m}$  could be a unit vector pointing in any direction within the plane of the interface. It must therefore be that<sup>5</sup>

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0. \quad (1.31)$$

Finally, we perform an analogous integral over the last Maxwell equation,  $\vec{\nabla} \times \vec{H} = 1/c \partial \vec{D} / \partial t + (4\pi/c) \vec{J}$ . The finiteness of  $\partial \vec{D} / \partial t$  means that its area integral over the loop goes to zero, but  $\vec{J}$  can have a non-zero area integral in general, since there can be a surface current density<sup>6</sup>  $\vec{K}$ . Thus we find

$$\oint \vec{H} \cdot d\vec{\ell} = (\vec{m} \times \vec{n}) \cdot (\vec{H}_2 - \vec{H}_1) \Delta \ell = \frac{4\pi}{c} \vec{m} \cdot \vec{K} \Delta \ell, \quad (1.32)$$

and since the left-hand side can be rewritten as  $\vec{m} \cdot [\vec{n} \times (\vec{H}_2 - \vec{H}_1)]$ , and the equation must hold for all choices of direction for the tangent vector  $\vec{m}$ , we conclude that

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c} \vec{K}. \quad (1.33)$$

To summarise, the boundary conditions we have derived above in (1.25), (1.26), (1.31) and (1.33) are

$$\begin{aligned} \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) &= 4\pi\sigma, & \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) &= 0, \\ \vec{n} \times (\vec{E}_2 - \vec{E}_1) &= 0, & \vec{n} \times (\vec{H}_2 - \vec{H}_1) &= \frac{4\pi}{c} \vec{K}. \end{aligned} \quad (1.34)$$

These give the junction conditions at the interface between medium 1 and medium 2, where  $\vec{n}$  is the unit normal vector pointing from 1 to 2 at the interface,  $\sigma$  is the surface charge density and  $\vec{K}$  is the surface current density. Note that the first line of (1.34) comprises conditions on the components of the fields *normal* to the interface, whilst the second line comprises conditions on the components *parallel* to the interface.

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<sup>5</sup>Note that although eqn (1.31) is written as a three-dimensional vector equation, the right-hand side in fact has only two non-vanishing components, since it is necessarily orthogonal to the unit vector  $\vec{n}$ . This is why we can deduce from eqn (1.30), which holds for all vectors  $\vec{m}$  lying in the plane perpendicular to  $\vec{n}$ , that eqn (1.31) must hold.

<sup>6</sup>Like the surface charge density we discussed previously, here the surface current resides entirely within the infinitesimal thickness of the interface between the two media. It is a current per unit length.

A special case of frequent interest arises for an electric field in free space, in the presence of a conducting surface. In the free-space region we have  $\vec{D} = \vec{E}$ , and the conductor can be viewed as the surface of a medium having infinite dielectric constant, which means that  $\vec{E} = 0$  there. Thus the pillbox integration of  $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$  becomes just the integral of  $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ , with  $\vec{E} = 0$  in “medium 1.” The upshot is that the first and third junction conditions in (1.34) become (dropping the “2” subscript in the free-space region outside the conductor)

$$\vec{n} \cdot \vec{E} = 4\pi\sigma, \quad \vec{n} \times \vec{E} = 0 \quad (1.35)$$

at the surface. The second equation says that there is no component of  $\vec{E}$  *tangent* to the conducting surface, and the first equation says that the *normal* component of the electric field at the conductor is equal to  $4\pi\sigma$ .

It should be emphasised that the expression in eqn (1.35) for the surface charge density on the conductor assumes that there is an electric field only on one side of the conductor. If there were electric fields on both sides of a conductor in a vacuum, then the surface charge density would be given by  $\sigma = (4\pi)^{-1} \vec{n} \cdot (\vec{E}_2 - \vec{E}_1)$ , where the unit normal vector points into region 2.

## 1.5 Gauge potentials

When solving Maxwell’s equations, it is often convenient to express the electric and magnetic fields in terms of potentials. This has the advantage that two of the four Maxwell equations are then explicitly solved from the outset, leaving just two more, which now become second order equations for the potentials.

Specifically, the two Maxwell equations that are solved by introducing potentials are the two that do not have  $\rho$  or  $\vec{J}$  as sources, namely

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \quad (1.36)$$

Notice that these two equations are exactly the same whether one is considering the free-space case (1.3) or the case where media are present (1.19).<sup>7</sup>

To introduce potentials we begin by considering  $\vec{\nabla} \cdot \vec{B} = 0$ . This can be solved by writing  $\vec{B}$  as the curl of a vector:

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (1.37)$$

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<sup>7</sup>These two Maxwell equations are known as *Bianchi identities*. By contrast, the remaining two Maxwell equations are known as the *Maxwell field equations*.

since the divergence of the curl of any vector vanishes identically. Crucially, the converse is also true (at least locally): if the divergence of a vector vanishes, then that vector can always be written as the curl of some vector field.

Passing now to the second equation in (1.36), we plug in (1.37) and deduce (after using the fact that the partial time derivative commutes with  $\vec{\nabla}$ ) that

$$\vec{\nabla} \times \left( \vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0. \quad (1.38)$$

If a vector has vanishing curl it can always be written (at least locally) as the gradient of a function, and so we can write  $\vec{E} + (1/c)\partial\vec{A}/\partial t = -\vec{\nabla}\phi$ .

To summarise, we can write  $\vec{E}$  and  $\vec{B}$  in terms of a scalar potential  $\phi$  and a vector potential  $\vec{A}$  as

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \quad (1.39)$$

The choice of the potentials  $\phi$  and  $\vec{A}$  that give rise to given  $\vec{E}$  and  $\vec{B}$  fields via (1.39) is not unique. Since the curl of a gradient vanishes identically, we get the same  $\vec{B}$  if the gradient of an arbitrary function  $\lambda$  is added to  $\vec{A}$ . Thus if we define

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda, \quad (1.40)$$

then  $\vec{A}'$  gives the same  $\vec{B}$  as does  $\vec{A}$ :

$$\vec{B}' \equiv \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}\lambda = \vec{\nabla} \times \vec{A} = \vec{B}. \quad (1.41)$$

Note that  $\lambda$  can be a completely arbitrary function, depending both on the spatial coordinates and on time.

It is now evident that if, at the same time, we transform  $\phi$  to

$$\phi' = \phi - \frac{1}{c} \frac{\partial \lambda}{\partial t}, \quad (1.42)$$

then we shall also find that  $\phi'$  and  $\vec{A}'$  give rise to the same  $\vec{E}$ , via (1.39), as do  $\phi$  and  $\vec{A}$ .

To summarise, the expressions (1.39) for the electromagnetic fields give the *same*  $\vec{E}$  and  $\vec{B}$  if we transform the potentials  $\phi$  and  $\vec{A}$  according to

$$\phi \longrightarrow \phi' = \phi - \frac{1}{c} \frac{\partial \lambda}{\partial t}, \quad \vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda, \quad (1.43)$$

where  $\lambda$  is an *arbitrary* function of  $\vec{r}$  and  $t$ . The transformations (1.43) are known as *gauge transformations*. The potentials  $\phi$  and  $\vec{A}$  are known as *gauge potentials*.

In this course, we shall mostly be concerned with the situation when the electric and magnetic fields are *static*, i.e. they are independent of time. It is evident from the Maxwell equations (1.3) that under these circumstances the electric and magnetic fields are totally decoupled from one another. Just because  $\vec{E}$  and  $\vec{B}$  are static, it does not necessarily mean that the potentials  $\phi$  and  $\vec{A}$  have to be taken to be time independent.<sup>8</sup> However, one would have to be quite perverse to choose to complicate a time-independent problem by opting to describe it in terms of time-dependent potentials! Thus in practice, in the static case, we always choose to take  $\phi$  and  $\vec{A}$  to be time-independent, and so (1.39) becomes simply

$$\vec{E} = -\vec{\nabla}\phi, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \quad (1.44)$$

The residual part of the gauge transformations (1.43) that preserves the time-independence of the gauge potentials is given by taking the gauge parameter  $\lambda$  to be of the form

$$\lambda(\vec{r}, t) = -c k t + \lambda(\vec{r}), \quad (1.45)$$

where  $k$  is a constant and  $\lambda(\vec{r})$  is an arbitrary function of position. (The inclusion of the factor of the speed of light  $c$  and the minus sign in the gauge transformation (1.45) is just for convenience, so that the final expression takes the simplified form (1.46) below.) Thus in the static case we have independent gauge transformations under which

$$\phi \longrightarrow \phi' = \phi + k, \quad \vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{r}). \quad (1.46)$$

The gauge transformation for  $\phi$  is just the familiar freedom to add an arbitrary constant to the electrostatic potential.

## 1.6 Electric field of a point charge; Coulomb's law

It was found experimentally long ago, by Cavendish, Coulomb and others, that the force between two charges  $q_1$  and  $q_2$  in free space was proportional to the product  $q_1 q_2$ ; was inversely proportional to the square of the distance between them (let us assume point charges, for simplicity); and was directed along the line joining the two charges. Furthermore, the force is attractive if  $q_1 q_2$  is negative, and repulsive if  $q_1 q_2$  is positive. If we work in Gaussian

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<sup>8</sup>One way to see this is from the gauge transformations in eqns (1.43): we could start with time-independent gauge potentials  $\phi(\vec{r})$  and  $\vec{A}(\vec{r})$  describing the static fields  $\vec{E}(\vec{r})$  and  $\vec{B}(\vec{r})$ , and then willfully transform to time-dependent gauge potentials  $\phi'(\vec{r}, t)$  and  $\vec{A}'(\vec{r}, t)$  by means of the gauge transformations in (1.43), by allowing the gauge parameter  $\lambda(\vec{r}, t)$  to have time dependence.

units, then as discussed in section 1.2, the magnitude of the force is simply equal to  $q_1 q_2$  divided by the square of the separation. All this is summarised in the equation

$$\vec{F} = q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}, \quad (1.47)$$

which gives the force on  $q_1$  due to  $q_2$ , where  $\vec{r}_1$  and  $\vec{r}_2$  are the position vectors of the two point charges  $q_1$  and  $q_2$ .

Coulomb also found that the force on a charge  $q$  was given by

$$\vec{F} = q \vec{E}, \quad (1.48)$$

and so we can read off from (1.47) that the electric field at the point  $\vec{r}$  due to a charge  $q_1$  located at the point  $\vec{r}_1$  is given by

$$\vec{E}(\vec{r}) = q_1 \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3}. \quad (1.49)$$

A very important and fundamental feature of electromagnetism is that it is described by a system of *linear* equations (see (1.3)), and so it obeys the principal of superposition. In particular, this means that if there are  $N$  point charges  $q_a$  located at positions  $\vec{r}_a$ , then the total electric field at  $\vec{r}$  is simply the sum of the individual contributions from each charge:

$$\vec{E}(\vec{r}) = \sum_{a=1}^N q_a \frac{\vec{r} - \vec{r}_a}{|\vec{r} - \vec{r}_a|^3}. \quad (1.50)$$

We can generalise this result to the case where there is a continuum of charge, with charge density  $\rho$ . In the infinitesimal volume  $dxdydz$  in the neighbourhood of the point  $\vec{r}$  there will be an infinitesimal charge  $\rho(\vec{r})dxdydz$ . For convenience, we may write the volume element  $dxdydz$  as  $d^3\vec{r}$ . Then, we simply generalise the discrete sum (1.50) to an integral, and obtain

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d^3\vec{r}'. \quad (1.51)$$

The inverse process, of passing from the continuum result (1.51) to the discrete sum (1.50), can be achieved by means of the Dirac delta function. In one dimension the Dirac delta function  $\delta(x - a)$  is a “spike” of zero width, infinite height, and total area=1 that has the property

$$f(a) = \int_{x_1}^{x_2} f(x) \delta(x - a) dx \quad (1.52)$$

for any function  $f(x)$ , provided that the integration interval  $[x_1, x_2]$  encompasses the point  $x = a$ . (The integral would give zero if  $a$  did not lie inside the integration interval.) We then define the three-dimensional delta function  $\delta^3(\vec{r} - \vec{a})$  as

$$\delta^3(\vec{r} - \vec{a}) \equiv \delta(x - a_1) \delta(y - a_2) \delta(z - a_3), \quad (1.53)$$



where  $a_1$ ,  $a_2$  and  $a_3$  are the  $x$ ,  $y$  and  $z$  components of the vector  $\vec{a}$ : i.e.  $\vec{a} = (a_1, a_2, a_3)$ . (In another commonly-used, although slightly cumbersome, notation, a vector  $\vec{a}$  would be written in terms of its Cartesian components  $a_1$ ,  $a_2$  and  $a_3$  as  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ , where  $\vec{i}$ ,  $\vec{j}$  and  $\vec{k}$  are unit vectors along the  $x$ ,  $y$  and  $z$  directions. The notation  $\vec{a} = (a_1, a_2, a_3)$  is equivalent, but more economical.) Clearly, the three-dimensional delta function has the property that

$$f(\vec{a}) = \int_V f(\vec{r}) \delta^3(\vec{r} - \vec{a}) d^3\vec{r}, \quad (1.54)$$

provided that the integration volume  $V$  encompasses the point  $\vec{r} = \vec{a}$  where the delta function has its “spike.” If the point  $\vec{a}$  does not lie within the volume  $V$  then the integral gives zero.

It should be noted that the Dirac delta function, and its three-dimensional generalisation that we discussed above, are both *symmetric* functions, in the sense that

$$\delta(-x) = \delta(x), \quad \delta^3(-\vec{r}) = \delta^3(\vec{r}). \quad (1.55)$$

Using the delta function, we can then write the charge density  $\rho$  for the set of charges in (1.50) as

$$\rho(\vec{r}) = \sum_{a=1}^N q_a \delta^3(\vec{r} - \vec{r}_a). \quad (1.56)$$

Substituting this into (1.51), and using (1.54), we indeed recover (1.50).

## 1.7 Gauss’s law

If we are considering electrostatics, i.e. the situation where there is a time-independent electric field and no magnetic field, the Maxwell equations (1.3) in free space reduce to

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{E} = 0. \quad (1.57)$$

By integrating the first equation over a volume  $V$ , and using the divergence theorem (1.22), we obtain Gauss’s law

$$\int_S \vec{E} \cdot d\vec{S} = 4\pi Q, \quad (1.58)$$

where  $S$  is the closed surface surrounding the volume  $V$ , and  $Q$  is the total charge contained within the volume  $V$ :

$$Q = \int_V \rho dV. \quad (1.59)$$

(Actually, historically, the Maxwell equation  $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$  was discovered experimentally in its equivalent integrated form (1.58).)

It is instructive to examine Gauss's law in the special case of a single point charge  $q$ . Since, when we set up a Cartesian coordinate system we can choose the origin to be at any arbitrary point, it is convenient to choose it so that the charge sits at the origin. Using (1.49) we see that the electric field of the charge will be given by

$$\vec{E} = \frac{q\vec{r}}{r^3}. \quad (1.60)$$

Let us check that this is consistent with the Maxwell equation  $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ , and its integrated form (1.58).

First, we calculate the divergence of  $\frac{\vec{r}}{r^3}$ . Clearly we have  $\vec{\nabla} \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ , and since  $r^2 = x^2 + y^2 + z^2$  we have, differentiating,  $2r \frac{\partial r}{\partial x} = 2x$ , etc., and hence  $\frac{\partial r}{\partial x} = \frac{x}{r}$ , etc. Thus we have

$$\vec{\nabla} \cdot \vec{r} = 3, \quad \vec{\nabla} r = \frac{\vec{r}}{r}, \quad (1.61)$$

and so<sup>9</sup>

$$\vec{\nabla} \cdot \left( \frac{\vec{r}}{r^3} \right) = \frac{\vec{\nabla} \cdot \vec{r}}{r^3} - \frac{3\vec{r}}{r^4} \cdot \frac{\vec{r}}{r} = \frac{3}{r^3} - \frac{3}{r^3} = 0. \quad (1.62)$$

This shows that for (1.60) we have in general  $\vec{\nabla} \cdot \vec{E} = 0$ . This is perfectly correct away from the origin, since we hope to find  $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$  and indeed  $\rho = 0$  away from the origin. However, at the origin the calculation (1.62) is not trustworthy, because there are denominators that go to zero at  $r = 0$ . The safe way to handle this is to consider the integrated form of the equation, in (1.58).

Let us take the volume  $V$  in (1.58) to be a sphere of radius  $R$  centred on the origin. Plugging in (1.60), we shall then have on the left-hand side

$$q \int_S \frac{\vec{r}}{r^3} \cdot d\vec{S} = q \int_S \frac{\vec{n} \cdot d\vec{S}}{R^2} = q \int d\Omega = 4\pi q, \quad (1.63)$$

where we have defined the unit vector  $\vec{n} \equiv \frac{\vec{r}}{r}$ , which is the unit outward normal on the sphere of radius  $r$ , and where  $d\Omega = \frac{\vec{n} \cdot d\vec{S}}{R^2}$  is the area element on the unit sphere (i.e. the solid angle element). In terms of spherical polar coordinates  $(r, \theta, \varphi)$ , we have  $d\Omega = \sin \theta d\theta d\varphi$ .

We obtained the result (1.63) by choosing to integrate over a sphere of radius  $R$  centred on the origin, but it is obvious that the result would be the same for *any* closed surface that surrounded the origin, and that instead we would get zero if the integration surface did not surround the origin. This can be seen as follows. Let the surface of the sphere of radius  $R$  be denoted by  $S$ , and now consider also some other closed surface  $S'$ , of arbitrary shape.

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<sup>9</sup>We are using the fact that for any scalar field  $f$  and vector field  $\vec{v}$ , we have  $\vec{\nabla} \cdot (f\vec{v}) = (\vec{\nabla} f) \cdot \vec{v} + f \vec{\nabla} \cdot \vec{v}$ .

For the sake of definiteness let's suppose that  $S'$  lies entirely outside  $S$ . We shall use  $\tilde{V}$  to denote the 3-volume between the surfaces  $S$  and  $S'$ . Now integrate the divergence of the electric field  $\vec{E}$  over the volume  $\tilde{V}$ , and use the divergence theorem:

$$\int_{\tilde{V}} \vec{\nabla} \cdot \vec{E} d^3\vec{r} = \int_{S'} \vec{E} \cdot d\vec{S} - \int_S \vec{E} \cdot d\vec{S}. \quad (1.64)$$

Note that the total surface bounding the volume  $\tilde{V}$  is the sum of the outer surface  $S'$  and the inner surface  $S$ . When applying the divergence theorem the area element should always be directed *outwards* from the volume, which is why the second integral on the right-hand side of (1.64) has a minus sign. Now in the volume integral on the left-hand side of (1.64) we can use the result (1.62) that  $\vec{\nabla} \cdot \vec{E} = 0$ , since the volume  $\tilde{V}$  in (1.64) does not include the origin. Hence we have

$$\int_{S'} \vec{E} \cdot d\vec{S} = \int_S \vec{E} \cdot d\vec{S}, \quad (1.65)$$

thus proving that the result in (1.63) is independent of how we choose the surface of integration, as long as it encloses the origin.

We have seen that the volume integral of  $\vec{\nabla} \cdot \vec{E}$  for the point charge at the origin will give  $4\pi q$  if the volume of integration includes the origin, whilst it will instead give zero if the volume does not include the origin. The conclusion from this is that the function  $\rho$  on the right-hand side of the Maxwell equation  $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$  must be a three-dimensional delta function centred on the origin, and that therefore the precise calculation of  $\vec{\nabla} \cdot \vec{E}$  gives

$$\vec{\nabla} \cdot \vec{E} = 4\pi q \delta^3(\vec{r}), \quad \text{for } \vec{E} = \frac{q\vec{r}}{r^3}. \quad (1.66)$$

In other words, a point charge  $q$  located at the origin is described by the charge density

$$\rho(\vec{r}) = q \delta^3(\vec{r}). \quad (1.67)$$

## 1.8 Electrostatic potential

In section 1.5 we introduced gauge potentials in terms of which the electric and magnetic fields could be expressed. For the case of electrostatics, we have the particularly simple situation that the electric field is written purely in terms of a scalar potential  $\phi$ , with

$$\vec{E} = -\vec{\nabla}\phi, \quad (1.68)$$

where an arbitrary constant  $k$  can clearly be added to  $\phi$  without altering  $\vec{E}$ . We can see by inspection that the second Maxwell equation in (1.57) is identically satisfied when we

write  $\vec{E} = -\vec{\nabla}\phi$ , and also that the first (vacuum) Maxwell equation in (1.57) becomes the Poisson equation

$$\nabla^2\phi = -4\pi\rho. \quad (1.69)$$

Solving any problem in electrostatics is therefore reduced to solving the Poisson equation (1.69) for a given charge density  $\rho$ , subject to given boundary conditions. This is called a *Boundary Value Problem*.

First, let us consider the potential due to a point charge  $q$  in an infinite free space. Taking the charge to be located at the origin for convenience, we therefore need to find a function  $\phi$  such that  $-\vec{\nabla}\phi$  is equal to  $\vec{E} = \frac{q\vec{r}}{r^3}$  as in (1.60). From the second equation in (1.61) we see that  $\vec{\nabla}(\frac{1}{r}) = -\frac{\vec{r}}{r^3}$ , and so we may take the electrostatic potential to be

$$\phi = \frac{q}{r}. \quad (1.70)$$

Of course having the charge located at the origin was an inessential convenience, and we can readily write down the answer in the case where the charge is instead located at the point  $\vec{r}_1$  just by shifting the origin of the Cartesian axes, giving

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_1|}. \quad (1.71)$$

Note that because of the residual gauge symmetry  $\phi \rightarrow \phi + \text{constant}$  we can add an arbitrary constant to  $\phi$  if we wish, but in fact the choice of this “integration constant” that has been made in writing (1.71) is already rather natural, since it means that  $\phi(\vec{r})$  goes to zero as  $\vec{r}$  goes to infinity.

Because of the linearity of the equations the generalisation to a system of  $N$  point charges  $q_a$  located at positions  $\vec{r}_a$  is immediate:

$$\phi(\vec{r}) = \sum_{a=1}^N \frac{q_a}{|\vec{r} - \vec{r}_a|}. \quad (1.72)$$

For the case of a continuous charge distribution, we similarly have

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'. \quad (1.73)$$

These expressions are analogous to equations (1.50) and (1.51) for the electric field. Indeed, one can easily explicitly see that calculating  $-\vec{\nabla}\phi$  for (1.72) or (1.73) gives (1.50) or (1.51) respectively. To do these calculations, one must be careful to note that  $\vec{\nabla}$  means the gradient

with respect to the coordinates in  $\vec{r}$ , and that  $\vec{r}_a$  in (1.72), or  $\vec{r}'$  in (1.73), are treated as constants in the differentiation. Thus one uses the result that

$$\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}. \quad (1.74)$$

We had previously obtained the result (1.66) that the electric field for a point charge  $q$  at the origin, namely  $\vec{E} = \frac{q\vec{r}}{r^3}$ , satisfies  $\vec{\nabla} \cdot \vec{E} = 4\pi q \delta^3(\vec{r})$ . Since  $\vec{E} = -\vec{\nabla}\phi$  with  $\phi = \frac{q}{r}$ , we therefore have

$$\nabla^2 \phi = -4\pi q \delta^3(\vec{r}). \quad (1.75)$$

More generally, if the charge  $q$  is located at  $\vec{r} = \vec{r}'$  so that the potential is given by  $\phi(\vec{r}) = q|\vec{r} - \vec{r}'|^{-1}$ , we shall therefore have

$$\nabla^2 \phi = -4\pi q \delta^3(\vec{r} - \vec{r}'). \quad (1.76)$$

From this we can read off a result that will be useful frequently in the future, namely that

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta^3(\vec{r} - \vec{r}'). \quad (1.77)$$

## 1.9 Electrostatic energy

The force on a charge  $q$  in an electric field  $\vec{E}$  is given by  $\vec{F} = q\vec{E}$ . It follows that the work done in moving it from a point  $P_1$  to a point  $P_2$  is given by

$$\Delta W = - \int_{P_1}^{P_2} \vec{F} \cdot d\vec{\ell} = -q \int_{P_1}^{P_2} \vec{E} \cdot d\vec{\ell}. \quad (1.78)$$

Using  $\vec{E} = -\vec{\nabla}\phi$  therefore gives

$$\Delta W = q \int_{P_1}^{P_2} \vec{\nabla}\phi \cdot d\vec{\ell} = q\phi(P_2) - q\phi(P_1). \quad (1.79)$$

Note that this result is independent of the path the charge follows as it moves from  $P_1$  to  $P_2$ . From eqn (1.79), we can read off the potential energy  $U$  of the charge  $q$  in the electrostatic field as being

$$U = q\phi. \quad (1.80)$$

If we consider a system of  $N$  charges  $q_a$  at points  $\vec{r}_a$  in free space, then the electrostatic energy of a particular charge  $q_a$  in to the potential due to the other  $N - 1$  charges will therefore be

$$U_a = q_a \sum_{b \neq a} \frac{q_b}{|\vec{r}_a - \vec{r}_b|}, \quad (1.81)$$

where the sum is taken over the remaining  $N - 1$  charges. The total potential energy will then be given by

$$U = \sum_{a=1}^N \sum_{b < a} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|}. \quad (1.82)$$

(The second summation is over  $b < a$  rather than over all  $b \neq a$  to avoid a double counting: the energy of charge 1 in the field of charge 2 is the same as the energy of charge 2 in the field of charge 1, but this energy counts *only once* to the total energy, not twice.) One way to view this is that we assemble the charges one at a time. Bringing up the first charge requires no energy, since there is no electrostatic potential (apart from its own) at that step. Bringing up the second charge requires doing work against the potential of the first charge that is already there. Bringing up the third charge requires doing work against the potentials of the first and second charges. And so on.

It is easy to see that another way of writing (1.82) is

$$U = \frac{1}{2} \sum_a \sum_{b \neq a} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|}. \quad (1.83)$$

where it is understood that, aside from the exclusion  $b \neq a$ , the indices  $a$  and  $b$  range over 1 to  $N$ . We can rewrite the summations like this because the quantity  $\frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|}$  that is being summed over is symmetric under exchanging the labels  $a$  and  $b$ . Note that the excluded terms where  $b = a$  would be infinite, since the denominator  $|\vec{r}_a - \vec{r}_b|$  would be zero. In some sense one could think of the excluded terms as representing the “self-energies” of the individual charges in their own electrostatic potentials. Note that although these terms would be infinite, they would in a sense be harmless even if they were included in the double summation, since they would be independent of the locations of the charges. What one really cares about when considering a potential energy is how it *changes* when the charges are moved around, since the utility of the concept of potential energy is to calculate forces by seeing how it changes when the charges are displaced. Thus an additive constant term (albeit infinite) in the expression for the potential energy, that is unaltered as the charges are moved, could just be neglected, or subtracted out, without really affecting anything physical.<sup>10</sup>

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<sup>10</sup>A very similar issue of infinities arises in the quantum theory of the electromagnetic field. This also is resolved, in a procedure known as regularisation and renormalisation, by subtracting out infinities while at the same time being careful to distinguish between quantities that are truly physical and those that are not.

Generalising (1.83) to the case of a continuous charge distribution clearly gives<sup>11</sup>

$$U = \frac{1}{2} \int \int \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r} d^3\vec{r}'. \quad (1.84)$$

In view of the fact that the potential  $\phi(\vec{r})$  is given by (1.73), we can rewrite (1.84) as

$$U = \frac{1}{2} \int \rho(\vec{r})\phi(\vec{r}) d^3\vec{r}. \quad (1.85)$$

We can also rewrite the energy purely in terms of the electric field. To do this, we use the Poisson equation (1.69) to write  $\rho$  in (1.85) as  $\rho = -\frac{1}{4\pi} \nabla^2 \phi$ , giving

$$\begin{aligned} U &= -\frac{1}{8\pi} \int \phi \nabla^2 \phi d^3\vec{r}, \\ &= -\frac{1}{8\pi} \int \vec{\nabla} \cdot (\phi \vec{\nabla} \phi) d^3\vec{r} + \frac{1}{8\pi} \int |\vec{\nabla} \phi|^2 d^3\vec{r}, \\ &= -\frac{1}{8\pi} \int_S \phi \vec{\nabla} \phi \cdot d\vec{S} + \frac{1}{8\pi} \int |\vec{\nabla} \phi|^2 d^3\vec{r}, \\ &= \frac{1}{8\pi} \int |\vec{\nabla} \phi|^2 d^3\vec{r}, \\ &= \frac{1}{8\pi} \int |\vec{E}|^2 d^3\vec{r}. \end{aligned} \quad (1.86)$$

Note that after performing the integration by parts here, we have dropped the surface term coming from the “sphere at infinity” since the fields are assumed to go to zero there; see the discussion below.

Let us pause here to comment on two of the concepts employed in the previous paragraph, namely integration by parts, and also the “sphere at infinity.” In the familiar case of a one-dimensional integral, integration by parts amounts to saying that in the integral

$$\int_a^b u(x) v'(x) dx, \quad (1.87)$$

where  $v'(x)$  means  $dv(x)/dx$ , we can write  $u v'$  as<sup>12</sup>

$$u v' = (u v)' - u' v, \quad (1.88)$$

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<sup>11</sup>Note that here the same issue of infinities from “self-energies” arises again, and here one cannot simply avoid the problem by saying  $\vec{r} \neq \vec{r}'$  (the analogue of the exclusion  $b \neq a$  in (1.83)), since  $\vec{r}$  can be very close to  $\vec{r}'$ , without being equal, and still the integrand would be very large and liable to lead to a divergent expression for the energy. The important point, again, is that the divergent term in the expression for the energy would be a pure constant that could be subtracted out without upsetting any physically-meaningful calculation.

<sup>12</sup>That is, we write  $u v'$  as the total derivative  $(u v)'$  minus the correction term  $u' v$ .

and so, plugging into the integral (1.87), we get

$$\begin{aligned}\int_a^b u v' dx &= \int_a^b (u v)' dx - \int_a^b u' v dx, \\ &= \left[ u(x) v(x) \right]_a^b - \int_a^b u' v dx.\end{aligned}\tag{1.89}$$

In the calculation in eqn (1.86) above, we did something analogous, but in three dimensions. Stated generally, for an arbitrary scalar field  $\psi$  and vector field  $\vec{v}$ , we write  $\psi \vec{\nabla} \cdot \vec{v}$  as the total derivative  $\vec{\nabla} \cdot (\psi \vec{v})$  minus the correction term  $\vec{\nabla} \psi \cdot \vec{v}$ . Integrating over a volume  $V$  bounded by the surface  $S$ , and using the divergence theorem, we have

$$\begin{aligned}\int_V \psi \vec{\nabla} \cdot \vec{v} dV &= \int_V \vec{\nabla} \cdot (\psi \vec{v}) dV - \int_V \vec{\nabla} \psi \cdot \vec{v} dV, \\ &= \int_S \psi \vec{v} \cdot d\vec{S} - \int_V \vec{\nabla} \psi \cdot \vec{v} dV.\end{aligned}\tag{1.90}$$

(In the case of (1.86) above,  $\psi$  is equal to  $\phi$  and  $\vec{v}$  is equal to  $\vec{\nabla} \phi$ .)

Note that we shall frequently be making use of this kind of three-dimensional integration by parts. It is helpful to keep in mind, therefore, that the notion of “integration by parts” is much broader than the special case in (1.89) of integration by parts in one dimension.<sup>13</sup>

The other thing we did in the derivation in (1.86) was to drop the boundary term after integrating by parts. The point here is that in our volume integrations in (1.86), we are integrating over all space. One way to make precise what is going on is to think of first integrating just over the volume  $V$  interior to a sphere of radius  $R$ . So the boundary surface  $S$  is then the surface of the sphere of radius  $R$ . At the end of the day, we shall send  $R$  to infinity, meaning that  $V$  will then be the whole of space, and  $S$  will be the “sphere at infinity.” We can see that in the calculation above it is legitimate to drop that surface integral over the sphere at infinity provided that the integrand, i.e.  $\phi \vec{\nabla} \phi$ , falls off sufficiently fast as  $R$  is sent to infinity.

This will be the case if we assume we are working in the gauge where  $\phi$  goes to zero at infinity. Our localised charge distribution will then have a scalar potential that falls off like  $\frac{Q}{r}$  at large distance (just like the potential of a single point charge), where  $Q$  here is

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<sup>13</sup>Note that in getting from the first to the second line of eqn (1.89) we used the *Fundamental Theorem of Calculus*, which says that for any function  $f(x)$  we have  $\int_a^b f'(x) dx = f(b) - f(a)$ . This is really just a one-dimensional analogue of the three-dimensional divergence theorem  $\int_V (\vec{\nabla} \cdot \vec{A}) dV = \int_S \vec{A} \cdot d\vec{S}$ . For the fundamental theorem of calculus, the one-dimensional integral of the derivative of  $f$  is equal to the “zero-dimensional integral” expressed in terms of the value of  $f$  at its endpoints (i.e. at the boundary of the one-dimensional line interval). For the divergence theorem, the three-dimensional volume integral of the divergence of  $\vec{A}$  is equal to the two-dimensional integral of  $\vec{A}$  over the boundary 2-surface  $S$ .



the total charge contained in the charge distribution. This means that  $\vec{\nabla}\phi$  will fall off like  $-\frac{Q\vec{r}}{r^3} = -\frac{Q\vec{n}}{r^2}$ , where  $\vec{n} = \frac{\vec{r}}{r}$  is the unit vector in the outward radial direction, and thus  $\phi\vec{\nabla}\phi$  will fall off like  $-\frac{Q^2\vec{n}}{r^3}$ . On the other hand, the area element  $d\vec{S} = r^2\vec{n}d\Omega$  of a sphere of radius  $r$  will grow like  $r^2$ . (As usual,  $d\Omega$  is the area element of a unit-radius sphere.) Thus on the spherical surface  $S$  at radius  $R$ , the integral  $\int_S \phi\vec{\nabla}\phi \cdot d\vec{S}$  will be falling off like  $\frac{Q^2}{R}$ , and this means it will go to zero as  $R$  goes to infinity.

If instead we were to choose a gauge where the potential  $\phi$  tended to a non-zero constant, say  $\phi_0$ , at infinity, then at large  $r$  we would have  $\phi = \phi_0 + \frac{Q}{r} + \mathcal{O}(\frac{1}{r^2})$ , and so at large  $r$   $\phi\vec{\nabla}\phi \rightarrow -\phi_0 Q \frac{\vec{n}}{r^2} + \mathcal{O}(\frac{1}{r^3})$ . Since  $\vec{n} \cdot d\vec{S} = r^2 d\Omega$ , the surface integral in the antepenultimate line of eqn (1.86) would therefore give the pure constant  $\frac{1}{2}\phi_0 Q$ . All that this means is that the expression in eqn (1.85) for the energy differs from the expression for the energy in the final line of (1.86) by a pure additive constant  $\frac{1}{2}\phi_0 Q$ . This could have been foreseen in any case by looking at the expression for the energy in eqn (1.85) and seeing what happens if  $\phi$  is shifted by an additive constant,  $\phi \rightarrow \phi + \phi_0$ . It is usually much more convenient just to work in the gauge where  $\phi$  goes to zero at infinity, thus avoiding these complications since the surface term in the penultimate line of (1.86) coming from the “sphere at infinity” then vanishes.

We saw in the final line of eqn (1.86) that by writing  $\vec{E} = -\vec{\nabla}\phi$ , the energy  $U$  can be written as

$$U = \frac{1}{8\pi} \int |\vec{E}|^2 d^3\vec{r}, \quad (1.91)$$

integrated over all space. This leads naturally to the definition of *energy density* in the electric field as

$$w = \frac{1}{8\pi} |\vec{E}|^2. \quad (1.92)$$

It is of interest to apply this result to the electric field at the surface of a conductor. Recall from section 1.4 that we found that the electric field is always normal to the surface of a conductor, and that there is a surface-charge density  $\sigma$  given by  $\vec{n} \cdot \vec{E} = 4\pi\sigma$  (see(1.35)). Furthermore, at the conductor  $\vec{n} \times \vec{E} = 0$ , so there is no tangential component of  $\vec{E}$ . This means that very close to the conductor  $\vec{E}$  must be very nearly parallel to the normal vector  $\vec{n}$ , so  $\vec{E} \approx (\vec{n} \cdot \vec{E}) \vec{n} = 4\pi\sigma \vec{n}$ . Thus in the neighbourhood of the conductor there is an energy density

$$w = \frac{1}{8\pi} |\vec{E}|^2 = 2\pi\sigma^2. \quad (1.93)$$

If an element of the surface of area  $\delta A$  is displaced outwards by a distance  $\delta x$ , the electro-

static energy will then change by an amount

$$\delta U = -w \delta A \delta x = -2\pi\sigma^2 \delta A \delta x, \quad (1.94)$$

and so there is an outward force  $\delta F$  given by  $\delta U = -\delta F \delta x$ . This implies an outward force per unit area (i.e. pressure) given by  $p = \delta F / \delta A$ , with

$$p = 2\pi\sigma^2 = \frac{1}{8\pi} |\vec{E}|^2. \quad (1.95)$$

Note that this formula for the pressure assumes that there is an electric field on only one side of the conducting surface. If there were electric fields  $\vec{E}_1$  on one side and  $\vec{E}_2$  on the other side, then there would be a pressure  $p_1 = \frac{1}{8\pi} |\vec{E}_1|^2$  exerted by the electric field  $\vec{E}_1$  and a pressure  $p_2 = \frac{1}{8\pi} |\vec{E}_2|^2$  exerted by the electric field  $\vec{E}_2$ . The pressure  $p_1$  would be directed outwards from the region 1 (i.e. towards region 2), and the pressure  $p_2$  would be directed outwards from region 2 (i.e. towards region 1). Thus the total pressure  $p = p_1 - p_2$  on the conductor (in the direction from region 1 to region 2) cannot be written purely in terms of total surface charge density  $\sigma = \frac{1}{4\pi} \vec{n} \cdot (\vec{E}_2 - \vec{E}_1)$  in the case when there are electric fields on both sides of the conductor. It is only in the case that there is an electric field on just one side that the formula (1.95) can be applied.

## 1.10 Capacitance

For a system of  $N$  conductors  $X_a$ , held at potentials  $V_a$  (for  $1 \leq a \leq N$ ), and carrying charges  $Q_a$ , the total electrostatic potential energy, following from (1.85), is given by

$$U = \frac{1}{2} \sum_{a=1}^N Q_a V_a. \quad (1.96)$$

Since Maxwell's equations are linear, it follows that the electrostatic potential must always be a linear function of the charge density, and so for this configuration of charged conductors there must exist a linear relation of the form

$$V_a = \sum_{b=1}^N \beta_{ab} Q_b, \quad (1.97)$$

for a certain set of constants  $\beta_{ab}$ . These constants will be characteristic of the geometry of the set of conductors, independent of the charges and the potentials. The set of  $N$  equations (1.97) can be inverted, to express the charges as functions of the potentials:

$$Q_a = \sum_{b=1}^N C_{ab} V_b. \quad (1.98)$$

Each diagonal element  $C_{aa}$  is called the *capacitance* of the corresponding conductor  $X_a$ , and the off-diagonal elements  $C_{ab}$  with  $a \neq b$  are called the *coefficients of induction*.

*In particular, the capacitance of a given conductor is equal to the charge on that conductor when it is held at unit potential, with all the other conductors held at zero potential.* To put this another way, the capacitance  $C$  of a given conductor is  $C = \frac{Q}{V}$ , where all the other conductors are held at zero potential,  $Q$  is the charge on the conductor under discussion, and  $V$  is its electrostatic potential.

Going back to the expression (1.96) for the energy  $U$  of the system of conductors, it follows from (1.98) that we may express it purely in terms of  $C_{ab}$  and the potentials  $V_a$ , as the quadratic form

$$U = \frac{1}{2} \sum_{a=1}^N \sum_{b=1}^N C_{ab} V_a V_b. \quad (1.99)$$

## 2 Uniqueness Theorem and Green Functions

### 2.1 Uniqueness theorem

Whenever one is solving a differential equation, such as the Poisson equation  $\nabla^2 \phi = -4\pi\rho$  that we encounter in electrostatics, the question arises as to what boundary conditions one must impose in order to obtain a unique and well-defined solution. Expressed more physically, one may ask how much boundary information must be specified in order to pin down the physics of the problem completely.

One answer for Poisson's equation is that the solution for the potential  $\phi$  inside a volume  $V$  will be uniquely determined if its value at all points on the (closed) surface  $S$  that bounds  $V$  is specified. For example, if we are solving  $\nabla^2 \phi = -4\pi\rho$  inside a sphere, then the solution will be uniquely determined if the value of  $\phi$  at every point on the surface of the sphere is specified. This type of boundary condition, in which  $\phi$  is specified on  $S$ , is known as a *Dirichlet boundary condition*.

An alternative possibility is to specify not  $\phi$  itself but its *normal derivative*  $\frac{\partial \phi}{\partial n}$  on the boundary, where

$$\frac{\partial \phi}{\partial n} \equiv \vec{n} \cdot \vec{\nabla} \phi. \quad (2.1)$$

That is, the derivative of  $\phi$  along the direction  $\vec{n}$  of the normal to the surface  $S$ . (Recall  $\vec{n}$  is the unit-length normal vector.) This boundary condition, where  $\frac{\partial \phi}{\partial n}$  is specified on  $S$ , is known as a *Neumann boundary condition*. Since the electric field is given in terms of the potential by  $\vec{E} = -\vec{\nabla} \phi$ , this means that for the Neumann boundary condition, the normal component of the electric field is specified on the boundary  $S$ . In this case the solution

for  $\phi$  is again unique, except for the (trivial) point that an arbitrary additive constant is undetermined. Thus, for the two cases the boundary data will be of the form

$$\text{Dirichlet:} \quad \phi(\vec{r})\Big|_S = f_D, \quad (2.2)$$

$$\text{Neumann:} \quad \frac{\partial \phi(\vec{r})}{\partial n}\Big|_S = f_N, \quad (2.3)$$

where  $f_D$  or  $f_N$  is a function, defined for all points *on the boundary*  $S$ , which characterises the boundary data one wishes to impose.

One can also consider mixed boundary conditions, which are Dirichlet on parts of the boundary and Neumann on the rest.

There is in addition a more general possibility known as *Robin boundary conditions*,<sup>14</sup> where the potential  $\phi$  on the boundary obeys

$$\text{Robin:} \quad \left(\frac{\partial \phi}{\partial n} + a\phi\right)\Big|_S = f_R, \quad (2.4)$$

where  $f_R$  is a freely-specifiable function on the boundary, and where  $a$  is a strictly non-negative, but otherwise freely specifiable, function on the boundary. So in this case, the functions  $f_R$  and  $a$  comprise the boundary data for the Robin boundary-value problem. The reason for the restriction that  $a \geq 0$  at all points on  $S$  will become evident shortly.

The uniqueness statements can be proven by supposing the contrary; that is, we suppose that for the given boundary condition on  $S$  and a given charge density  $\rho$ , there exist two *different* solutions to  $\nabla^2 \phi = -4\pi\rho$ . Let these solutions be  $\phi_1$  and  $\phi_2$ . The idea will be to try to prove that actually  $\phi_1 = \phi_2$ , and so the solution is unique. With

$$\nabla^2 \phi_1 = -4\pi\rho \quad \text{and} \quad \nabla^2 \phi_2 = -4\pi\rho, \quad (2.5)$$

it follows by subtraction that the function  $\psi$  defined by

$$\psi \equiv \phi_1 - \phi_2 \quad (2.6)$$

will satisfy Laplace's equation

$$\nabla^2 \psi = 0 \quad (2.7)$$

in the volume  $V$ .

Let us consider the cases of pure Dirichlet or pure Neumann boundary conditions first. Since  $\phi_1$  and  $\phi_2$  by definition satisfy identical boundary conditions on  $S$ , either (2.2) in the Dirichlet case or (2.3) in the Neumann case, it follows that  $\psi$  will satisfy either  $\psi = 0$  (Dirichlet) or  $\frac{\partial \psi}{\partial n} = 0$  (Neumann) on  $S$ .

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<sup>14</sup>Pronounced roughly like ‘‘Roe-ban;’’ Victor Gustave Robin was a 19th century French mathematician.

We now multiply (2.7) by  $\psi$ , integrate over  $V$ , and then perform an integration by parts:

$$\begin{aligned}
0 &= \int_V \psi \nabla^2 \psi \, dV, \\
&= \int_V \left[ \vec{\nabla} \cdot (\psi \vec{\nabla} \psi) - \vec{\nabla} \psi \cdot \vec{\nabla} \psi \right] dV, \\
&= \int_S \psi \vec{\nabla} \psi \cdot d\vec{S} - \int_V |\vec{\nabla} \psi|^2 \, dV, \\
&= \int_S \psi \frac{\partial \psi}{\partial n} \, dS - \int_V |\vec{\nabla} \psi|^2 \, dV. \tag{2.8}
\end{aligned}$$

(The first term on the penultimate line comes by using the divergence theorem (1.22).) Now the area element  $d\vec{S}$  in the surface integral is in the direction of the unit normal  $\vec{n}$  (i.e.  $d\vec{S} = \vec{n} \, dS$ ), and so we see that since either  $\psi = 0$  (Dirichlet) or  $\frac{\partial \psi}{\partial n} \equiv \vec{n} \cdot \vec{\nabla} \psi = 0$  (Neumann) at all points on  $S$ , we are left with

$$\int_V |\vec{\nabla} \psi|^2 \, dV = 0. \tag{2.9}$$

The integrand is everywhere non-negative, and so the integral can only be zero if the integrand vanishes everywhere in  $V$ . But if  $|\vec{\nabla} \psi|^2 = 0$  everywhere in  $V$  it follows that

$$\vec{\nabla} \psi = 0 \tag{2.10}$$

everywhere in  $V$ , and so we conclude that  $\psi = \text{constant}$  everywhere in  $V$ . In other words, we have proved that  $\psi = k$  and hence

$$\phi_1 = \phi_2 + k, \tag{2.11}$$

where  $k$  is a constant.

In the case of Dirichlet boundary conditions we know that  $\phi_1 = \phi_2$  on  $S$ , and so the constant  $k$  must be zero. This proves that  $\phi_1 = \phi_2$  everywhere in  $V$ , thus establishing that the solution is unique.

In the case of Neumann boundary conditions, where only the normal derivative is specified on  $S$ , it is clear that the constant  $k$  in eqn (2.11) can never be determined. This is of no consequence, since  $\phi$  and  $\phi + k$  give rise to the same physical  $\vec{E}$  field in any case. So in the Neumann case too, the solution for the electric field  $\vec{E}$  is unique.

Note that the results above can apply not only to the problem of solving for  $\phi$  inside a finite volume  $V$  with finite-sized closed boundary  $S$ , but also in the case where the volume  $V$  is infinite. A typical example would be when there is a finite-sized surface  $S_1$  (for example a spherical conductor) and the volume  $V$  is taken to be the entire infinite space outside it. In this case there is no actual boundary at infinity, but we can treat the problem by

imagining that we introduce a spherical boundary surface  $S_2$  at some very large radius  $R$ , and eventually we send  $R$  to infinity. When  $R$  is large but finite, we have a finite volume  $V$  bounded by the disconnected sum of the two surfaces  $S_1$  (in the middle) and  $S_2$  (at large distance).

The uniqueness arguments discussed above can then be applied to this situation, with the surface integral in (2.8) becoming the sum of two integrals, one over the component  $S_1$  of the total boundary and the other over the component  $S_2$ . Dirichlet or Neumann boundary conditions are specified on  $S_1$ , and so that contribution to the surface integral will vanish. The surface integral over  $S_2$  will become zero in the limit that the radius  $R$  is sent to infinity, provided that  $\phi$  goes to zero sufficiently fast at infinity. Thus in practice we think of  $S_2$  as “the sphere at infinity,” and we impose the boundary condition that  $\phi$  goes to zero sufficiently rapidly at infinity, thereby ensuring that the  $S_2$  component of the surface integral in (2.8) will vanish too. This ensures that again we are left with just the volume integral (2.9), and so the uniqueness proof goes through as before.

Note also that we can allow multiple disconnected surfaces at finite distance, provided that Dirichlet or Neumann boundary conditions are imposed on all of them.

In summary, therefore, we have a uniqueness proof too in the case where the volume  $V$  is infinite, provided that we not only impose Dirichlet or Neumann boundary conditions on any boundary surfaces at finite distance, but we also impose a fall-off condition on the potential at infinity.

Note that we have established uniqueness of the solution subject to the imposition of *either* Dirichlet *or* Neumann boundary conditions at each point on the boundary. It could be taken to Dirichlet for some parts of the boundary, and Neumann for the others, but at any given point one can specify only one of Dirichlet or Neumann.

In the case of Robin boundary conditions, the difference  $\psi = \phi_1 - \phi_2$  between two solutions of  $\nabla^2\phi = -4\pi\rho$  obeying the *same* Robin boundary condition (2.4) (and so the same functions  $f_R$  and  $a$  for each of  $\phi_1$  and  $\phi_2$ ) will therefore satisfy  $\frac{\partial\psi}{\partial n} + a\psi = 0$  on the boundary surface  $S$ ; in other words  $\frac{\partial\psi}{\partial n} = -a\psi$  on  $S$ . The vanishing of the last line in the uniqueness proof following from integrating  $\psi\nabla^2\psi = 0$  as in (2.8) will give (after multiplying by  $-1$  for convenience)

$$\int_S a\psi^2 dS + \int_V |\vec{\nabla}\psi|^2 dV = 0. \quad (2.12)$$

In this case, then, the surface integral is not immediately zero like it was in the Dirichlet or Neumann cases, but instead we have that the integrand in the surface integral is non-

negative everywhere on  $S$ . (Recall that the function  $a$  in (2.4) was required to be everywhere non-negative on the boundary  $S$ .) Thus, by an extension of the argument we used in the Dirichlet and Neumann cases, we now have two integrals, each of which has a pointwise non-negative integrand, summing up to zero. This can only be true if each integrand vanishes at each point within its integration domain, so the volume integral tells us that  $\vec{\nabla}\psi = 0$  in  $V$  and hence  $\psi = k$  where  $k$  is a constant in  $V$ , while the surface integral tells us that  $\psi = 0$  on  $S$  (unless the non-negative function  $a$  is actually zero everywhere on  $S$ , in which case the Robin boundary condition would degenerate to a Neumann boundary condition). Thus we see that  $k$  must equal zero and hence we conclude that  $\psi = 0$  everywhere, implying  $\phi_1 = \phi_2$  and hence uniqueness of the solution. Notice that in this case with Robin boundary conditions the uniqueness of the solution is complete, just like in the case of Dirichlet boundary conditions, rather than being like the Neumann case where the uniqueness was qualified by the fact that two solutions for the potential could differ by an undetermined additive constant.

With any of the Dirichlet, Neumann, mixed or Robin boundary conditions specified, the problem is said to be *well posed*. This means that these boundary conditions are neither too weak, leaving the problem underdetermined and not fully pinned down, nor are they too strong, leaving the problem overdetermined and therefore admitting no solution.

An example of an overdetermined problem would be if one tried to impose *both* Dirichlet *and* Neumann boundary conditions at each point on  $S$ . In other words, if one tried to specify both the potential *and* its normal derivative at each point on  $S$ . Specifying both  $\phi$  and  $\frac{\partial\phi}{\partial n}$  on  $S$  is known as specifying *Cauchy* boundary conditions. That this would be an overdetermination is obvious from the fact that Dirichlet conditions alone are sufficient to give a unique solution. And, on the other hand, Neumann conditions alone are sufficient to give another unique solution. Except in the unlikely event that one picked precisely the “matching” set of Neumann conditions that would reproduce the solution with the Dirichlet conditions, there will be a conflict between the two, implying that no solution would exist.

## 2.2 Green’s theorem

In section 1.8 we gave the expression (1.73) for the electrostatic potential due to a distribution of charge with charge density  $\rho$ :

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \quad (2.13)$$

This result assumes that the charge distribution exists in otherwise free space, with no conductors or other boundaries present. In practice, as we have already remarked, a typical realistic situation is one where there are other conductors, etc., on which boundary conditions are specified.

To handle the case where there are boundaries, the following procedure can be useful. We first derive a simple result known as *Green's theorem*, and then apply it to the case of interest.

Let  $\phi(\vec{r})$  and  $\psi(\vec{r})$  be two scalar functions. We can then consider

$$\begin{aligned}\vec{\nabla} \cdot (\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi) &= \vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi - \vec{\nabla} \psi \cdot \vec{\nabla} \phi - \psi \nabla^2 \phi, \\ &= \phi \nabla^2 \psi - \psi \nabla^2 \phi.\end{aligned}\tag{2.14}$$

Integrating this over a volume  $V$  bounded by surface  $S$ , and using the divergence theorem (1.22), we therefore find

$$\int_V (\phi(\vec{r}) \nabla^2 \psi(\vec{r}) - \psi(\vec{r}) \nabla^2 \phi(\vec{r})) d^3 \vec{r} = \int_S (\phi(\vec{r}) \vec{\nabla} \psi(\vec{r}) - \psi(\vec{r}) \vec{\nabla} \phi(\vec{r})) \cdot d\vec{S}.\tag{2.15}$$

This is Green's theorem. Actually, we shall apply it to functions of a primed position vector  $\vec{r}'$  rather than  $\vec{r}$ , so we shall use

$$\int_V [\phi(\vec{r}') \nabla'^2 \psi(\vec{r}') - \psi(\vec{r}') \nabla'^2 \phi(\vec{r}')] d^3 \vec{r}' = \int_S [\phi(\vec{r}') \vec{\nabla}' \psi(\vec{r}') - \psi(\vec{r}') \vec{\nabla}' \phi(\vec{r}')] \cdot d\vec{S}',\tag{2.16}$$

where  $\vec{\nabla}'$  denotes the gradient with respect to the primed position vector coordinates, and  $d\vec{S}'$  denotes the area element using the primed variables also.

We now apply (2.16) to our electrostatics problem by taking  $\phi$  to be the electrostatic potential satisfying Poisson's equation (1.69), but written using the primed coordinates (i.e.  $\nabla'^2 \phi(\vec{r}') = -4\pi \rho(\vec{r}')$ ), and taking

$$\psi(\vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}.\tag{2.17}$$

Note that the unprimed position vector  $\vec{r}$  will just behave as a constant parameter in this discussion, since all the differentiations and integrations will be with respect to the primed variable  $\vec{r}'$ . Thus (2.16) becomes

$$\begin{aligned}&\int_V \left[ \phi(\vec{r}') \nabla'^2 \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'|} \nabla'^2 \phi(\vec{r}') \right] d^3 \vec{r}' \\ &= \int_S \left[ \phi(\vec{r}') \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}' \phi(\vec{r}') \right] \cdot d\vec{S}'.\end{aligned}\tag{2.18}$$



Note from (1.77) that we shall have

$$\nabla'^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi\delta^3(\vec{r}' - \vec{r}). \quad (2.19)$$

(Both the function  $|\vec{r} - \vec{r}'|^{-1}$  and the delta function  $\delta^3(\vec{r} - \vec{r}')$  are *symmetric* under the exchange of  $\vec{r}$  and  $\vec{r}'$ .) This can be used in the first term on the left-hand side of (2.18), and for the second term we can use  $\nabla'^2\phi(\vec{r}') = -4\pi\rho(\vec{r}')$ . Thus we obtain

$$\phi(\vec{r}) = \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' + \frac{1}{4\pi} \int_S \left[ \frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}'\phi(\vec{r}') - \phi(\vec{r}') \vec{\nabla}'\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \right] \cdot d\vec{S}', \quad (2.20)$$

where the term with the delta function that arose upon using (2.19) has been used in order to perform the volume integration on that term, thus giving the  $\phi(\vec{r})$  after having divided everything by  $(-4\pi)$ . The first term on the right-hand side of (2.20) is of the same form as the expression (2.13) that held in free space. The surface integrals in (2.20) represent the contribution from charge distributions on the boundary  $S$  that we are now including.

Rewriting eqn (2.20) in terms of the normal derivative introduced in eqn (2.1) (now placing a prime on the unit normal vector because the surface integral is over the primed area element  $d\vec{S}'$ ), we have

$$\phi(\vec{r}) = \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' + \frac{1}{4\pi} \int_S \left[ \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial\phi(\vec{r}')}{\partial n'} - \phi(\vec{r}') \frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \right] dS', \quad (2.21)$$

We can interpret (2.20) or (2.21) as giving the expression for the potential everywhere in  $V$  in terms of the charge density  $\rho$  in  $V$  and the potential (and its normal derivative) on  $S$ . However, we cannot view (2.20) in itself as providing the answer we are seeking for how to solve for the potential in a general electrostatics problem. It can be seen from (2.21) that we would need to feed in the information about  $\phi$  on the boundary *and also* about  $\partial\phi/\partial n$  on the boundary in order to obtain the expression for  $\phi$  in  $V$ . But we saw in the discussion of the uniqueness theorem that we are not allowed to specify independently the values of  $\phi$  and also of its normal derivative on the boundary; that would give an overdetermined problem that admitted no solution.

Thus we can only regard (2.21) as an integral equation which will tell us what  $\phi$  is everywhere, once we know what it and its normal derivative are on the boundary. To solve the general boundary-value problem we will need to introduce another tool, which is called the *Green function*.

## 2.3 Green functions and the boundary-value problem

Although (2.20) solves the desired Poisson equation (1.69) with the desired boundary conditions, it is unsatisfactory as a solution of the boundary-value problem because one would have to know both  $\phi$  and its normal derivative on the boundary, whereas in fact these are not independent pieces of information and so they cannot be independently specified. The difficulty would be overcome if we could somehow arrange that only one of the two surface-integral terms in (2.20) were present. This can be achieved by changing the choice for the function  $\psi$  in (2.17) that we inserted into Green's theorem (2.15) in order to obtain (2.20). Instead of taking  $\psi$  to be simply given by (2.17), we need to be a little more ingenious. The function we require is known as a *Green function*.<sup>15</sup>

The key point about the function  $\psi = |\vec{r} - \vec{r}'|^{-1}$  that we needed in deriving the result (2.20) was that it satisfied (2.19). In fact there is a much broader class of functions that satisfy equation (2.19). This is an inhomogeneous equation with the delta function providing a source on the right-hand side, and so we can add to the solution (2.17) an arbitrary solution of the homogeneous equation. Thus we may take  $\psi$  in (2.15) to be any function of the form

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}'), \quad (2.22)$$

where  $F(\vec{r}, \vec{r}')$  is any solution of the homogeneous equation

$$\nabla'^2 F(\vec{r}, \vec{r}') = 0. \quad (2.23)$$

(i.e.  $F(\vec{r}, \vec{r}')$  is an *harmonic* function.) Thus we have

$$\nabla'^2 G(\vec{r}, \vec{r}') = -4\pi\delta^3(\vec{r} - \vec{r}'). \quad (2.24)$$

The idea now is that we will *choose*  $F(\vec{r}, \vec{r}')$  so that  $G(\vec{r}, \vec{r}')$ , which is called a *Green function*, satisfies appropriate boundary conditions.

It is worth emphasising, because it will be seen to be important later, that when we say that  $F(\vec{r}, \vec{r}')$  should satisfy eqn (2.23), what we actually require is that  $F(\vec{r}, \vec{r}')$  should satisfy eqn (2.23) *inside the volume*  $V$ . It doesn't matter whether or not  $F(\vec{r}, \vec{r}')$  satisfies eqn (2.23) outside the volume  $V$ , because we are only concerned with solving our electrostatics

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<sup>15</sup>One sometimes sees this referred to as “a Green's function.” This is a grammatical abomination, which is as wrong as calling an apple pie “an apple's pie,” or speaking of “a Bessel's function” or “a Laplace's transform.” There are numerous offenders in the physics community. J.D. Jackson used to be one of them, but although he referred to “a Green's function” in the first edition of *Classical Electrodynamics*, he had reformed by the time the second edition appeared.

problem *inside* the volume  $V$ . In fact, as we shall see, in typical situations the function  $F(\vec{r}, \vec{r}')$  that does the job usually does *not* satisfy eqn (2.23) outside the volume  $V$ , and indeed that is precisely how it manages to do what we want it to do.

To see how all this works, we first note that there is an analogous result to (2.20) where we take  $\psi(\vec{r}')$  in (2.16) to be  $G(\vec{r}, \vec{r}')$  rather than  $|\vec{r} - \vec{r}'|^{-1}$ , namely

$$\begin{aligned} & \int_V \left[ \phi(\vec{r}') \nabla'^2 G(\vec{r}, \vec{r}') - G(\vec{r}, \vec{r}') \nabla'^2 \phi(\vec{r}') \right] d^3 \vec{r}' \\ &= \int_S \left[ \phi(\vec{r}') \vec{\nabla}' G(\vec{r}, \vec{r}') - G(\vec{r}, \vec{r}') \vec{\nabla}' \phi(\vec{r}') \right] \cdot d\vec{S}'. \end{aligned} \quad (2.25)$$

Using (2.24) together with  $\nabla'^2 \phi(\vec{r}') = -4\pi\rho(\vec{r}')$ , we therefore find

$$\phi(\vec{r}) = \int_V \rho(\vec{r}') G(\vec{r}, \vec{r}') d^3 \vec{r}' + \frac{1}{4\pi} \int_S \left[ G(\vec{r}, \vec{r}') \vec{\nabla}' \phi(\vec{r}') - \phi(\vec{r}') \vec{\nabla}' G(\vec{r}, \vec{r}') \right] \cdot d\vec{S}'. \quad (2.26)$$

In other words, we have

$$\phi(\vec{r}) = \int_V \rho(\vec{r}') G(\vec{r}, \vec{r}') d^3 \vec{r}' + \frac{1}{4\pi} \int_S \left[ G(\vec{r}, \vec{r}') \frac{\partial \phi(\vec{r}')}{\partial n'} - \phi(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \right] dS'. \quad (2.27)$$

Consider first the case where we wish to specify Dirichlet boundary conditions for the potential on the surface  $S$ . We achieve this by choosing the harmonic function  $F(\vec{r}, \vec{r}')$  in (2.22) so that  $G(\vec{r}, \vec{r}')$  vanishes when  $\vec{r}'$  lies in the surface  $S$ . Thus, denoting this *Dirichlet Green function* by  $G_D(\vec{r}, \vec{r}')$ , we have

$$G_D(\vec{r}, \vec{r}') = 0 \quad \text{when} \quad \vec{r}' \in S. \quad (2.28)$$

Using  $G_D(\vec{r}, \vec{r}')$  in (2.27) we therefore obtain

$$\phi(\vec{r}) = \int_V \rho(\vec{r}') G_D(\vec{r}, \vec{r}') d^3 \vec{r}' - \frac{1}{4\pi} \int_S \phi(\vec{r}') \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} dS'. \quad (2.29)$$

This has achieved the goal of giving an expression for  $\phi(\vec{r})$  everywhere in the volume  $V$ , expressed in terms of the given charge density  $\rho(\vec{r})$  and the values of  $\phi(\vec{r})$  on the boundary surface  $S$ . Thus, we may say that the Dirichlet boundary-value problem is solved, albeit somewhat formally.

One might worry that (2.29) has done little more than replace one difficult problem (solving  $\nabla^2 \phi = 4\pi\rho$  for  $\phi$ ) by another equivalently difficult problem (solving for the harmonic function  $F(\vec{r}, \vec{r}')$  that is needed in order to ensure the Green function satisfies the all-important boundary condition (2.28)). However, this is not quite true, and moreover, a very important advantage of adopting this Green-function approach is that solving just once for the Green function for the given geometry then allows us to solve many different

boundary-value problems with different choices for the charge density  $\rho(\vec{r})$  and different choices for the boundary value of the potential  $\phi(\vec{r})$ .

The point is the following. Having once solved for  $G_D(\vec{r}, \vec{r}')$  in the given geometry (i.e. for the specified boundary surface  $S$ ), one can now construct the solution for  $\phi(\vec{r})$  for *any* choice of charge density  $\rho(\vec{r})$  and for *any* choice of the boundary-value potential on the surface  $S$ . Thus finding the Dirichlet Green function just once for the chosen geometry allows us to solve *any* Dirichlet boundary-value problem for that geometry.

The solution to the Neumann problem goes rather similarly, although with a minor subtlety. One might think that now one should choose  $F(\vec{r}, \vec{r}')$  in (2.22) so that the normal derivative of the Green function vanished on  $S$ ,

$$\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n'} = 0 \quad \text{when } \vec{r}' \in S. \quad (2.30)$$

But this would in general lead to a contradiction, since integrating (2.24) over  $V$  and using the divergence theorem (1.22) gives

$$\int_S \vec{\nabla}' G_N(\vec{r}, \vec{r}') \cdot d\vec{S}' = -4\pi, \quad \text{i.e.} \quad \int_S \frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n'} dS' = -4\pi, \quad (2.31)$$

and so we cannot impose the boundary condition (2.30) on  $S$ . The simplest way around this is to impose instead

$$\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n'} = -\frac{4\pi}{A} \quad \text{when } \vec{r}' \in S, \quad (2.32)$$

where  $A = \int_S dS$  is the area of the boundary  $S$ .<sup>16</sup>

Substituting (2.32) into (2.27), we therefore find

$$\phi(\vec{r}) = \langle \phi \rangle_S + \int_V \rho(\vec{r}') G_N(\vec{r}, \vec{r}') d^3\vec{r}' + \frac{1}{4\pi} \int_S G_N(\vec{r}, \vec{r}') \frac{\partial \phi(\vec{r}')}{\partial n'} dS', \quad (2.33)$$

where  $\langle \phi \rangle_S$  denotes the average value of  $\phi$  over the surface  $S$ ,

$$\langle \phi \rangle_S = \frac{1}{A} \int_S \phi dS. \quad (2.34)$$

Note that  $\langle \phi \rangle$  is just a pure constant, so it is not in any case important as far as the calculation of physical quantities such as the electric field  $\vec{E} = -\vec{\nabla}\phi$  is concerned. It is just a manifestation of the usual minor inconvenience that the additive constant part of the potential is not pinned down when Neumann boundary conditions are imposed. If

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<sup>16</sup>Note that any other choice for  $\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n'}$  when  $\vec{r}'$  lies in  $S$  would be equally acceptable, provided that it gave the same result  $-4\pi$  when integrated over the whole of  $S$ .

another choice, different from (2.32), had been made for the values of  $\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n'}$  when  $\vec{r}'$  lies in  $S$  (still obeying the requirement that (2.31) hold), the final result would be that  $\langle \phi \rangle_S$  in (2.33) would be replaced by a different constant, corresponding to some differently-weighted averaging of  $\phi$  over  $S$ . This would again be of no importance when calculating physical quantities like the electric field.

The solutions (2.29) or (2.33) for  $\phi$  in terms of the Dirichlet or Neumann Green function provide at least a formal solution to the boundary-value problem. How useful they are in practice depends upon the details of the geometry of the problem. It all comes down to the question of whether one can solve explicitly for the Green function  $G_D(\vec{r}, \vec{r}')$  or  $G_N(\vec{r}, \vec{r}')$ . For a boundary  $S$  of some generic type it will certainly be impossible in closed form. In certain special cases one can obtain closed-form expressions. We shall meet examples shortly where this can be done, in the case of an infinite planar boundary  $S$ , and in the case of a spherical boundary.

It is worth making a few closing remarks about the physical interpretation of the Green function. The simplest example is when

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}, \quad (2.35)$$

which is, as we have seen, the Green function for the Dirichlet problem where the only boundary is the “sphere at infinity.” We can recognise (2.35) as being the electric potential at the point  $\vec{r}'$  due to a unit-strength point charge at the point  $\vec{r}$ . In fact (2.35) is symmetrical under the exchange of  $\vec{r}$  and  $\vec{r}'$ , and so it can be equivalently viewed as the potential at  $\vec{r}$  due to a unit charge at  $\vec{r}'$ .

In the more general case, the Green function is of the form (2.22), where  $F(\vec{r}, \vec{r}')$  is an harmonic function. This means that  $G(\vec{r}, \vec{r}')$  again has the interpretation of being the potential at  $\vec{r}'$  due to a unit charge at  $\vec{r}$ , but now in the more complicated situation where  $G(\vec{r}, \vec{r}')$  vanishes on  $S$  (in the Dirichlet case), or its normal derivative obeys (2.31) (in the Neumann case).

One can show quite generally that in the case of Dirichlet boundary conditions, the Green function  $G_D(\vec{r}, \vec{r}')$  is necessarily symmetrical under the exchange of  $\vec{r}$  and  $\vec{r}'$ :

$$G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r}). \quad (2.36)$$

(This can be seen by using Green’s theorem (2.15); see homework 2.) In the case of Neumann boundary conditions, symmetry under the exchange of  $\vec{r}$  and  $\vec{r}'$  is not automatic, but it can always be imposed. (These properties of  $G_D(\vec{r}', \vec{r})$  and  $G_N(\vec{r}', \vec{r})$  are explored in homework 2.)

## 2.4 Dirichlet and Neumann Green functions for infinite planar conductor

In some cases, if the boundary surface in a boundary value problem has particularly simple and symmetrical geometry, it is possible to solve for the Green function in closed form, by using the *Method of Images*. (Recall from the previous discussion that solving for the Green function amounts to solving for the potential due to a unit point charge in the volume  $V$ , subject to the appropriate boundary condition on the boundary surface  $S$ .)

Suppose, for example, that we wish to construct the Dirichlet Green function in the volume  $V$ , bounded by the surface  $S$ . This means that we need to find the solution  $G_D(\vec{r}, \vec{r}')$  for the potential inside  $V$ , viewed as being a function of  $\vec{r}'$ , due to a point charge of unit strength located at a point  $\vec{r}$  that is also inside  $V$ , subject to the boundary condition that this potential vanish when  $\vec{r}'$  lies on the boundary  $S$ . (Because of the symmetry  $G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r})$ , we can equally well exchange the roles of  $\vec{r}$  and  $\vec{r}'$  in the previous sentence.)

If the geometry of the volume  $V$  and its boundary  $S$  is suitably symmetrical, it may be possible to “mock up” the same zero-potential surface by considering a totally free space, with no actual boundary surfaces anywhere, but with one or more additional *image charges* judiciously introduced in the region of the total space that lies *outside* the volume  $V$ . (i.e. the additional image charges are on the “other side” of the surface  $S$  in the actual physical problem, disconnected from the volume  $V$ .) Suppose that by introducing image charges in this way, one can arrange that the total potential due to the original charge plus the image charges is zero on the “virtual” surface  $S$  in the mocked-up situation. It is then clear, by invoking the uniqueness theorem (see section 2.1), that the potential at all points in  $V$  must be the same in the image-charge “mock-up” and in the original problem with the boundary condition that the potential should vanish on the actual boundary surface  $S$ . Of course, the requirement that we want the potential to vanish on  $S$  is equivalent to saying that for the purposes of this calculation (i.e. of finding the Dirichlet Green function  $G_D(\vec{r}, \vec{r}')$ ), we can think of  $S$  as being a grounded conductor.

The challenge is to figure out how to achieve the “virtual” zero-potential surface by means of image charges. In practice, there are very few cases where it can be done. We shall discuss two of them now.

The simplest example where the method of images can be employed is in the case of an infinite planar conductor. Let us suppose, for convenience, that Cartesian axes are chosen so that the conductor lies in the plane  $z = 0$ . We shall take the volume  $V$  that is of physical interest to be the whole of the half-space  $z > 0$ . As is required for the Dirichlet Green

function, the conductor is fixed at zero potential. That is, we wish to find the potential at a point  $\vec{r}$  in the upper half-space  $z > 0$  due to a unit-strength point charge located at a point  $\vec{r}'$  in the upper half-space, in the case that the whole plane at  $z = 0$  is an infinite grounded planar conductor.

We can solve this problem by turning to the “mock-up” problem where the infinite grounded conductor is removed, and we instead place an image charge at the point in the lower half-plane that is located at the mirror reflection point of the actual charge in the upper half-plane. To be specific, if a point charge  $q$  is located at some point in  $V$ , then it is obvious that if an image charge  $-q$  is placed “behind” the conductor at precisely the mirror-image location, then by symmetry it must be the case that the total potential of original plus image charge, now taken to be in a completely free space with no conductor at all, will vanish on the plane  $z = 0$ . Therefore, the potential at any point in  $V$  in the original problem with conductor will be given by the total potential in the image-charge “mock up.”

To be more precise, let us suppose that the original charge  $q$  is located at

$$\vec{r}_1 = (x_1, y_1, z_1), \quad z > 0. \quad (2.37)$$

The image charge  $-q$  will then be located at

$$\vec{r}_2 = (x_2, y_2, z_2) = (x_1, y_1, -z_1). \quad (2.38)$$

Therefore, the total potential is given by

$$\begin{aligned} \phi(\vec{r}) &= \frac{q}{|\vec{r} - \vec{r}_1|} - \frac{q}{|\vec{r} - \vec{r}_2|}, \\ &= \frac{q}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} - \frac{q}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z + z_1)^2}}. \end{aligned} \quad (2.39)$$

Clearly this potential indeed vanishes on the surface  $z = 0$ , and so therefore by the uniqueness theorem  $\phi(\vec{r})$  describes the potential, at all points with  $z > 0$ , of the single charge  $q$  in the presence of the infinite conductor at  $z = 0$ .

#### 2.4.1 Dirichlet Green function for infinite planar boundary

We are now in a position to construct the Dirichlet and Neumann Green functions for this case. Recall that the Dirichlet Green function  $G_D(\vec{r}, \vec{r}')$  is defined to be the potential at  $\vec{r}'$  due to a unit strength charge at  $\vec{r}$ , subject to the condition that  $G_D(\vec{r}, \vec{r}')$  should vanish when  $\vec{r}'$  is on the boundary  $S$ . (Or, because of the symmetry of  $G_D$ , we can exchange the

roles of  $\vec{r}$  and  $\vec{r}'$ .) Thus we can read off from (2.39) that the Dirichlet Green function in the case of the infinite planar boundary at  $z = 0$  is given by

$$G_D(x, y, z; x', y', z') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}},$$

Note that this is indeed symmetric under the exchange of the primed and the unprimed variables. Note also that it is indeed of the form given in (2.22), namely

$$G_D(x, y, z; x', y', z') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}'), \quad (2.40)$$

with

$$F(\vec{r}, \vec{r}') = -\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}. \quad (2.41)$$

The first term in (2.40) is a solution of the inhomogeneous equation  $\nabla'^2 |\vec{r} - \vec{r}'|^{-1} = -4\pi\delta^3(\vec{r} - \vec{r}')$  (and, since it is symmetric, also  $\nabla^2 |\vec{r} - \vec{r}'|^{-1} = -4\pi\delta^3(\vec{r} - \vec{r}')$ ), while the second term is a solution of the homogeneous equation  $\nabla'^2 F(\vec{r}, \vec{r}') = 0$  (and also therefore  $\nabla^2 F(\vec{r}, \vec{r}') = 0$ ) *in the region  $V$  of interest*, i.e. when  $z$  and  $z'$  lie in the upper half-space.<sup>17</sup> As discussed in chapter 2.3, the solution  $F(\vec{r}, \vec{r}')$  is added in order to ensure that the total expression  $G_D(\vec{r}, \vec{r}') = |\vec{r} - \vec{r}'|^{-1} + F(\vec{r}, \vec{r}')$  obeys the required Dirichlet boundary condition on the plane  $z = 0$ .

To use (2.40) in order to solve Dirichlet boundary-value problems in this geometry, we just plug it into the general expression (2.29). This requires that we evaluate the normal derivative of  $G_D$  on the boundary, which in the present case means that we need

$$-\left. \frac{\partial G_D(x, y, z; x', y', z')}{\partial z'} \right|_{z'=0} = -\frac{2z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}. \quad (2.42)$$

(The normal derivative in (2.29) is directed *outwards* from the volume  $V$ , which means in this case in the *negative* direction along the  $z'$  axis.) Suppose, for simplicity, we consider the situation where there is no charge distribution  $\rho$  in the problem, and so the entire contribution to the potential  $\phi(\vec{r})$  comes from the boundary contribution in (2.29). Then we find

$$\phi(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\phi(x', y', 0)}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}. \quad (2.43)$$

Thus we have constructed the solution of the source-free equation  $\nabla^2 \phi = -4\pi\rho = 0$ , in which the boundary value of  $\phi$  on the infinite planar surface  $z' = 0$  is specified.

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<sup>17</sup>  $F(\vec{r}, \vec{r}')$  actually satisfies  $\nabla^2 F(\vec{r}, \vec{r}') = \nabla'^2 F(\vec{r} - \vec{r}') = 4\pi\delta(x-x')\delta(y-y')\delta(z+z')$  in general, since it is the potential due to a negative unit charge at  $(x', y', -z')$  in free space. In the region of space we are interested in, namely  $z \geq 0$ , it is indeed therefore harmonic, obeying  $\nabla'^2 F(\vec{r}, \vec{r}') = \nabla^2 F(\vec{r} - \vec{r}') = 0$ .



It should be stressed here that the potential  $\phi$  appearing in (2.43) is the potential that we are wishing to solve for in the general Dirichlet boundary-value problem in the upper half-space. It has nothing whatever to do with the potential  $\phi$  that we found in eqn (2.39), which was the potential at  $\vec{r}$  due to a point charge  $q$  above the grounded plane. When one is solving potential theory problems in electrostatics one almost always uses the symbol  $\phi$  for the potential. The potential  $\phi$  in (2.39) was an intermediate step in arriving at the Dirichlet Green function written in (2.40). Having got that result, we have now wiped the slate clean, “cleared all the variables,” and are now free to use the symbol  $\phi$  for a different purpose, which is what we are doing in (2.43).

#### 2.4.2 Neumann Green function for infinite planar boundary

We can also easily construct the Neumann Green function  $G_N(\vec{r}, \vec{r}')$  for this geometry. In this case, it is defined to be the potential at  $\vec{r}$  due to a unit strength charge at  $\vec{r}'$ , subject to the condition that the normal derivative of  $G_N$  should vanish on the plane  $z' = 0$ .<sup>18</sup> This time, we can suspect that a small modification of the image-charge trick should give us the required result. Indeed this works, and all we need to do is to replace the minus sign in front of the second term in (2.40) by a plus sign, to give

$$G_N(x, y, z; x', y', z') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}.$$

It is easily seen that this satisfies the required condition that

$$\left. \frac{\partial G_N(x, y, z; x', y', z')}{\partial z'} \right|_{z'=0} = 0. \quad (2.44)$$

We now plug the Neumann Green function into (2.33), in order to solve the general class of boundary-value problems in which the normal derivative of  $\phi$  is specified on the infinite planar surface  $z = 0$ . Suppose again, for simplicity, we consider the case where  $\rho = 0$ . Plugging (2.44) into (2.33) then gives

$$\phi(x, y, z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} \left( \left. \frac{\partial \phi(x', y', z')}{\partial z'} \right|_{z'=0} \right). \quad (2.45)$$

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<sup>18</sup>In our general discussion for the Green function for Neumann boundary conditions, we had the requirement (2.32) that the normal derivative should equal  $-4\pi/A$ , where  $A$  was the area of the boundary. In the present case this area is infinite, and so we can simply require that the normal derivative of  $G_N$  should vanish.

(The minus sign arises because the outward normal derivative at  $z' = 0$  is  $-\partial/\partial z'$ .) Of course, since  $\vec{E} = -\vec{\nabla}\phi$ , we may write (2.46) as

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{E_z(x', y', 0)}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}. \quad (2.46)$$

## 2.5 Dirichlet Green function for spherical conductor

### 2.5.1 Method of images for spherical conductor

A slightly more subtle example where the method of images can be employed is in the case of a spherical conductor. Suppose a conducting sphere of radius  $a$  is held at zero potential, and that a charge  $q$  is placed outside the sphere, at a distance  $b$  from its origin. It turns out that this situation can be “mocked up” by considering instead entirely free space containing the original charge and also a certain charge  $q'$  placed at a certain distance  $c$  from the origin of the sphere, on the line joining the charge  $q$  and the origin.

A quick way to derive this result is as follows. Imagine that the sphere is centred on the origin of Cartesian coordinates, and that the charge  $q$  is placed at distance  $b$  along the  $z$  axis, i.e. at  $(x, y, z) = (0, 0, b)$ . The claim is that the image charge  $q'$  should also lie on the  $z$  axis, at some point  $(x, y, z) = (0, 0, c)$ . If this does indeed give rise to a spherical surface of radius  $a$  that has zero potential, then in particular it must be that the potential is zero at the two points  $(0, 0, a)$  and  $(0, 0, -a)$  on the sphere (the north pole and the south pole). Since these two points are aligned on the same axis as the charges, it is particularly easy to write down the conditions that the potential should be zero at each pole:

$$\text{S. pole : } \frac{q}{b+a} + \frac{q'}{a+c} = 0, \quad \text{N. pole : } \frac{q}{b-a} + \frac{q'}{a-c} = 0. \quad (2.47)$$

These two conditions determine  $q'$  and  $c$ , giving

$$q' = -\frac{aq}{b}, \quad c = \frac{a^2}{b}. \quad (2.48)$$

Observe that since  $b > a$ , we have  $c = a(a/b) < a$ . Thus, as one would expect, the image charge is *inside* the spherical surface. It remains to verify that the potential then vanishes for an arbitrary point on the sphere. The problem has rotational symmetry around the  $z$  axis, so it suffices to consider a point  $P$  at angle  $\theta$  from the  $z$ -axis. If the distance from  $q$  to  $P$  is  $\ell_q$ , and the distance from  $q'$  to  $P$  is  $\ell_{q'}$ , then the cosine rule gives

$$\ell_q^2 = a^2 + b^2 - 2ab \cos \theta, \quad \ell_{q'}^2 = a^2 + c^2 - 2ac \cos \theta. \quad (2.49)$$

After using (2.48) we see that  $\ell_{q'} = (a/b) \ell_q$  and that indeed we have

$$\frac{q}{\ell_q} + \frac{q'}{\ell_{q'}} = 0 \quad (2.50)$$

for all  $\theta$ , and so the potential vanishes everywhere on the sphere  $x^2 + y^2 + z^2 = a^2$ .

It is useful also to give the result in a more general fashion, in which the original charge  $q$  is placed at an arbitrary point  $\vec{r}_1$  located outside the sphere, rather than lying specifically on the  $z$  axis.

Clearly, if the charge  $q$  lies at  $\vec{r}_1$  then the charge  $q'$  must lie at a point  $\vec{r}_2$  along the same direction, and since the second relation in (2.48) can be written as  $c = (a^2/b^2)b$ , we must have

$$\vec{r}_2 = \frac{a^2}{r_1^2} \vec{r}_1, \quad q' = -\frac{aq}{r_1}. \quad (2.51)$$

Thus, the potential at  $\vec{r}$  outside the zero-potential sphere at  $r = a$  due to a charge  $q$  located at  $\vec{r}_1$  outside the sphere is given by

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_1|} - \frac{qa/r_1}{|\vec{r} - (a^2/r_1^2)\vec{r}_1|}. \quad (2.52)$$

If we define  $\gamma$  to be the angle between  $\vec{r}$  and  $\vec{r}_1$ , so that  $\vec{r} \cdot \vec{r}_1 = rr_1 \cos \gamma$ , then (2.52) can be expressed as

$$\phi(\vec{r}) = \frac{q}{(r^2 + r_1^2 - 2rr_1 \cos \gamma)^{1/2}} - \frac{qa}{(r^2 r_1^2 + a^4 - 2a^2 rr_1 \cos \gamma)^{1/2}}. \quad (2.53)$$

Using the expression for the surface charge density on a conductor,  $\sigma = \vec{n} \cdot \vec{E}/(4\pi)$  (see (1.35)), we have

$$\sigma = -\frac{1}{4\pi} \frac{\partial \phi}{\partial r} \Big|_{r=a} = -\frac{q(r_1^2 - a^2)}{4\pi a(a^2 + r_1^2 - 2a r_1 \cos \gamma)^{3/2}}. \quad (2.54)$$

If (2.54) is integrated over the area of the sphere, it gives a total charge  $q' = -qa/r_1$ . In other words, the total induced charge on the surface of the sphere is equal to the *image* charge. This is in accordance with Gauss's law.

### Some simple generalisations:

Because of the linearity of the Maxwell equations, it is straightforward to generalise the above result in a variety of ways. For example, instead of taking the conducting sphere to be at zero potential, we could consider a situation where it is held at a non-zero potential  $V$  (relative to zero at infinity, still). All that need be done is to add another term to the potential (2.52), corresponding to the introduction of a point charge at the origin. Thus if we now take

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_1|} - \frac{qa/r_1}{|\vec{r} - (a^2/r_1^2)\vec{r}_1|} + \frac{Q}{r}, \quad (2.55)$$

then the potential on the surface of the sphere becomes  $\phi = Q/a$ . Choosing  $Q = Va$  therefore gives the required result.

As another generalisation, we can calculate the solution for a grounded sphere placed in a previously-uniform electric field.<sup>19</sup> Without loss of generality, let us take the electric field to be directed along the  $z$  axis. The uniform field can be achieved via a limiting process in which two point charges  $\pm Q$  are placed at  $z = \mp b$  respectively. Close to the origin, there will therefore be an approximately uniform electric field  $E_0 \approx 2Q/b^2$  directed along  $z$ . Eventually, we take  $b$  to infinity, whilst holding  $E_0 = 2Q/b^2$  fixed, and the approximation becomes exact.

In the presence of the grounded sphere, each of the above charges will have its image charge, with  $+Q$  at  $-b$  having an image charge  $-Qa/b$  at  $z = a^2/b$ , and  $-Q$  at  $+b$  having an image charge  $+Qa/b$  at  $z = -a^2/b$ . If we use spherical polar coordinates to write

$$\vec{r} = (x, y, z) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta), \quad (2.56)$$

then from (2.53) we deduce that the total potential for the system we are considering will be

$$\begin{aligned} \phi(r, \theta, \varphi) = & \frac{Q}{(r^2 + b^2 + 2rb \cos \theta)^{1/2}} - \frac{Q}{(r^2 + b^2 - 2rb \cos \theta)^{1/2}} \\ & - \frac{Qa/b}{(r^2 + a^4/b^2 + 2a^2r/b \cos \theta)^{1/2}} + \frac{Qa/b}{(r^2 + a^4/b^2 - 2a^2r/b \cos \theta)^{1/2}}. \end{aligned} \quad (2.57)$$

Expanding as a power series in  $1/b$ , we find

$$\phi = -\frac{2Q}{b^2} \left( r - \frac{a^3}{r^2} \right) \cos \theta - \frac{Q}{b^4} \left( r^3 - \frac{a^7}{r^4} \right) (5 \cos^2 \theta - 3) \cos \theta + \dots \quad (2.58)$$

The first term, with the  $-2Q/b^2$  prefactor, remains finite and non-zero in the limit when we send  $b$  and  $Q$  to infinity while holding  $E_0 = 2Q/b^2$  fixed. The second term has a prefactor  $-Q/b^4$ , which is equal to  $E_0/(2b^2)$ , and so this goes to zero as  $b$  goes to infinity. Similarly, all the higher terms represented by the ellipses will vanish as  $b$  is sent to infinity. In this limit, we therefore find that

$$\phi = -E_0 \left( r - \frac{a^3}{r^2} \right) \cos \theta. \quad (2.59)$$

The first term in (2.59) can be written using Cartesian coordinates as  $\phi = -E_0 z$ , and so it just describes the purely uniform electric field  $\vec{E} = -\vec{\nabla} \phi = (0, 0, E_0)$  that would occur in the absence of the grounded sphere. The second term describes an electric dipole contribution to the potential, arising from the two pairs of charges plus images.

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<sup>19</sup>A more mathematically precise statement is an electric field that is asymptotically uniform at large distance.

If we did not perform the limiting procedure, but simply kept  $b$  and  $Q$  finite, the higher-order terms would remain in (2.58). These would correspond to higher terms in an expansion of the field in multipole moments, with the second displayed term in (2.58) being associated with an octopole. We shall discuss these kinds of expansions in a systematic way later in the course.

### 2.5.2 Dirichlet Green function for spherical boundary

We can use the results in section 2.5 to construct the Dirichlet Green function for the boundary-value problem where the potential is specified on the surface of a sphere. We just need to set  $q = 1$  and  $\vec{r}_1 = \vec{r}'$  in (2.52), leading to

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{a/r'}{|\vec{r} - (a^2/r'^2)\vec{r}'|}. \quad (2.60)$$

As in the previous planar example, here the first term is a solution of the inhomogeneous equation  $\nabla^2|\vec{r} - \vec{r}'|^{-1} = -4\pi\delta^3(\vec{r} - \vec{r}')$ , while the second term satisfies the homogeneous equation  $\nabla^2 F(\vec{r}, \vec{r}') = 0$  in the region of interest (outside the sphere), and is added in order to ensure that  $G_D(\vec{r}, \vec{r}')$  satisfies the Dirichlet boundary condition on the sphere of radius  $a$ .

If we introduce  $\gamma$  as the angle between  $\vec{r}$  and  $\vec{r}'$ , then (2.60) can be written as

$$G_D(\vec{r}, \vec{r}') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}} - \frac{1}{\sqrt{a^2 + r^2 r'^2/a^2 - 2rr' \cos \gamma}}. \quad (2.61)$$

Written in this form, it is manifest that  $G_D(\vec{r}, \vec{r}')$  is symmetric under the exchange of  $\vec{r}$  and  $\vec{r}'$ . It is also manifest that  $G_D(\vec{r}, \vec{r}')$  vanishes, as it should, if  $\vec{r}$  or  $\vec{r}'$  lies on the surface of the sphere.

To use this expression in the general boundary-value integral in (2.29), we need to calculate the normal derivative with respect to  $\vec{r}'$ , evaluated on the sphere at  $r' = a$ . Bearing in mind that the *outward* normal from the volume  $V$  (external to the sphere) is directed *inwards* towards the centre of the sphere, we therefore need<sup>20</sup>

$$\left. \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} \right|_{r'=a} = - \left. \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial r'} \right|_{r'=a} = - \frac{r^2 - a^2}{a [r^2 + a^2 - 2ar \cos \gamma]^{3/2}}. \quad (2.62)$$

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<sup>20</sup>Note that if we were instead solving for the potential *inside* the sphere subject to Dirichlet boundary conditions on the surface of the sphere, the required Green function would be exactly the same expression as before, as given in eqn (2.60). However now, the normal derivative  $\left. \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} \right|_S$  would be given by  $+\left. \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial r'} \right|_{r'=a}$  rather than  $-\left. \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial r'} \right|_{r'=a}$ , since now the unit normal pointing outwards from the volume  $V$  would be directed radially outwards rather than radially inwards.

Substituting into (2.29) (and taking the charge density  $\rho = 0$  for simplicity), we obtain

$$\begin{aligned}\phi(\vec{r}) &= \frac{a(r^2 - a^2)}{4\pi} \int \frac{\phi(a, \theta', \varphi')}{[r^2 + a^2 - 2ar \cos \gamma]^{3/2}} d\Omega', \\ &= \frac{a(r^2 - a^2)}{4\pi} \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\varphi' \frac{\phi(a, \theta', \varphi')}{[r^2 + a^2 - 2ar \cos \gamma]^{3/2}},\end{aligned}\quad (2.63)$$

where we are using spherical polar coordinates  $(r', \theta', \varphi')$ . The area element on the sphere of radius  $a$  is written as  $\vec{n} \cdot d\vec{S}' = a^2 d\Omega'$ , where  $d\Omega' = \sin \theta' d\theta' d\varphi'$  is the area element on the unit sphere (i.e. the solid angle element). The expression (2.63) gives the result for the potential everywhere outside a spherical surface of radius  $a$ , on which the potential is specified to be  $\phi(a, \theta', \varphi')$ .

Note that the integration in (2.63) is actually rather complicated, even if  $\phi(a, \theta', \varphi')$  itself is a simple function, because of the  $\cos \gamma$  appearing in the denominator. Using spherical polar coordinates, the Cartesian components of  $\vec{r}$  and  $\vec{r}'$  are

$$\vec{r} = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta), \quad \vec{r}' = (r' \sin \theta' \cos \varphi', r' \sin \theta' \sin \varphi', r' \cos \theta'), \quad (2.64)$$

and so  $\cos \gamma$ , which is defined by  $\vec{r} \cdot \vec{r}' = rr' \cos \gamma$ , is given by

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'). \quad (2.65)$$

Consider, as an example, the case where one hemisphere of the boundary surface is held at a constant potential  $V$ , while the other hemisphere is held at potential  $-V$ . Since we are using standard spherical polar coordinates, it is natural to orient things so that the two hemispheres correspond to the parts of the sphere with  $z > 0$  and  $z < 0$  respectively. In other words, we have

$$\begin{aligned}\phi(a, \theta, \varphi) &= +V & \text{for } 0 \leq \theta < \frac{\pi}{2}, \\ \phi(a, \theta, \varphi) &= -V & \text{for } \frac{\pi}{2} < \theta \leq \pi.\end{aligned}\quad (2.66)$$

Equation (2.63) therefore gives

$$\begin{aligned}\phi(\vec{r}) &= \frac{aV(r^2 - a^2)}{4\pi} \int_0^{2\pi} d\varphi' \left[ \int_0^{\pi/2} \frac{\sin \theta'}{(r^2 + a^2 - 2ar \cos \gamma)^{3/2}} d\theta' \right. \\ &\quad \left. - \int_{\pi/2}^\pi \frac{\sin \theta'}{(r^2 + a^2 - 2ar \cos \gamma)^{3/2}} d\theta' \right],\end{aligned}\quad (2.67)$$

where  $\cos \gamma$  is given by (2.65). By making the change of variables  $\theta' \rightarrow \pi - \theta'$  and  $\varphi' \rightarrow \varphi' + \pi$  in the second integral, this can be written as

$$\phi(\vec{r}) = \frac{aV(r^2 - a^2)}{4\pi} \int_0^{2\pi} d\varphi' \int_0^{\pi/2} \left[ \frac{\sin \theta'}{(r^2 + a^2 - 2ar \cos \gamma)^{3/2}} - \frac{\sin \theta'}{(r^2 + a^2 + 2ar \cos \gamma)^{3/2}} \right] d\theta' \quad (2.68)$$

Unfortunately, the integrations are too complicated to admit a useful explicit closed-form result.<sup>21</sup>

We can easily integrate (2.68) if we ask for the potential  $\phi$  only in the special case where we are on the positive  $z$  axis, i.e. for  $\theta = 0$ . It then follows from (2.65) that  $\cos \gamma = \cos \theta'$ , and then elementary integration of (2.68) gives, for  $z > a$ ,

$$\phi(z) = V \left( 1 - \frac{z^2 - a^2}{z\sqrt{z^2 + a^2}} \right). \quad (2.69)$$

In the absence of a closed-form expression for the general off-axis potential, one could resort to making a power-series expansion of the integrand in (2.68) in powers of  $\cos \gamma$ , and then performing the integrations term by term. This is a somewhat clumsy approach, and so instead we shall postpone further discussion of this example until a little later in the course, when we shall have developed an approach which will allow us to obtain the power series expression for the off-axis potential very easily.

One might wonder whether the Neumann Green function for the case of a spherical volume  $V$  bounded by  $S$  could also be constructed, as we did in the case of the semi-infinite half-space  $z \geq 0$ , by making some small change to the method of images argument that led to the spherical Dirichlet Green function. It is easy to see that the “obvious” guess, of changing the sign on the second term in eqn (2.60), does not work. As far as I am aware, there is no simple closed-form expression for the Neumann Green function for the spherical volume. One can however, as we shall see later, construct an expression for the Neumann Green function as an infinite sum over modes.

### 3 Solution of Laplace’s Equation in Cartesian Coordinates

#### 3.1 Introduction

The boundary-value problem in electrostatics is formulated as the problem of solving Poisson’s equation  $\nabla^2 \phi = -4\pi\rho$  in a volume  $V$  bounded by a surface  $S$  on which appropriate boundary conditions are imposed. Quite commonly, we are interested in the situation where  $\rho = 0$  in  $V$ , so that the potential  $\phi$  in  $V$  is governed entirely by the boundary conditions that it, or its normal derivative, satisfy on  $S$ .

The geometry of the boundary surface  $S$  typically dictates what type of coordinate system is best adapted to the problem. For example, if  $S$  is formed by one or more planar

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<sup>21</sup>This illustrates an important point, that although we may say that the boundary-value problem for the spherical boundary has been “solved” once we obtained an explicit closed-form result for the Green function, it does not necessarily mean that we can present an explicit closed-form expression for the solution.

surfaces, then Cartesian coordinates are likely to be the most convenient choice. If, on the other hand, the boundary  $S$  is spherical, then spherical polar coordinates will probably be the best choice. For a boundary of cylindrical shape, cylindrical polar coordinates will be most convenient.

All three of these coordinate systems share the special property that when using them the Laplacian operator  $\nabla^2$  is *separable*. This would not be true for some arbitrary choice of coordinate system. The defining property of a separable coordinate system is that Laplace's equation, which is itself a second-order partial differential equation, can be factored into a system of second-order *ordinary* differential equations. This is of enormous benefit when one tries to construct solutions.

We shall describe in detail the process of separation of variables in the three cases of Cartesian, spherical polar, and cylindrical polar, coordinates. In each case, the solution of the factored ordinary differential equations requires an understanding of certain classes of special functions. In the Cartesian case, the relevant special functions are just the familiar sine and cosine trigonometric functions, or their hyperbolic cousins  $\sinh$  and  $\cosh$  (or, equivalently, real exponential functions). In the case of spherical polar coordinates, the Legendre and associated Legendre functions arise, whilst in the case of cylindrical polar coordinates it is Bessel functions that arise. We shall also look briefly at a fourth example, involving the use of oblate spheroidal coordinates, where again one can separate variables and in fact it allows one to obtain an elegant solution to a nice electrostatics problem.

We begin in this section with a review of the separation of variables for the Laplace equation in Cartesian coordinates.

### 3.2 Separation of variables in Cartesian coordinates

The Laplace equation in Cartesian coordinates is simply

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (3.1)$$

The separation of variables is achieved by first looking for solutions where  $\phi$  has the factorised form<sup>22</sup>

$$\phi(x, y, z) = X(x) Y(y) Z(z). \quad (3.2)$$

Substituting this into (3.1), and dividing out by  $\phi$ , yields

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0. \quad (3.3)$$

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<sup>22</sup>The general solution of (3.1) does not have this factorised form; the important point, though, is that the general solution can be written as a linear combination of factorised solutions.



The first term depends only on  $x$ , the second term depends only on  $y$ , and the third term depends only on  $z$ . Moving the second and third terms to the right-hand side of the equation, we therefore have that a function purely depending on  $x$  is equal to a function depending only on  $y$  and  $z$ . This is only possible if each side of the equation is actually just a constant, depending on none of  $x$ ,  $y$  or  $z$ . Arguing similarly, where we instead move the first and third terms to the right-hand side, or the first and second terms to the right-hand side, it therefore follows that each term in eqn (3.3) must separately be a constant, with the three constants summing to zero. Therefore either two of the constants are positive with the third negative, or two are negative with the third positive. Let us for now take the constants in the first two terms to be negative, and the last to be positive, so that we may write

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0, \quad \frac{d^2 Y}{dy^2} + \beta^2 Y = 0, \quad \frac{d^2 Z}{dz^2} - \gamma^2 Z = 0, \quad (3.4)$$

with  $\alpha$ ,  $\beta$  and  $\gamma$  being real constants subject to

$$\gamma^2 = \alpha^2 + \beta^2. \quad (3.5)$$

The solutions for  $X$ ,  $Y$  and  $Z$  will therefore be of the forms

$$X = A_1 \sin \alpha x + B_1 \cos \alpha x, \quad Y = A_2 \sin \beta y + B_2 \cos \beta y, \quad Z = A_3 \sinh \gamma z + B_3 \cosh \gamma z. \quad (3.6)$$

Equivalently, the solutions for  $X$  and  $Y$  can be taken to be linear combinations of  $e^{\pm i\alpha x}$  and  $e^{\pm i\beta y}$  respectively, while  $Z$  can be written in terms of  $e^{\pm \gamma z}$ .

Note that in the above separation of variables we introduced the three constants  $\alpha$ ,  $\beta$  and  $\gamma$ , and these are subject to the constraint  $\gamma^2 = \alpha^2 + \beta^2$ . Thus there are two independent *separation constants*, which we may take to be, for example,  $\alpha$  and  $\beta$ . In fact the process of separating variables in three dimensions will *always* involve the introduction of two independent separation constants. The analogous separation of variables in  $n$  dimensions will always involve the introduction of  $(n - 1)$  independent separation constants.

The general solution to (3.1) can now be written as a sum over all the factorised solutions of the form (3.2) that we have now constructed. Since  $\alpha$  and  $\beta$  are at this stage arbitrary constants, we could write the general solution in the form of a sum over all possible factorised solutions,

$$\phi(x, y, z) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta e^{i(\alpha x + \beta y)} \left( a(\alpha, \beta) e^{\sqrt{\alpha^2 + \beta^2} z} + b(\alpha, \beta) e^{-\sqrt{\alpha^2 + \beta^2} z} \right), \quad (3.7)$$

where  $a(\alpha, \beta)$  and  $b(\alpha, \beta)$  are arbitrary functions of the separation constants  $\alpha$  and  $\beta$ .

The general solution to (3.1) is expressed as an integral over a continuum of the basic factorised solutions, as in (3.7). This is not usually, of itself, a particularly useful thing to do. However, as soon as we also impose boundary conditions on the solution, the continuous integrals will be reduced to a discrete sum over factorised solutions.

**Example: A rectangular hollow box:**

Suppose, for example, we wish to solve Laplace's equation inside a hollow rectangular box, with sides of length  $a$ ,  $b$  and  $c$  in the  $x$ ,  $y$  and  $z$  directions respectively. We may set up the axes so that the origin is at one corner of the box, so that the faces are located at  $x = 0$  and  $x = a$ ; at  $y = 0$  and  $y = b$ ; and at  $z = 0$  and  $z = c$ . We must then impose boundary conditions on the surface that forms the boundary of the interior of the box, i.e. on the six faces of box. We shall take a simple example where we impose Dirichlet boundary conditions of the following form: Suppose that the faces are all held at zero potential, except for the face at  $z = c$ , on which the potential is specified to be

$$\phi(x, y, c) = V(x, y), \quad (3.8)$$

for some specified voltage profile function  $V(x, y)$ .

Since the potential vanishes at  $x = 0$  for all  $y$  and  $z$ , it follows that we must arrange for  $X(x)$  to vanish at  $x = 0$ . Since the general solution for  $X(x)$  is

$$X(x) = A_1 \sin \alpha x + B_1 \cos \alpha x, \quad (3.9)$$

this means we must have  $B_1 = 0$ , and so  $X(x) = A_1 \sin \alpha x$ . The potential also vanishes at  $x = a$  for all  $y$  and  $z$ , and this means that we must have  $X(a) = 0$ . This implies that  $\alpha$  must be restricted to take only a discrete (but infinite) set of values,

$$\alpha = \frac{m\pi}{a}, \quad (3.10)$$

where  $m$  is any integer. Without loss of generality we may assume that  $m$  is a *positive* integer, since the negative values will just reproduce the same set of functions (multiplied by  $-1$ ), and the case  $m = 0$  just gives  $X = 0$ .

In the same way, the vanishing of  $\phi$  at  $y = 0$  and  $y = b$  implies that  $Y(y)$  must be proportional to  $\sin \beta y$ , and that  $\beta$  must be of the form

$$\beta = \frac{n\pi}{b}, \quad (3.11)$$

where  $n$  is any positive integer.

The vanishing of  $\phi$  at  $z = 0$  implies that  $B_3 = 0$  and so  $Z(z)$  is proportional to  $\sinh \gamma z$ . Since  $\gamma$  is given in terms of  $\alpha$  and  $\beta$  by (3.5), it follows that the general solution for  $\phi$  that satisfies all the boundary conditions except the one on the remaining face at  $z = c$  can be written as

$$\phi(x, y, z) = \sum_{m \geq 1} \sum_{n \geq 1} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \left( \pi z \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \right), \quad (3.12)$$

where  $A_{mn}$  are arbitrary constants.

The constants  $A_{mn}$  are determined by matching  $\phi$  to the given boundary condition (3.8) at  $z = c$ . This amounts to constructing a two-dimensional Fourier series expansion for the function  $V(x, y)$ . Recall that we had obtained the expression (3.12) for  $\phi(x, y, z)$  everywhere inside the box, expressed as a double summation. It remains for us to determine the expansion coefficients  $A_{mn}$ , by matching  $\phi(x, y, z)$  to the given boundary potential  $V(x, y)$  at  $z = c$ . In other words, we must find  $A_{mn}$  such that

$$V(x, y) = \sum_{m \geq 1} \sum_{n \geq 1} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \left( \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \right), \quad (3.13)$$

This amounts to solving for the coefficients  $a_{mn}$  such that

$$V(x, y) = \sum_{m \geq 1} \sum_{n \geq 1} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (3.14)$$

where we have defined  $a_{mn}$  by

$$A_{mn} = \frac{a_{mn}}{\sinh \left( \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \right)}. \quad (3.15)$$

To determine the coefficients  $a_{mn}$  in (3.14), we recognise that this is an example of a two-dimensional expansion in terms of the complete sets of functions  $\sin \frac{m\pi x}{a}$  and  $\sin \frac{n\pi y}{b}$ , and so to read off  $a_{mn}$  we just need to multiply by  $\sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b}$  and integrate. We just need to note that

$$\begin{aligned} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{p\pi x}{a} dx &= \frac{1}{2} \int_0^a \left( \cos \frac{(m-p)\pi x}{a} - \cos \frac{(m+p)\pi x}{a} \right) dx, \\ &= \left[ \frac{a}{2(m-p)\pi} \sin \frac{(m-p)\pi x}{a} - \frac{a}{2(m+p)\pi} \sin \frac{(m+p)\pi x}{a} \right]_0^a = 0 \end{aligned} \quad (3.16)$$

when  $m \neq p$ , whilst when  $m = p$  we have

$$\int_0^a \sin^2 \frac{m\pi x}{a} dx = \int_0^a \left( \frac{1}{2} - \frac{1}{2} \cos \frac{2m\pi x}{a} \right) dx = \frac{1}{2}a - \frac{a}{4m\pi} \left[ \sin \frac{2m\pi x}{a} \right]_0^a = \frac{1}{2}a. \quad (3.17)$$

Thus we have

$$\int_0^a \sin \frac{m\pi x}{a} \sin \frac{p\pi x}{a} dx = \frac{1}{2}a \delta_{mp}, \quad (3.18)$$

where the Kronecker delta  $\delta_{mp}$  equals 1 if  $m = p$  and equals 0 if  $m \neq p$ .

Multiplying (3.14) by  $\sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b}$  and integrating  $\int_0^a dx \int_0^b dy$ , we therefore find

$$\begin{aligned} \int_0^a dx \int_0^b dy V(x, y) \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} &= \frac{ab}{4} \sum_{m \geq 1} \sum_{n \geq 1} a_{mn} \delta_{mp} \delta_{nq}, \\ &= \frac{ab}{4} a_{pq}, \end{aligned} \quad (3.19)$$

where we have made use of (3.18) (and the analogous result for the integral over  $y$ ). Using this, we deduce that

$$a_{mn} = \frac{4}{ab} \int_0^a dx \int_0^b dy V(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (3.20)$$

and hence, using (3.15), we have solved for the coefficients  $A_{mn}$  in the series expansion (3.12) that gives the solution for the potential everywhere inside the rectangular box.

### 3.3 Two-dimensional problems and complex methods

A special case that is interesting to examine in greater detail is a situation where, using Cartesian coordinates, everything is independent of the  $z$  coordinate. (More generally, where everything is independent of any one specific direction; but we may as well rotate our Cartesian coordinate system so that this direction is the  $z$  direction.) An example would be where one considers a cylindrical conductor of infinite length along the  $z$  direction, and asks for the potential everywhere outside it.

Thus we may consider the general situation where there is a translational symmetry along the  $z$  direction, implying that the potential  $\phi(x, y, z)$  is actually just  $\phi(x, y)$ . Thus solving Laplace's equation  $\nabla^2 \phi = 0$  just reduces to solving the two-dimensional Laplace equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi(x, y) = 0. \quad (3.21)$$

Thus the problem of finding the potential outside the cylindrical conductor of infinite length along the  $z$  direction becomes instead the problem of finding the potential in two dimensions outside a conducting circle. There are some very special techniques available for solving (3.21), which work only because we are in two dimensions. They are based upon the use of complex analysis.

Since in this subsection we are exclusively concerned with two-dimensional problems in the  $(x, y)$  plane, we will not need the symbol  $z$  for its usual purpose of denoting the Cartesian coordinate along the third direction. We can therefore use the symbol  $z$  to mean something completely different from what we have meant up until now, namely, we use  $z$  to mean the complex variable

$$z = x + \mathrm{i} y. \quad (3.22)$$

A general complex function  $f$  can be represented as  $f(x, y)$ . A *holomorphic* function is one that depends on  $x$  and  $y$  only in the special form  $f = f(z)$ . That is, it depends on  $x$  and  $y$  only as a function of the single complex variable  $z$ , but not  $\bar{z}$ . We can treat  $z$  and its complex conjugate  $\bar{z}$  as if they were independent variables, with  $z$  given by (3.22) and  $\bar{z}$  given by its complex conjugate,

$$\bar{z} = x - \mathrm{i} y. \quad (3.23)$$

Thus we have

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2\mathrm{i}}, \quad (3.24)$$

and so we can calculate the derivatives  $\partial_z \equiv \partial/\partial z$  and  $\partial_{\bar{z}} \equiv \partial/\partial \bar{z}$  by using the chain rule:

$$\begin{aligned} \partial_z &= \frac{\partial x}{\partial z} \partial_x + \frac{\partial y}{\partial z} \partial_y = \frac{1}{2}(\partial_x - \mathrm{i} \partial_y), \\ \partial_{\bar{z}} &= \frac{\partial x}{\partial \bar{z}} \partial_x + \frac{\partial y}{\partial \bar{z}} \partial_y = \frac{1}{2}(\partial_x + \mathrm{i} \partial_y), \end{aligned} \quad (3.25)$$

where  $\partial_x$  means  $\partial/\partial x$  and  $\partial_y$  means  $\partial/\partial y$ . Thus the two-dimensional Laplace operator in (3.21) can be written as

$$(\partial_x^2 + \partial_y^2) = 4\partial_z \partial_{\bar{z}}. \quad (3.26)$$

A complex function  $f(x, y)$  is said to be *holomorphic* (or, equivalently, *complex analytic*) if it obeys  $\partial_{\bar{z}} f = 0$ . If we write  $f$  in terms of its real and imaginary parts as

$$f(x, y) = \chi(x, y) + \mathrm{i} \sigma(x, y), \quad (3.27)$$

then from the expression for  $\partial_{\bar{z}}$  in (3.25), it follows that if  $f$  is holomorphic then

$$(\partial_x + \mathrm{i} \partial_y)(\chi + \mathrm{i} \sigma) = 0. \quad (3.28)$$

The real and imaginary parts must vanish separately, and since  $x$ ,  $y$ ,  $\chi$  and  $\sigma$  are all real, we get

$$\partial_x \chi = \partial_y \sigma, \quad \partial_y \chi = -\partial_x \sigma. \quad (3.29)$$

These are the Cauchy-Riemann equations.

Notice that if we define  $\vec{\nabla}$  in this subsection to mean the two-dimensional gradient operator  $\vec{\nabla} = (\partial_x, \partial_y)$ , then it follows from (3.29) that

$$(\vec{\nabla}\chi) \cdot (\vec{\nabla}\sigma) = 0. \quad (3.30)$$

That is, for a holomorphic function  $f$  the lines of  $\chi = \text{constant}$  intersect the lines of  $\sigma = \text{constant}$  at 90 degrees. Also, crucially, any holomorphic function satisfies  $\nabla^2 f = 0$ , since, as we see from eqn (3.26),  $\nabla^2 f = 4\partial_z \partial_{\bar{z}} f$ , and by definition  $\partial_{\bar{z}} f = 0$  for a holomorphic function. This means that the real and imaginary parts of  $f$  each independently satisfy Laplace's equation:

$$\nabla^2 \chi(x, y) = 0, \quad \nabla^2 \sigma(x, y) = 0. \quad (3.31)$$

(This can also be seen by using the Cauchy-Riemann equations in the form (3.29).)

This last observation lies at the heart of how we can use the methods of complex analysis in order to obtain solutions to the two-dimensional Laplace equation in some quite useful and non-trivial situations. The key further observation is that if we make a holomorphic change of complex variable, from  $z$  to  $w$  where

$$w = w(z), \quad (3.32)$$

then if  $\Psi(w)$  is a holomorphic function of  $w$ , it follows that  $\psi(z) \equiv \Psi(w(z))$  will be a holomorphic function of  $z$ . Thus if we write the real and imaginary parts of  $w$  and  $z$  as

$$w = u + \mathrm{i}v, \quad z = x + \mathrm{i}y, \quad (3.33)$$

then if we are able to construct a (real) solution  $\hat{\phi}(u, v)$  of Laplace's equation in the two-dimensional space with coordinates  $(u, v)$ , and if we can figure out the corresponding complex holomorphic function  $\Psi(w)$  of which  $\hat{\phi}(u, v)$  is the real part, then this allows us to construct the complex holomorphic function  $\psi(z) = \Psi(w(z))$  in the two-dimensional space that has coordinates  $(x, y)$ . The real part  $\phi(x, y)$  of  $\psi(z)$  will give us the solution of Laplace's equation in the two-dimensional  $(x, y)$  space that maps into the solution  $\hat{\phi}(u, v)$  in the  $(u, v)$  space.

The trick now is to find a clever mapping  $w = w(z)$  that maps an easily-solved potential theory problem in the  $(u, v)$  space into the more complicated problem that we really want to solve in the  $(x, y)$  space. Mappings of this sort are known as *conformal mappings*.

An example at this point is worth a thousand words. Suppose we wish to find the potential in the space between two infinitely long cylindrical conductors, each extending along the  $z$  direction (old  $z$ , the third Cartesian coordinate!). Thus this is really just a

two-dimensional problem of the kind we have been discussing, in which we wish to find the two-dimensional potential between two conducting circles. This would be very easy if the cylinders were concentric, but we want to consider the more complicated situation where the axes of the two cylinders do not coincide. The trick in this example is to find a mapping  $w = w(z)$  that maps two concentric circles in the  $(u, v)$  plane into two non-concentric circles in the  $(x, y)$  plane.

There is a whole body of knowledge about conformal mappings, and how to map simple two-dimensional shapes into more complicated, but useful, two-dimensional shapes. Drawing from this knowledge, let us consider as an example the conformal mapping

$$w = \frac{z - \frac{1}{2}}{\frac{1}{2}z - 1}. \quad (3.34)$$

Using  $z = x + iy$  we have therefore have

$$w = \frac{2z - 1}{z - 2} = \frac{(2x - 1) + 2iy}{(x - 2) + iy}, \quad (3.35)$$

and so

$$|w|^2 = \frac{(2x - 1)^2 + 4y^2}{(x - 2)^2 + y^2}. \quad (3.36)$$

Consider the unit circle, centred on the origin, in the  $w$  plane, i.e. the circle  $|w| = 1$ . Using (3.36) we can work out what shape this describes in the  $z$  plane. Simple algebra shows that we shall have

$$x^2 + y^2 = 1. \quad (3.37)$$

In other words, the unit circle  $|w| = 1$  in the  $w$  plane has mapped into the unit circle centred on the origin in the  $z$  plane, i.e.  $|z| = 1$ .

Now look at the circle  $|w| = \frac{1}{2}$  in the  $w$  plane. Again using (3.36), simple algebra shows that this corresponds to

$$\left(x - \frac{2}{5}\right)^2 + y^2 = \frac{4}{25}. \quad (3.38)$$

This is a circle of radius  $\frac{2}{5}$ , centred on the point  $(x, y) = (\frac{2}{5}, 0)$  in the  $(x, y)$  plane.

What we have established, therefore, is that the pair of concentric circles  $|w| = 1$  and  $|w| = \frac{1}{2}$  in the  $(u, v)$  plane has been mapped into the pair of non-concentric circles (3.37) and (3.38) in the  $(x, y)$  plane.

Note that there is nothing particularly special about the two radii we chose for the original circles in the  $(u, v)$  plane. Any circle in the  $(u, v)$  plane will map into a circle in the  $(x, y)$  plane. In general, if one considers a circle in the  $(u, v)$  plane that is centred on the origin, it will map into a circle in the  $(x, y)$  plane will be centred on some point on the  $x$

axis, displaced from the origin. One could also choose different values for the constants in the transformation (3.34) relating  $w$  and  $z$ , and this would alter the specifics of the circle in the  $(x, y)$  plane that results from a particular circle in the  $(u, v)$  plane. Our specific example is chosen just in order to illustrate the general idea.

Suppose now we consider the easy problem in the  $(u, v)$  plane, in which the circle at  $|w| = \frac{1}{2}$  is held at potential  $V$  and the circle at  $|w| = 1$  is held at potential 0. The problem is rotationally symmetric, and so we can easily see that the solution is

$$\hat{\phi}(u, v) = c \log(u^2 + v^2)^{1/2}, \quad c = -\frac{V}{\log 2}. \quad (3.39)$$

This can be written in terms of  $w$  as

$$\hat{\phi}(u, v) = c \log |w|. \quad (3.40)$$

Now we need to find the holomorphic function  $\Phi(w)$  whose real part is given by  $\hat{\phi}(u, v)$ .

A classic exercise in any complex analysis course is where the lecturer presents you with a real function and tells you it is the real part of a holomorphic function. You are then asked to integrate up the Cauchy-Riemann equations (3.29) in order to find the corresponding imaginary part of the holomorphic function, and then, putting this together with the real part, to write down the holomorphic function the lecturer first thought of.<sup>23</sup> Of course you will only succeed in doing this if the lecturer has played fair and given you a real function that is capable of arising as the real part of a holomorphic function. In other words, the function the lecturer first gives you must itself satisfy Laplace's equation. Note, by the way, that although the discussion in this paragraph, and in footnote 23, is about a holomorphic function  $f(z) = \chi(x, y) + i\sigma(x, y)$ , in our specific discussion we would be applying it to a holomorphic function  $\Psi(w)$  whose real part is  $\hat{\phi}(u, v)$ .

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<sup>23</sup>Actually there is an easier way to find the holomorphic function  $f(z)$  whose real part  $\chi(x, y)$  is given, without needing to integrate anything at all. The procedure, which does not seem to be well known, is described in section 5.5 of my online lecture notes *Methods of Theoretical Physics I*, where it is referred to as the Oppenheim method: If  $\chi(x, y)$  is the real part of  $f(z)$ , then  $f(z)$  itself is given by

$$f(z) = 2\chi\left(\frac{z+a}{2}, \frac{z-a}{2i}\right) + b, \quad (3.41)$$

where  $a$  and  $b$  are constants. The constant  $a$  can be chosen almost arbitrarily; the only requirement is that  $\chi(x, y)$  should be analytic at the point  $z \equiv x + iy = a$ . (So unless  $\chi(x, y)$  is non-analytic at  $x = y = 0$ , one can simply choose  $a = 0$ .) The real part of  $b$  is determined from the known properties of  $\chi(x, y)$ . The imaginary part of  $b$  is undeterminable; the same issue arises if one solves for the imaginary part  $\sigma(x, y)$  by the usual procedure of integrating the Cauchy-Riemann equations, since  $\sigma(x, y)$  can only be determined up to a constant of integration. The reason why (3.41) works is very simple; see my Methods lecture notes.



In our case we do indeed know that  $\hat{\phi}$  satisfies Laplace's equation, so we are guaranteed to be able to carry out the above procedure. Actually, this particular example is so simple that we can see the answer almost by inspection; the holomorphic function is just

$$\Psi(w) = c \log w. \quad (3.42)$$

We can check this by verifying that its real part is indeed  $\hat{\phi}$  in (3.40):

$$\Re(\Psi) = \frac{1}{2}(\Psi + \bar{\Psi}) = \frac{c}{2}(\log w + \log \bar{w}) = \frac{c}{2} \log(w\bar{w}) = \frac{c}{2} \log |w|^2 = c \log |w|. \quad (3.43)$$

(It is very easy to derive (3.42) by using the Oppenheim method described in footnote 23. This is an example where the constant  $a$  must be taken to be non-zero, since  $\hat{\phi}(u, v) = c \log |w|$  is non-analytic at  $w = 0$ .)

Now, we use the conformal mapping (3.34) in order to map the problem to the  $z$  plane, i.e. the  $(x, y)$  plane. We thus have the holomorphic function  $\psi(z) = \Psi(w(z))$  and so

$$\psi(z) = c \log \frac{z - \frac{1}{2}}{\frac{1}{2}z - 1}. \quad (3.44)$$

The solution to the potential between the two non-concentric circles is then given by the real part of  $\psi(z)$ , namely

$$\begin{aligned} \phi(x, y) &= c \log \left| \frac{z - \frac{1}{2}}{\frac{1}{2}z - 1} \right| = \frac{c}{2} \log \left| \frac{z - \frac{1}{2}}{\frac{1}{2}z - 1} \right|^2, \\ &= \frac{c}{2} \log \left[ \frac{(2x - 1)^2 + 4y^2}{(x - 2)^2 + y^2} \right], \end{aligned} \quad (3.45)$$

where, from eqns (3.39),  $c = -V/\log 2$ . That is, this is the potential between a circle of radius  $\frac{2}{5}$ , centred on  $(x, y) = (\frac{2}{5}, 0)$  and held at potential  $V$ , and a circle of radius 1, centred on the origin and held at potential 0.

The calculation above illustrated just one relatively simple example of how one can use a conformal mapping to turn a tricky two-dimensional potential theory problem into an easily-solvable one, by mapping the complicated geometry of the original problem into a much simpler geometry. It is a technique that can be quite useful in a variety of instances. Anyone interested can easily delve further into the subject for themselves.

## 4 Separation of variables in spherical polar coordinates

The spherical polar coordinates  $(r, \theta, \varphi)$  are related to Cartesian coordinates  $(x, y, z)$  by

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (4.1)$$

In terms of  $(r, \theta, \varphi)$ , Laplace's equation (3.1) becomes<sup>24</sup>

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \nabla_{(\theta, \varphi)}^2 \phi = 0, \quad (4.2)$$

where  $\nabla_{(\theta, \varphi)}^2$  is the two-dimensional Laplace operator on the surface of the unit-radius sphere,

$$\nabla_{(\theta, \varphi)}^2 \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (4.3)$$

It is sometimes useful to note that the radial part of the Laplacian in (4.2) can be rewritten so that the Laplacian becomes

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial^2 (r\phi)}{\partial r^2} + \frac{1}{r^2} \nabla_{(\theta, \varphi)}^2 \phi = 0, \quad (4.4)$$

Laplace's equation can be separated by first looking for factorised solutions for  $\phi(r, \theta, \varphi)$ , of the form

$$\phi(r, \theta, \varphi) = \frac{1}{r} R(r) Y(\theta, \varphi). \quad (4.5)$$

Substituting into (4.4), dividing by  $\phi$ , and multiplying by  $r^2$ , we get

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{1}{Y} \nabla_{(\theta, \varphi)}^2 Y = 0. \quad (4.6)$$

The last term in (4.6) depends only on  $\theta$  and  $\varphi$ , while the first term depends only on  $r$ , and so since the first term is equal to minus the last term, it means that each of the terms must be a constant, so

$$\nabla_{(\theta, \varphi)}^2 Y = -\lambda Y, \quad \frac{d^2 R}{dr^2} = \frac{\lambda}{r^2} R, \quad \lambda = \text{constant}. \quad (4.7)$$

The key point now is that one can show that the harmonics  $Y(\theta, \varphi)$  on the sphere are well-behaved only if the separation constant  $\lambda$  takes a certain discrete infinity of non-negative values. The most elegant way to show this is by making use of the symmetry properties of the sphere, and we shall look at that approach in a bit more detail later in the course.<sup>25</sup> For now, we shall follow the more “traditional,” if more pedestrian, approach of examining the

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<sup>24</sup>This can be seen by a brute-force calculation using the chain rule to convert from derivatives with respect to  $x, y$  and  $z$  to derivatives with respect to  $r, \theta$  and  $\varphi$ . Thus,  $\partial/\partial x = (\partial r/\partial x)\partial/\partial r + (\partial \theta/\partial x)\partial/\partial \theta + (\partial \varphi/\partial x)\partial/\partial \varphi$ , etc. There are also more elegant ways to do the calculation if one is familiar with general-coordinate tensor analysis.

<sup>25</sup>The essential point is that the surface of the unit sphere can be defined as  $x^2 + y^2 + z^2 = 1$ , and this is invariant under transformations of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow M \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

conditions under which singular behaviour of the eigenfunction solutions of the differential equation can be avoided.

To study the eigenvalue problem  $\nabla_{(\theta,\phi)}^2 Y = -\lambda Y$  in detail, we make a further separation of variables by first looking for  $Y(\theta, \varphi)$  of the factorised form  $Y(\theta, \varphi) = \Theta(\theta) \Phi(\varphi)$ . Substituting this in, and multiplying by  $\sin^2 \theta Y^{-1}$ , we get

$$\frac{1}{\Theta} \sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \lambda \sin^2 \theta + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0. \quad (4.8)$$

By now-familiar arguments, the last term depends only on  $\varphi$ , while the first two depend only on  $\theta$ . Consistency for all  $\theta$  and  $\varphi$  therefore implies that the last term must be a constant, which we shall call  $-m^2$ , and so we have

$$\frac{d^2 \Phi}{d\varphi^2} + m^2 \Phi = 0, \quad (4.9)$$

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + (\lambda \sin^2 \theta - m^2) \Theta = 0, \quad m^2 = \text{constant}. \quad (4.10)$$

The solution to the  $\Phi$  equation is  $\Phi \sim e^{\pm i m \varphi}$ . The constant  $m^2$  could, *a priori*, be positive or negative, but we must recall that the coordinate  $\varphi$  is periodic on the sphere, with period  $2\pi$ . The periodicity implies that the eigenfunctions  $\Phi$  should be periodic too, and hence it must be that  $m^2$  is non-negative. In order that we have  $\Phi(\varphi + 2\pi) = \Phi(\varphi)$  it must furthermore be the case that  $m$  is an integer.

As a side remark, note that the statement about periodicity in the preceding paragraph assumes that we are wanting to solve  $\nabla^2 \phi = 0$  in a region that includes the whole  $2\pi$  of rotation around the  $z$  axis. If we were instead solving the equation in a wedge-shaped region, with  $\varphi$  restricted to the interval  $0 \leq \varphi \leq \alpha$ , say, then we might want to impose boundary conditions such as  $\phi(r, \theta, 0) = 0$  and  $\phi(r, \theta, \alpha) = 0$ . In such a case, we would need the azimuthal functions to be of the form  $\Phi(\varphi) \sim \sin(n\pi\varphi/\alpha)$  where  $n$  is an integer, and so then the separation constant  $m$  would be of the form  $m = n\pi/\alpha$ . We shall not pursue such possibilities in what follows, and it will be understood that  $m$  is an integer.

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where  $M$  is any constant  $3 \times 3$  orthogonal matrix, satisfying  $M^T M = \mathbf{1}$ . This shows that the sphere is invariant under the orthogonal group  $O(3)$ , and hence the eigenfunctions  $Y$  must fall into representations under  $O(3)$ . This group is in fact the rotation group in three dimensions (together with reflections). The calculation of the allowed values for  $\lambda$ , and the forms of the associated eigenfunctions  $Y$ , then follow from group-theoretic considerations. Anticipating the result that we shall see by other means, the eigenvalues  $\lambda$  take the form  $\lambda_\ell = \ell(\ell + 1)$ , where  $\ell$  is any non-negative integer. The eigenfunctions are classified by  $\ell$  and a second integer  $m$ , with  $-\ell \leq m \leq \ell$ , and are the well-known spherical harmonics  $Y_{\ell m}(\theta, \phi)$ . The fact that  $\lambda$  depends on  $\ell$  but not  $m$  means that the eigenvalue  $\lambda_\ell = \ell(\ell + 1)$  has a degeneracy  $(2\ell + 1)$ .

## 4.1 Series solution of the Legendre equation

To analyse the eigenvalue equation (4.10) for  $\Theta$ , it is advantageous to define a new independent variable  $x$ , related to  $\theta$  by  $x = \cos \theta$ . (Do not confuse this variable  $x$  with the Cartesian coordinate  $x$ !) At the same time, let us now use  $y$  instead of  $\Theta$  as our symbol for the dependent variable. Equation (4.10) therefore becomes

$$\frac{d}{dx} \left( (1-x^2) \frac{dy}{dx} \right) + \left( \lambda - \frac{m^2}{1-x^2} \right) y = 0. \quad (4.11)$$

This equation is called the *Associated Legendre Equation*, and it will become necessary to study its properties, and solutions, in some detail in order to be able to construct solutions of Laplace's equation in spherical polar coordinates. In fact, as we shall see, it is convenient first to study the simpler equation when  $m = 0$ , which corresponds to the case where the harmonics  $Y(\theta, \varphi)$  on the sphere are independent of the azimuthal angle  $\varphi$ . The equation (4.11) in the case  $m = 0$  is called the *Legendre Equation*.

Taking  $m = 0$  for now, the associated Legendre equation (4.11) reduces to the Legendre equation

$$[(1-x^2)y']' + \lambda y = 0, \quad (4.12)$$

which we can also write as

$$(1-x^2)y'' - 2xy' + \lambda y = 0. \quad (4.13)$$

Note that here we are denoting a derivative with respect to  $x$  by a prime, so that  $dy/dx$  is written as  $y'$ , and so on. We can construct the solutions to (4.13) by applying a procedure known as the *Frobenius Method*, in which  $y(x)$  is obtained as a power series in  $x$ .

The general theory of how to construct series solutions to ordinary differential equations is quite involved. However, in the present case things are very simple, because the point  $x = 0$  around which we wish to expand the power series is a so-called *ordinary point* of the differential equation. (i.e. if the equation is written in the form  $y'' + P(x)y' + Q(x)y = 0$  by dividing out by  $(1-x^2)$ , the functions  $P = -2x/(1-x^2)$  and  $Q = \lambda/(1-x^2)$  are regular and analytic around  $x = 0$ .) This means that the two independent solutions to (4.13) can both be expanded in Taylor series around  $x = 0$ .<sup>26</sup>

<sup>26</sup>In more general cases where one is expanding around a singular point of the equation (say at  $x = b$ ), one needs to consider a series of the form  $y(x) = \sum_{n \geq 0} a_n(x-b)^{n+\sigma}$ , where  $\sigma$  is a constant that may be non-integer. It is rather common to see people automatically considering this more general type of expansion, even when they are expanding around an ordinary point of the equation. Although this is not actually wrong,

Thus, we begin by writing the series expansion

$$y = \sum_{n \geq 0} a_n x^n. \quad (4.14)$$

Clearly we shall have

$$y' = \sum_{n \geq 0} n a_n x^{n-1}, \quad y'' = \sum_{n \geq 0} n(n-1) a_n x^{n-2}. \quad (4.15)$$

Substituting into equation (4.13), we find

$$\sum_{n \geq 0} n(n-1) a_n x^{n-2} + \sum_{n \geq 0} (\lambda - n(n+1)) a_n x^n = 0. \quad (4.16)$$

Since we want to equate terms order by order in  $x$ , it is useful to shift the summation variable by 2 in the first term, by writing  $n = m + 2$ ;

$$\sum_{n \geq 0} n(n-1) a_n x^{n-2} = \sum_{m \geq -2} (m+2)(m+1) a_{m+2} x^m = \sum_{m \geq 0} (m+2)(m+1) a_{m+2} x^m. \quad (4.17)$$

(The last step, where we have dropped the  $m = -2$  and  $m = -1$  terms in the summation, clearly follows from the fact that the  $(m+2)(m+1)$  factor gives zero for these two values of  $m$ .) Finally, relabelling  $m$  as  $n$  again, we get from (4.16)

$$\sum_{n \geq 0} \left[ (n+2)(n+1) a_{n+2} + (\lambda - n(n+1)) a_n \right] x^n = 0. \quad (4.18)$$

Since this must hold for all values of  $x$ , it follows that the coefficient of each power of  $x$  must vanish separately, giving

$$(n+2)(n+1) a_{n+2} + (\lambda - n(n+1)) a_n = 0 \quad (4.19)$$

for all  $n \geq 0$ . Thus we have the recursion relation

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} a_n. \quad (4.20)$$

We see from (4.20) that all the coefficients  $a_n$  with  $n \geq 2$  can be solved for, in terms of  $a_0$  and  $a_1$ .<sup>27</sup> In fact, since the recursion relation involves a step of 2 (i.e. it gives  $a_{n+2}$  in terms of  $a_n$ ), all the  $a_n$  for *even*  $n$  can be solved for in terms of  $a_0$ , while all the  $a_n$  for *odd*  $n$  it is rather inconvenient, since it represents an over-parameterisation of the problem and therefore it obscures the essential simplicity of the procedure. Possibly they do it because they are unaware of the theorem that both solutions of a second-order ordinary differential equation are analytic in the neighbourhood of an ordinary point.

<sup>27</sup>Later, we shall give the explicit solution to the recursion relation (4.20).

can be solved for in terms of  $a_1$ . Since the equation is linear, we can take the even- $n$  series and the odd- $n$  series as the two linearly independent solutions of the Legendre equation, which we can call  $y_{\text{even}}(x)$  and  $y_{\text{odd}}(x)$ :

$$\begin{aligned} y_{\text{even}}(x) &= a_0 + a_2 x^2 + a_4 x^4 + \cdots, \\ y_{\text{odd}}(x) &= a_1 x + a_3 x^3 + a_5 x^5 + \cdots. \end{aligned} \quad (4.21)$$

The first solution involves only the even  $a_n$ , and thus has only even powers of  $x$ , whilst the second involves only the odd  $a_n$ , and has only odd powers of  $x$ . We can conveniently consider the two solutions separately, by taking either  $a_1 = 0$ , to discuss  $y_{\text{even}}$ , or else taking  $a_0 = 0$ , to discuss  $y_{\text{odd}}$ .

Starting with  $y_{\text{even}}$ , we can take  $a_0 \neq 0$  and  $a_1 = 0$ , so we therefore have from (4.20) that  $a_2 = -\frac{1}{2}\lambda a_0$ ,  $a_3 = 0$ ,  $a_4 = \frac{1}{12}(6 - \lambda)a_2$ ,  $a_5 = 0$ , etc. In the expression for  $a_4$ , we can substitute the expression already found for  $a_2$ , and so on. Thus we shall get

$$\begin{aligned} a_2 &= -\frac{1}{2}\lambda a_0, & a_4 &= -\frac{1}{24}\lambda(6 - \lambda)a_0, & \dots \\ a_3 &= a_5 = a_7 = \cdots = 0. \end{aligned} \quad (4.22)$$

The series solution in this case is therefore given by

$$y_{\text{even}} = a_0 \left( 1 - \frac{1}{2}\lambda x^2 - \frac{1}{24}\lambda(6 - \lambda)x^4 + \cdots \right). \quad (4.23)$$

To discuss the solution  $y_{\text{odd}}$  instead, we can take  $a_0 = 0$  and  $a_1 \neq 0$ . The recursion relation (4.20) now gives  $a_2 = 0$ ,  $a_3 = \frac{1}{6}(2 - \lambda)a_1$ ,  $a_4 = 0$ ,  $a_5 = \frac{1}{20}(12 - \lambda)a_3$ ,  $a_6 = 0$ , etc, and so we find

$$\begin{aligned} a_3 &= \frac{1}{6}(2 - \lambda)a_1, & a_5 &= \frac{1}{120}(2 - \lambda)(12 - \lambda)a_1, & \dots \\ a_2 &= a_4 = a_6 = \cdots = 0. \end{aligned} \quad (4.24)$$

The series solution in this case therefore has the form

$$y_{\text{odd}} = a_1 \left( x + \frac{1}{6}(2 - \lambda)x^3 + \frac{1}{120}(2 - \lambda)(12 - \lambda)x^5 + \cdots \right). \quad (4.25)$$

To summarise, we have produced two linearly independent solutions to our differential equation (4.13), which are given by (4.23) and (4.25). The fact that they are linearly independent is obvious, since the first is an even function of  $x$  whilst the second is an odd function.

So far in the discussion, the separation constant  $\lambda$  has been allowed to be completely arbitrary. As we shall now see, it must be restricted to take a discrete infinite set of values

in order to have solutions that are non-singular in the coordinate range  $-1 \leq x \leq 1$ . (Recall that  $x = \cos \theta$ , and so  $x$  ranges from 1 down to  $-1$  as  $\theta$  ranges from  $\theta = 0$  (at the north pole of the sphere) to  $\theta = \pi$  (at the south pole of the sphere).) Although the solutions of the Legendre equation that we have constructed are valid for any value of  $\lambda$ , we shall see that  $\lambda$  must be restricted to a discrete set of values if one insists that the power series expansions should converge for all  $x$  in the interval  $-1 \leq x \leq 1$ .

One way to test the convergence of an infinite series is by applying the *ratio test*. The statement of this test is that the series converges if the absolute value of the ratio of successive terms in the series is of magnitude less than 1, in the limit as one looks further and further down the series. If, on the other hand, the absolute value of the ratio in this limit is greater than 1, then the series diverges. If the absolute value of the ratio becomes equal to 1, then the test is inconclusive.

Since the two independent series  $y_{\text{evn}}$  and  $y_{\text{odd}}$  that we obtained correspond to the even powers of  $x$  and the odd powers of  $x$  in (4.14), the absolute value of the ratio of successive terms in either series will be

$$\begin{aligned} R_n &= \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \left| \frac{a_{n+2}}{a_n} \right| x^2, \\ &= \left| \frac{n(n+1) - \lambda}{(n+1)(n+2)} \right| x^2. \end{aligned} \tag{4.26}$$

(We used the recursion relation (4.20) in getting to the second line.) In the limit when  $n \rightarrow \infty$ , holding  $\lambda$  fixed, we therefore have

$$R_\infty = x^2. \tag{4.27}$$

Thus the series (4.14) converges if  $|x| < 1$  and diverges if  $|x| > 1$ . The test is inconclusive for  $x = \pm 1$ .

Recalling that  $x = \cos \theta$  here, we see that the values  $x = \pm 1$  are in fact attained in the physical problems we are studying, since these values correspond to  $\theta = 0$  and  $\theta = \pi$  (the north and south poles of the sphere). It is therefore important to establish how the series expansion behaves at  $x = \pm 1$ .

We shall not present a detailed discussion of the convergence here, but we simply state the result, which is that in fact, for generic values of  $\lambda$ , the series (4.14) diverges at  $x = \pm 1$ ; i.e. both solutions (even and odd) diverge. (We shall illustrate this in a concrete example, below.) Thus, in order to obtain regular solutions for the Legendre equation, we must instead arrange, by judicious choice of the values for  $\lambda$ , to make the series terminate.

Looking at the recursion relation (4.20), it is evident that we can arrange for a termination if  $\lambda$  is chosen to be given by

$$\lambda = \ell(\ell + 1), \quad \ell = \text{integer} \geq 0. \quad (4.28)$$

Then, we shall have that  $a_{\ell+2}$  is equal to zero, and then all the higher coefficients of the form  $a_{\ell+4}$ ,  $a_{\ell+6}$ , etc., will also vanish. Thus, we obtain a terminating polynomial solution, of degree  $\ell$ , when  $\lambda$  satisfies (4.28). Obviously, since it is a sum of a finite number of terms, the polynomial solution is non-singular for all  $x$  in the interval  $-1 \leq x \leq 1$ .

Note that if  $\ell$  is an even integer then it is the *even* series that terminates to give a finite polynomial solution, but the odd series does not terminate. Conversely, if  $\ell$  is an odd integer then it is the *odd* series that terminates, while the even series does not. Thus we only ever get one terminating polynomial solution, for each integer  $\ell$ .

As an example to illustrate the divergent behaviour if the series does not terminate, consider the odd series  $y_{\text{odd}}(x)$ , with  $\ell = 0$ , i.e.  $\lambda = 0$ . From (4.20) we then have

$$a_{n+2} = \frac{n a_n}{n+2}, \quad (4.29)$$

(with  $n$  odd), for which the solution can easily be seen to be  $a_n = a_1/n$ . Thus the series (4.14) becomes

$$y = a_1 \left( x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots \right) = a_1 \sum_{p \geq 0} \frac{x^{2p+1}}{2p+1}, \quad (4.30)$$

which can be recognised as the power-series expansion of

$$y = \frac{1}{2}a_1 \log \left( \frac{1+x}{1-x} \right). \quad (4.31)$$

This function clearly diverges logarithmically at  $x = \pm 1$ . For all other values of  $\lambda$  that lead to non-terminating series, one similarly finds a logarithmic divergence at  $x = \pm 1$ .

To summarise, we have established that if  $\lambda$  is given by (4.28), there exists one polynomial solution to the Legendre equation, and it is in particular regular for all  $-1 \leq x \leq 1$ . This solution is called the *Legendre Polynomial*  $P_\ell(x)$ , satisfying

$$(1-x^2) \frac{d^2 P_\ell(x)}{dx^2} - 2x \frac{dP_\ell(x)}{dx} + \ell(\ell+1) P_\ell(x) = 0. \quad (4.32)$$

By convention, the Legendre polynomial  $P_\ell(x)$  is normalised so that

$$P_\ell(1) = 1. \quad (4.33)$$



The first few are therefore given by

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x, & P_2(x) &= \frac{1}{2}(3x^2 - 1), \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x), & P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned} \quad (4.34)$$

With  $\lambda$  given by (4.28), and with the corresponding Legendre polynomial normalised according to (4.33), it is not hard to solve the recursion relation (4.20) explicitly, giving the result that

$$P_\ell(x) = \sum_{k=0}^{[\ell/2]} \frac{(-1)^k (2\ell - 2k)!}{2^\ell k! (\ell - k)! (\ell - 2k)!} x^{\ell-2k}, \quad (4.35)$$

where  $[\ell/2]$  is the integer part of  $\ell/2$ .<sup>28</sup>

The expression (4.35) for the  $\ell$ 'th Legendre polynomial is somewhat unwieldy, and it is often useful to have alternative ways of writing  $P_\ell(x)$ . We shall give two such alternative expressions in the next two subsections.

## 4.2 Rodrigues' formula

First, noting that  $\frac{d}{dx} x^p = px^{p-1}$ ,  $\frac{d^2}{dx^2} x^p = p(p-1)x^{p-2}$ , etc., and so  $\frac{d^\ell}{dx^\ell} x^p = \frac{p! x^{p-\ell}}{(p-\ell)!}$ , we observe that (4.35) can be written as

$$P_\ell(x) = \frac{d^\ell}{dx^\ell} \sum_{k=0}^{[\ell/2]} \frac{(-1)^k}{2^\ell k! (\ell - k)!} x^{2\ell-2k}. \quad (4.36)$$

The summation can then be extended up to  $k = \ell$ , since the extra terms that are added in the summation will involve powers of  $x$  of degree less than  $\ell$ , and these will be killed off by the  $\frac{d^\ell}{dx^\ell}$ . Inserting a factor of  $\ell!$  inside the sum, and dividing it out again in the prefactor, then gives

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \sum_{k=0}^{\ell} \frac{(-1)^k \ell!}{k! (\ell - k)!} x^{2\ell-2k}, \quad (4.37)$$

and we can recognise the sum as the binomial expansion of  $(x^2 - 1)^\ell$ . Thus we arrive at *Rodrigues' formula* for  $P_\ell(x)$ :

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell. \quad (4.38)$$

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<sup>28</sup>To see this, read off the coefficients  $a_n$  in (4.14) by equating the coefficients of each power of  $x$  with those in (4.35), and then show that these expressions for  $a_n$  indeed satisfy the recursion relation (4.20). This proves that (4.35) indeed satisfies the Legendre equation (4.32). The only slightly tricky point is establishing that the normalisation in (4.35) is indeed such that  $P_\ell(x)$  satisfies (4.33). We give a simple proof of this in the next section.

Having shown that the expressions (4.35) and (4.38) agree, we can now easily confirm that  $P_\ell(x)$  so defined does indeed satisfy the normalisation (4.33). To do this, write (4.38) as

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \left[ (x-1)^\ell (x+1)^\ell \right], \quad (4.39)$$

and note that when we differentiate the product  $(x-1)^\ell (x+1)^\ell$  a total of  $\ell$  times, the *only* term that survives after then setting  $x = 1$  is the term where all  $\ell$  derivatives land on  $(x-1)^\ell$ . (This is because if we differentiate  $(x-1)^\ell$  fewer than  $\ell$  times, there will be one or more undifferentiated factors of  $(x-1)$ , and these will give zero when we set  $x = 1$ .) Since  $\frac{d^\ell}{dx^\ell} (x-1)^\ell = \ell!$ , we therefore see that indeed

$$P_\ell(1) = \frac{1}{2^\ell \ell!} \ell! (1+1)^\ell = 1. \quad (4.40)$$

### 4.3 The generating function

Another very useful way of representing the Legendre polynomials is by means of a *Generating Function*. The claim is that the Legendre polynomials  $P_\ell(x)$  satisfy

$$G(x, t) \equiv (1 - 2xt + t^2)^{-1/2} = \sum_{\ell \geq 0} t^\ell P_\ell(x), \quad (4.41)$$

where, for convergence of the series, we must have  $|t| < 1$ . We can see how this is working by looking at the first few terms in the power-series expansion of the left-hand side in powers of  $t$ , giving

$$G(x, t) = 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \frac{1}{2}(5x^3 - 3x)t^3 + \frac{1}{8}(35x^4 - 30x^2 + 3)t^4 + \cdots \quad (4.42)$$

Equating this with the right-hand side of (4.41), and comparing the coefficients of each power of  $t$ , we read off

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad (4.43)$$

and so on, which is precisely what we were finding in (4.34).

To prove that (4.41) correctly generates *all* the Legendre polynomials, we note that if  $P_\ell(x)$  satisfies the Legendre equation (4.32) for all  $\ell$ , then multiplying by  $t^\ell$  and summing over  $\ell$  implies that  $H$  defined by

$$H \equiv \sum_{\ell \geq 0} t^\ell [(1-x^2) P_\ell'' - 2x P_\ell' + \ell(\ell+1) P_\ell] \quad (4.44)$$

should vanish. In fact,  $H$  will vanish if and only if  $P_\ell(x)$  satisfies the Legendre equation (4.32) for all integers  $\ell \geq 0$ . Now, looking at (4.41) we can see that  $H$  can be written as

$$H = (1 - x^2) \frac{\partial^2 G(x, t)}{\partial x^2} - 2x \frac{\partial G(x, t)}{\partial x} + t \frac{\partial^2 (t G(x, t))}{\partial t^2}. \quad (4.45)$$

(The three terms here correlate exactly with the three terms on the right-hand side of (4.44).) It is now just a simple exercise to calculate the  $G$  derivatives in (4.45), using the definition of  $G$  in (4.41), to show that indeed we have  $H = 0$ , which proves that the functions  $P_\ell(x)$  defined by (4.41) satisfy the Legendre equation. They are clearly polynomials, because the power-series expansion of the left-hand side of (4.41) in powers of  $t$  will clearly have  $x$ -dependent coefficients that are polynomial in  $x$ , as we saw in the first few terms of the expansion, in (4.42).

Finally, we must check the normalisation, i.e. that  $P_\ell(1) = 1$ . This is very easy; we just set  $x = 1$  in (4.41), to get

$$(1 - 2t + t^2)^{-1/2} = \sum_{\ell \geq 0} t^\ell P_\ell(1). \quad (4.46)$$

But the left-hand side is just  $(1 - t)^{-1}$ , which has the binomial expansion

$$\frac{1}{1 - t} = 1 + t + t^2 + t^3 + t^4 + \dots = \sum_{\ell \geq 0} t^\ell, \quad (4.47)$$

and so by comparing with the right-hand side in (4.46) we immediately get  $P_\ell(1) = 1$ .

#### 4.4 Expansion in Legendre polynomials

Recall that our goal is to construct solutions of Laplace's equation (4.2) written in spherical polar coordinates, and that we have established that for azimuthally-symmetric solutions the relevant functions in the  $\theta$  direction are expressed in terms of Legendre polynomials. We have seen that the factorised azimuthally-symmetric solutions for the potential will be of the form  $\phi(r, \theta) = r^{-1} R_\ell(r) P_\ell(\cos \theta)$ , where  $P_\ell$  are the Legendre polynomials and  $R_\ell(r)$  will be the solutions of the radial equation in (4.7), now with the separation constant  $\lambda$  determined to be equal to  $\ell(\ell + 1)$ . The general azimuthally-symmetric solution will then involve a sum over all such factorised solutions, with  $\ell$ -dependent constant coefficients that will be determined by the boundary conditions.

Constructing a general azimuthally-symmetric solution will therefore require that we expand a general function of  $\theta$  as a sum over Legendre polynomials  $P_\ell(\cos \theta)$ . In terms of

$x = \cos \theta$ , the first task then is to expand a general function  $f(x)$  in the form

$$f(x) = \sum_{\ell \geq 0} a_\ell P_\ell(x). \quad (4.48)$$

We can establish the following properties of the Legendre polynomials. Firstly,

$$\int_{-1}^1 dx P_\ell(x) P_n(x) = 0, \quad \text{if } \ell \neq n. \quad (4.49)$$

This can be seen by taking the Legendre equation (4.32) and multiplying it by  $P_n(x)$ , then subtracting the same expression but with  $\ell$  and  $n$  interchanged, and finally, integrating over the interval  $-1 \leq x \leq 1$ . This gives

$$\begin{aligned} \int_{-1}^1 dx \left[ P_n(x) \frac{d}{dx} \left( (1-x^2) \frac{dP_\ell(x)}{dx} \right) - P_\ell(x) \frac{d}{dx} \left( (1-x^2) \frac{dP_n(x)}{dx} \right) \right. \\ \left. [\ell(\ell+1) - n(n+1)] P_\ell(x) P_n(x) \right] = 0. \end{aligned} \quad (4.50)$$

The two terms on the top line can be written as

$$\frac{d}{dx} \left[ P_n(x) (1-x^2) \frac{dP_\ell(x)}{dx} - P_\ell(x) (1-x^2) \frac{dP_n(x)}{dx} \right], \quad (4.51)$$

as can be seen by distributing the derivative and comparing with the first two terms in eqn (4.50). Thus we can integrate the first two terms in (4.50), giving

$$\begin{aligned} \left[ P_n(x) (1-x^2) \frac{dP_\ell(x)}{dx} - P_\ell(x) (1-x^2) \frac{dP_n(x)}{dx} \right]_{-1}^1 \\ + [\ell(\ell+1) - n(n+1)] \int_{-1}^1 dx P_\ell(x) P_n(x) = 0. \end{aligned} \quad (4.52)$$

The first line vanishes, because the  $(1-x^2)$  factor vanishes at the  $x = \pm 1$  endpoints, leaving

$$[\ell(\ell+1) - n(n+1)] \int_{-1}^1 dx P_\ell(x) P_n(x) = 0. \quad (4.53)$$

Thus if  $\ell \neq n$  (and recalling that  $\ell$  and  $n$  are both non-negative integers), we see that the orthogonality condition (4.49) must hold.

We also need to know what the integral in (4.49) gives when  $\ell = n$ . Here is a slightly unorthodox derivation, which is actually quite elegant. Using the generating function (4.41) twice over, once for  $\sum_\ell t^\ell P_\ell(x)$  and once for  $\sum_n t^n P_n(x)$ , we can deduce that

$$\int_{-1}^1 dx (1-2xt+t^2)^{-1/2} (1-2xt+t^2)^{-1/2} = \sum_{\ell \geq 0} \sum_{n \geq 0} t^\ell t^n \int_{-1}^1 dx P_\ell(x) P_n(x), \quad (4.54)$$

and so

$$\int_{-1}^1 \frac{dx}{1-2xt+t^2} = \sum_{\ell \geq 0} \sum_{n \geq 0} t^\ell t^n \int_{-1}^1 dx P_\ell(x) P_n(x), \quad (4.55)$$

Using the already-established orthogonality result (4.49), we therefore find

$$\int_{-1}^1 \frac{dx}{1-2xt+t^2} = \sum_{\ell \geq 0} t^{2\ell} \int_{-1}^1 dx (P_\ell(x))^2. \quad (4.56)$$

Performing the integral on the left-hand side gives

$$\frac{1}{t} \log \left( \frac{1+t}{1-t} \right) = \sum_{\ell \geq 0} t^{2\ell} \int_{-1}^1 dx (P_\ell(x))^2, \quad (4.57)$$

and expanding the logarithm in a Taylor series implies

$$\sum_{\ell \geq 0} \frac{2t^{2\ell}}{2\ell+1} = \sum_{\ell \geq 0} t^{2\ell} \int_{-1}^1 dx (P_\ell(x))^2. \quad (4.58)$$

Equating the coefficients of each power of  $t$  then gives<sup>29</sup>

$$\int_{-1}^1 dx (P_\ell(x))^2 = \frac{2}{2\ell+1}. \quad (4.61)$$

In summary, we have shown that

$$\int_{-1}^1 dx P_\ell(x) P_n(x) = \frac{2}{2\ell+1} \delta_{\ell,n}, \quad (4.62)$$

where  $\delta_{\ell,n}$  is the Kronecker delta symbol, which equals 1 if  $\ell$  and  $n$  are equal, and is 0 otherwise.

With this result, we can now determine the coefficients  $a_\ell$  in the generalised Fourier expansion (4.48). Multiplying this equation by  $P_n(x)$ , integrating over the interval  $-1 \leq x \leq 1$ , and using (4.62), we find

$$\int_{-1}^1 dx f(x) P_n(x) = \sum_{\ell \geq 0} a_\ell \int_{-1}^1 dx P_\ell(x) P_n(x) = \sum_{\ell \geq 0} \frac{2}{2\ell+1} \delta_{\ell,n} a_\ell = \frac{2}{2n+1} a_n, \quad (4.63)$$

and hence the coefficients in (4.48) are given by

$$a_\ell = \frac{1}{2} (2\ell+1) \int_{-1}^1 dx f(x) P_\ell(x). \quad (4.64)$$

<sup>29</sup>With a little more work, one can consider the result where we use a different auxiliary variable  $s$ , instead of  $t$ , in the generating function for  $P_n(x)$ :

$$(1-2xs+s^2)^{-1/2} = \sum_{n \geq 0} s^n P_n(x).$$

We then have

$$\int_{-1}^1 dx (1-2xt+t^2)^{-1/2} (1-2xs+s^2)^{-1/2} = \sum_{\ell \geq 0} \sum_{n \geq 0} t^\ell s^n \int_{-1}^1 dx P_\ell(x) P_n(x). \quad (4.59)$$

This gives

$$\frac{1}{\sqrt{st}} \left[ 2 \log(1+\sqrt{st}) - \log(1-st) \right] = \sum_{p \geq 0} \frac{2}{2p+1} (st)^p, \quad (4.60)$$

and so matching the powers of  $s$  and  $t$  with those on the right-hand side of (4.59), one can deduce immediately the orthogonality (4.49) and the result (4.61).

#### 4.5 Azimuthally-symmetric solutions of Laplace's equation

With these preliminaries, we can now return to the original problem, of finding solutions to Laplace's equation in spherical polar coordinates. For now, we are restricting attention to the case where the problem has azimuthal symmetry, meaning that it is independent of the azimuthal coordinate  $\varphi$ . This means that we take  $m = 0$  in (4.8), and so the original potential  $\phi$  has been separated by finding factorised solutions of the form

$$\phi(r, \theta) = \frac{1}{r} R(r) \Theta(\theta), \quad (4.65)$$

with  $R$  and  $\Theta$  satisfying

$$\frac{d^2 R}{dr^2} = \frac{\lambda}{r^2} R, \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \lambda \Theta = 0. \quad (4.66)$$

(See equations (4.7) and (4.8).) As we now know, the equation for  $\Theta$  becomes the Legendre equation (4.11) after making the change of variable  $\cos \theta = x$ , and its regular solutions are the Legendre polynomials  $P_\ell(x)$ , occurring when  $\lambda = \ell(\ell + 1)$ .

It is easy to see, by looking for trial solutions of the form  $R(r) = r^\alpha$ , that the two linearly independent solutions of the radial equation in (4.66) are given by

$$R = r^{\ell+1} \quad \text{and} \quad R = r^{-\ell}. \quad (4.67)$$

Thus from eqn (4.65) we see that for a given value of  $\ell$  there are two factorised solutions of Laplace's equation, namely

$$\phi(r, \theta) = r^\ell P_\ell(\cos \theta) \quad \text{and} \quad \phi(r, \theta) = r^{-\ell-1} P_\ell(\cos \theta). \quad (4.68)$$

By summing such solutions for all integers  $\ell \geq 0$ , with arbitrary constant coefficients for each term, we arrive at the general azimuthally-symmetric solution

$$\phi(r, \theta) = \sum_{\ell \geq 0} \left( A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos \theta), \quad (4.69)$$

where  $A_\ell$  and  $B_\ell$  are arbitrary constants. These will be determined by the boundary conditions that specify a particular physical configuration.

#### Exterior Problem:

Suppose, for example, we are considering the *exterior* problem, in which the potential is specified on a spherical surface at  $r = a$ , and one wishes to calculate the potential at all

points outside this surface. We may assume that the potential goes to zero at infinity, and so we must have

$$A_\ell = 0. \quad (4.70)$$

The  $B_\ell$  coefficients are determined in the manner we established in the previous subsection. Note that after transforming back to the variable  $\theta$  from the variable  $x = \cos \theta$ , the integral (4.62) becomes

$$\int_0^\pi d\theta \sin \theta P_\ell(\cos \theta) P_n(\cos \theta) = \frac{2}{2\ell + 1} \delta_{\ell,n}, \quad (4.71)$$

and so the coefficients  $B_\ell$  in (4.69) are given in terms of the boundary values  $\phi(a, \theta)$  of the potential by

$$B_\ell = \frac{1}{2}(2\ell + 1) a^{\ell+1} \int_0^\pi d\theta \sin \theta \phi(a, \theta) P_\ell(\cos \theta). \quad (4.72)$$

#### Interior Problem:

A related case arises for the *interior* problem, where the potential is specified on a closed spherical surface  $r = a$  and one wishes to solve for the potential everywhere inside this surface. Assuming there are no charges inside the sphere (i.e. it is just a vacuum inside), then the coefficients  $B_\ell$  in the general expansion (4.69) must now be zero, since we require that the potential be regular at  $r = 0$ . The  $A_\ell$  coefficients will now be given by

$$A_\ell = \frac{1}{2}(2\ell + 1) a^{-\ell} \int_0^\pi d\theta \sin \theta \phi(a, \theta) P_\ell(\cos \theta). \quad (4.73)$$

#### Region Between Two Spherical Surfaces:

A third related example arises if one wishes to solve for  $\phi$  in the free-space region between two concentric spherical surfaces with  $r = a$  and  $r = b$ . Let us assume that  $b > a$ . Now, both the  $A_\ell$  and  $B_\ell$  coefficients will in general be non-vanishing. The boundary values for  $\phi$  must be specified on both the surfaces, and so one again has the correct total number of equations to solve for all of the coefficients. Thus one will have

$$\begin{aligned} A_\ell a^\ell + B_\ell a^{-\ell-1} &= \frac{1}{2}(2\ell + 1) \int_0^\pi d\theta \sin \theta \phi(a, \theta) P_\ell(\cos \theta), \\ A_\ell b^\ell + B_\ell b^{-\ell-1} &= \frac{1}{2}(2\ell + 1) \int_0^\pi d\theta \sin \theta \phi(b, \theta) P_\ell(\cos \theta), \end{aligned} \quad (4.74)$$

which can be solved for all  $A_\ell$  and  $B_\ell$ .

#### Example with Neumann boundary condition:

Of course one can easily consider other classes of problem too. For example, one may consider Neumann cases where it is the normal derivative of  $\phi$ , rather than  $\phi$  itself, that is specified on the spherical boundary or boundaries. Consider, for example, the exterior problem in this case. From (4.69), again assuming that  $\phi$  goes to zero at infinity and so  $A_\ell = 0$ , one has

$$\left. \frac{\partial \phi(r, \theta)}{\partial r} \right|_{r=a} = - \sum_{\ell \geq 0} (\ell + 1) B_\ell a^{-\ell-2} P_\ell(\cos \theta). \quad (4.75)$$

Multiplying by  $P_n(\cos \theta)$  and integrating, we therefore find

$$B_\ell = - \frac{(2\ell + 1)}{2(\ell + 1)} a^{\ell+2} \int_0^\pi d\theta \sin \theta \left( \left. \frac{\partial \phi(r, \theta)}{\partial r} \right|_{r=a} \right) P_\ell(\cos \theta). \quad (4.76)$$

Analogous results can be obtained for the interior problem and for the case of the region between two concentric spheres.

#### 4.6 Some useful techniques for azimuthally-symmetric problems

The procedure for solving Laplace's equation described in the previous subsection is straightforward, but evaluating the integrals in order to read off the expansion coefficients in (4.48) can sometimes be a little involved. There is a very useful “trick” which can often be employed to obtain the answer in a more elegant way, with considerably less effort. We describe this, and a couple of further techniques, below.

##### Solution by off-axis extrapolation:

In an electrostatics problem with azimuthal symmetry, it is often relatively easy to obtain the expression for the potential along the axis of symmetry by elementary means. Thus, if we make the natural choice and align the  $z$  axis along the axis of azimuthal symmetry, this means that we may have a situation where the potential  $\phi(z)$ , on the  $z$  axis, is easily calculated. In terms of spherical polar coordinates, points along the  $z$  axis correspond to  $\theta = 0$  (for the positive  $z$  axis), or  $\theta = \pi$  (for the negative  $z$  axis).

Consider an “exterior” azimuthally-symmetric electrostatics problem, for which we wish to find  $\phi(r, \theta)$  in an exterior region, say  $r \geq a$ , subject to the assumption that  $\phi$  goes to zero at infinity. From (4.48), the  $A_\ell$  coefficients will be zero, and the general such solution to Laplace's equation will take the form

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta). \quad (4.77)$$

Suppose now that we already know what the solution for  $\phi$  is on the  $z$  axis; call this  $\phi(z)$ . For simplicity we shall first consider first the positive  $z$  axis, which corresponds to  $\theta = 0$ .



Recalling that in the transformation from Cartesian to spherical polar coordinates we have  $z = r \cos \theta$ , we note that  $\theta = 0$  and  $r = z$  on the positive  $z$  axis, and therefore the potential on the positive  $z$  axis, following from (4.77), will be

$$z > 0 : \quad \phi(z) = \phi(z, 0) = \sum_{\ell \geq 0} \frac{B_\ell}{z^{\ell+1}} P_\ell(1) = \sum_{\ell \geq 0} \frac{B_\ell}{z^{\ell+1}}, \quad (4.78)$$

where, in getting to the final expression, we have used the normalisation property  $P_\ell(1) = 1$  of the Legendre polynomials.

If we have already, by some other means, calculated the potential  $\phi(z)$  on the positive  $z$  axis then we only need to take this expression, expand it as a power series in inverse powers of  $z$ , and equate the coefficients of each inverse power of  $z$  to the corresponding coefficients  $B_\ell$  in (4.78). Having thereby determined the  $B_\ell$  coefficients we can then substitute them into (4.79) in order to obtain an expression for the potential  $\phi(r, \theta)$  in the entire upper half space corresponding to  $0 \leq \theta < \frac{1}{2}\pi$ . Thus, we have performed an extrapolation of a result for the potential on the positive  $z$  axis into the entire upper half space.

There may in fact be circumstances where the expansion coefficients are different in the regions above and below the  $z = 0$  plane (i.e. in the regions  $0 \leq \theta < \frac{1}{2}\pi$  and  $\frac{1}{2}\pi < \theta \leq \pi$ ). To allow for this possibility, we shall write different expansions in the two regions:

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta), \quad 0 \leq \theta < \frac{1}{2}\pi, \quad (4.79)$$

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{\tilde{B}_\ell}{r^{\ell+1}} P_\ell(\cos \theta), \quad \frac{1}{2}\pi < \theta \leq \pi. \quad (4.80)$$

We can then carry out an analogous extrapolation in the lower half space. On the negative  $z$  axis we have  $\theta = \pi$  and so  $r = -z = |z|$  ( $r$  is of course always, by definition, non-negative). Thus from (4.80) we shall have<sup>30</sup>

$$z < 0 : \quad \phi(z) = \phi(|z|, \pi) = \sum_{\ell \geq 0} \frac{\tilde{B}_\ell}{|z|^{\ell+1}} P_\ell(-1) = \sum_{\ell \geq 0} \frac{\tilde{B}_\ell}{|z|^{\ell+1}} (-1)^\ell = - \sum_{\ell \geq 0} \frac{\tilde{B}_\ell}{z^{\ell+1}}. \quad (4.81)$$

Note that we have made use of the fact that  $P_\ell(-x) = (-1)^\ell P_\ell(x)$ , and so  $P_\ell(-1) = (-1)^\ell$ . Thus if we already know by some other means the expression for the potential on the negative  $z$  axis, we just have to expand it as a power series in inverse powers of  $z$  and then, matching it with (4.81), read off the coefficients  $\tilde{B}_\ell$ . Substituting into (4.80) then gives an expression for the potential in the entire lower half space.

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<sup>30</sup>For a first reading of this material, it is probably advantageous to forget about the subtleties of negative  $z$  versus positive  $z$ , and just focus on the previous discussion for positive  $z$  only. Otherwise, there is a risk that the essential simplicity of the procedure becomes obscured in the minutiae of the minus signs.

The crucial thing that allows this procedure of off-axis extrapolation to work is that the number of arbitrary coefficients in the expansion (4.79) (or (4.80)), namely one coefficient per inverse power of  $r$ , is exactly the same as the number of coefficients in the expansion of a potential on the  $z$  axis in inverse powers of  $z$ , namely, one coefficient per inverse power. Thus the knowledge of these coefficients in the expansion of the potential on the  $z$  axis fully determines the coefficients in (4.79) or (4.80).<sup>31</sup>

An analogous discussion can be given for the small- $r$  region, where we match expansions of the form

$$\phi(r, \theta) = \sum_{\ell \geq 0} A_\ell r^\ell P_\ell(\cos \theta), \quad 0 \leq \theta < \frac{1}{2}\pi, \quad (4.82)$$

$$\phi(r, \theta) = \sum_{\ell \geq 0} \tilde{A}_\ell r^\ell P_\ell(\cos \theta), \quad \frac{1}{2}\pi < \theta \leq \pi \quad (4.83)$$

to a small- $z$  expansion of the on-axis potential. Again, one needs in general to allow for the possibility that the coefficients  $A_\ell$  determined from the expansion on the positive  $z$  axis may be different from the coefficients  $\tilde{A}_\ell$  determined from the expansion on the negative  $z$  axis.

Note that in practice, the potentials one is working with often have a very simple symmetry between the positive- $z$  region and the negative- $z$  region. For example, one may have  $\phi(z) = \phi(-z)$ , or else one may have  $\phi(z) = -\phi(-z)$ . In either of these cases, an easy way to sort out the relation between the positive- $z$  expansion and the negative- $z$  expansion is to work out first the extrapolation from  $\phi(z)$  to  $\phi(r, \theta)$  for positive  $z$ , and then check to see whether the resulting Legendre polynomial expansion for  $\phi(r, \theta)$  gives the right result on the negative  $z$  axis. If it agrees with the known expression for  $\phi(z)$  on the negative  $z$  axis then all is well. If the signs of the coefficients are incorrect, then one has to write a different expansion for  $\phi(r, \theta)$  in the negative- $z$  region, with the signs changed appropriately.

As an example, consider the problem that we discussed in section (2.5.2), when illustrating the use of the Dirichlet Green function for the sphere. The potential on the upper hemisphere of a sphere of radius  $a$  was taken to be the constant  $+V$ , and on the lower hemisphere the potential was  $-V$ . The exact expression for the potential in the region  $r > a$  was obtained in the form of the integral (2.68), but this cannot be evaluated explicitly in any useful form. We observed at the time that the expression for  $\phi$  on the (positive)  $z$  axis took the form (2.69). We should really be a little more careful now, and consider the expression

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<sup>31</sup>An off-axis extrapolation like this would not work in a situation where there is no azimuthal symmetry, as we shall see later.

on the negative  $z$  axis also. If we do this, it is straightfoward to see that the general on-axis expression for  $\phi$ , valid both for  $z > a$  and for  $z < -a$ , is

$$\phi(z) = V \left( \text{sign}(z) - \frac{z^2 - a^2}{z\sqrt{z^2 + a^2}} \right), \quad (4.84)$$

where  $\text{sign}(z)$  equals 1 if  $z > 0$ , whilst it equals  $-1$  if  $z < 0$ . Note that (4.84) has the property  $\phi(z) = -\phi(-z)$ , i.e. that it changes sign under  $z \rightarrow -z$ , as it should since the boundary data on the sphere at  $r = a$  changes sign under  $z \rightarrow -z$ . The potential (4.84) can be written as

$$\phi(z) = \text{sign}(z) V \left[ 1 - \left( 1 - \frac{a^2}{z^2} \right) \left( 1 + \frac{a^2}{z^2} \right)^{-1/2} \right]. \quad (4.85)$$

It is a simple matter to expand the inverse square root using the binomial theorem, and thereby obtain the expansion for  $\phi(z)$  as an infinite series in powers of  $1/z$ . The first few terms are

$$\phi(z) = \text{sign}(z) \left[ \frac{3V}{2} \frac{a^2}{z^2} - \frac{7V}{8} \frac{a^4}{z^4} + \frac{11V}{16} \frac{a^6}{z^6} - \frac{75V}{128} \frac{a^8}{z^8} + \dots \right], \quad (4.86)$$

and the general result is

$$\begin{aligned} \phi(z) &= \text{sign}(z) V \left[ 1 - \left( 1 - \frac{a^2}{z^2} \right) \sum_{n \geq 0} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \left( \frac{a}{z} \right)^{2n} \right] \\ &= \text{sign}(z) V \sum_{n \geq 1} \frac{(-1)^{n+1} (4n-1)(2n-2)!}{2^{2n-1} n! (n-1)!} \left( \frac{a}{z} \right)^{2n}. \end{aligned} \quad (4.87)$$

Reading off the coefficients in the expansion (4.86) or (4.87), and matching them against the expansion coefficients  $B_\ell$  and  $\tilde{B}_\ell$  in (4.78) and (4.81), we can therefore immediately conclude that  $B_{2n} = \tilde{B}_{2n} = 0$ , that the odd coefficients are non-zero, with  $B_{2n+1} = \tilde{B}_{2n+1}$ , and that the general off-axis series for the potential is given by

$$\begin{aligned} \phi(r, \theta) &= V \sum_{n \geq 1} \frac{(-1)^{n+1} (4n-1)(2n-2)!}{2^{2n-1} n! (n-1)!} \left( \frac{a}{r} \right)^{2n} P_{2n-1}(\cos \theta) \\ &= \frac{3V}{2} \frac{a^2}{r^2} P_1(\cos \theta) - \frac{7V}{8} \frac{a^4}{r^4} P_3(\cos \theta) + \frac{11V}{16} \frac{a^6}{r^6} P_5(\cos \theta) - \frac{75V}{128} \frac{a^8}{r^8} P_7(\cos \theta) + \dots, \end{aligned} \quad (4.88)$$

valid in the region  $r > a$ . As a check, we may observe that the potential indeed changes sign under reflection in the  $z = 0$  plane, i.e. under  $\theta \rightarrow \pi - \theta$ , because all the  $P_\ell(\cos \theta)$  have this property when  $\ell$  is odd. Thus in this case, the result we obtain by off-axis extrapolation in the upper half space is already also valid in the lower half space.

The technique of off-axis extrapolation is clearly very powerful. Indeed, it can really be viewed as the default approach that one should try first, when solving electrostatics problems where there is azimuthal symmetry.

### Solution by inversion:

It is sometimes the case that one already has the result for the series expansion in, say, the exterior region  $r > a$ , and one now wishes to find the series expansion in the interior region  $r < a$ . This can, of course, be done by going back and re-solving the problem from scratch, in the region  $r < a$ , where now it will be the  $A_\ell$  coefficients in (4.48), rather than the  $B_\ell$  coefficients, that are non-zero. Sometimes, an easier method is to use the following procedure, known as *Inversion*.

Suppose we have already found that the solution in the exterior region is given by

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta), \quad (4.89)$$

where the coefficients  $B_\ell$  are known. On general grounds, we know that the solution in the interior region will be of the general form

$$\phi(r, \theta) = \sum_{\ell \geq 0} A_\ell r^\ell P_\ell(\cos \theta). \quad (4.90)$$

(We are assuming there are no free charges in the interior region.)

Under the assumption that there are no charges that could give rise to singularities at the  $r = a$  interface, it follows that we can use either of the expressions (4.89) or (4.90) at  $r = a$  itself, and so equating the two, we find

$$\sum_{\ell \geq 0} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos \theta) = \sum_{\ell \geq 0} A_\ell a^\ell P_\ell(\cos \theta). \quad (4.91)$$

Since this equation must hold for all  $\theta$ , and since the Legendre polynomials are linearly independent, it follows that the coefficients of each  $P_\ell$  separately must be equal on the two sides of the equation. Therefore we can deduce that

$$A_\ell = \frac{B_\ell}{a^{2\ell+1}}, \quad (4.92)$$

and hence the series expansion (4.90) in the interior region is given by

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{B_\ell r^\ell}{a^{2\ell+1}} P_\ell(\cos \theta), \quad (4.93)$$

in terms of the already-determined coefficients  $B_\ell$  of the exterior region.

One could, of course, equally well apply the inversion procedure the other way around, and instead find the expansion in the exterior region if the expansion in the interior region is already known.

The example we discussed previously, of the potential outside the two hemispheres held at potentials  $V$  and  $-V$ , provides a nice illustration of the method of inversion. The general solution for the potential outside the two hemispheres was found to be given by (4.88). Using the inversion relation (4.92), it therefore follows that all we need to do to obtain the solution inside the two hemispheres is to apply the replacement rule

$$\left(\frac{a}{r}\right)^{\ell+1} P_{\ell}(\cos \theta) \longrightarrow \left(\frac{r}{a}\right)^{\ell} P_{\ell}(\cos \theta) \quad (4.94)$$

to the term in  $P_{\ell}(\cos \theta)$  in (4.88), for each value of  $\ell$ . This therefore gives the interior solution

$$\begin{aligned} \phi(r, \theta) &= V \sum_{n \geq 1} \frac{(-1)^{n+1} (4n-1)(2n-2)!}{2^{2n-1} n! (n-1)!} \left(\frac{r}{a}\right)^{2n-1} P_{2n-1}(\cos \theta) \\ &= \frac{3V}{2} \frac{r}{a} P_1(\cos \theta) - \frac{7V}{8} \frac{r^3}{a^3} P_3(\cos \theta) + \frac{11V}{16} \frac{r^5}{a^5} P_5(\cos \theta) - \frac{75V}{128} \frac{r^7}{a^7} P_7(\cos \theta) + \dots \end{aligned} \quad (4.95)$$

The method of inversion is clearly a very convenient way of solving the interior problem, once the exterior problem has been solved, or *vice versa*. In fact, all that is really going on here is that if one knows the solution for  $\phi(r, \theta)$  in the region  $a \leq r \leq \infty$  *exterior* to the surface  $r = a$ , then in particular one knows the potential on the surface at  $r = a$ . This expression,  $\phi(a, \theta)$ , can then be taken as the Dirichlet boundary data for solving the *interior* problem in the region  $0 \leq r \leq a$ .

One must, however, be careful about the circumstances under which the inversion procedure can be applied. It is *essential* that the series expansion that is valid in the region  $r > a$  should also be valid actually at  $r = a$ , and likewise that the solution valid for  $r < a$  should be valid at  $r = a$ . This is an issue of *convergence* of the series, and in turn this is related to the question of the *analyticity* of the solution.

In general, the exterior and interior series will be convergent on the surface  $r = a$  itself *as long as there are no singularities anywhere on the  $r = a$  surface*. This is true in the example with the two hemispheres that we considered, where the potential is perfectly finite at all points on the  $r = a$  surface (it is either  $+V$  or  $-V$ ). However, an example where it may *not* be true is if we considered a situation where there was an infinite charge density somewhere on the  $r = a$  surface, such as would result from point charges or other kinds of singular charge densities. In fact the singularities need not be so extreme as in this example, and still the method of inversion may fail. For example, a function such as  $f(x) = (x - x_0)^{5/2}$  is singular at  $x = x_0$ , in the sense that it is not analytic there. (The third derivative, and above, do not exist at  $x = x_0$ , and so  $f(x)$  does not admit a Taylor expansion around  $x = x_0$ .)

In summary, while the method of inversion can sometimes be useful, it must be applied with great care. It is often in practice safer to recalculate from scratch in the inner region (for example, using the off-axis extrapolation method discussed previously), rather than applying inversion to the solution in the outer region.

## 4.7 The spherical harmonics

So far in our discussion of solutions of Laplace's equation in spherical polar coordinates, we have focused on situations with azimuthal symmetry, for which the associated Legendre equation (4.11) reduced to the Legendre equation (4.13). Now, we have to restore the integer  $m$ , and consider the associated Legendre equation itself.

For convenience, we present again the Associated Legendre Equation:

$$\frac{d}{dx} \left( (1-x^2) \frac{dy}{dx} \right) + \left( \lambda - \frac{m^2}{1-x^2} \right) y = 0. \quad (4.96)$$

Luckily, it turns out that we can construct the relevant solutions of this equation rather simply, in terms of the Legendre polynomials that we have already studied.

To begin, we change variables from  $y(x)$  to  $w(x)$ , where  $y(x) = (1-x^2)^{m/2} w(x)$ , and substitute this into (4.96). After simple algebra we find, after extracting an overall factor of  $(1-x^2)^{m/2}$ , that  $w$  must satisfy

$$(1-x^2) w'' - 2(m+1)x w' + [\lambda - m(m+1)] w = 0. \quad (4.97)$$

(We are using a prime to denote differentiation  $\frac{d}{dx}$  here.) Now suppose that we have a solution  $u$  of the ordinary Legendre equation:

$$(1-x^2) u'' - 2x u' + \lambda u = 0. \quad (4.98)$$

Next, we differentiate this  $m$  times. Let us use the notation  $\partial^m$  as a shorthand for  $\frac{d^m}{dx^m}$ . It is useful to note that we have the following lemma, which is just a consequence of Leibnitz' rule for the differentiation of a product, iterated  $m$  times:

$$\begin{aligned} \partial^m(fg) &= f(\partial^m g) + m(\partial f)(\partial^{m-1} g) + \frac{m(m-1)}{2!}(\partial^2 f)(\partial^{m-2} g) \\ &\quad + \frac{m(m-1)(m-2)}{3!}(\partial^3 f)(\partial^{m-3} g) + \cdots + m(\partial^{m-1} f)(\partial g) + (\partial^m f)g. \end{aligned} \quad (4.99)$$

We only need the first two or three terms in this expression if we apply it to the products in (4.98), and so we easily find that

$$(1-x^2) \partial^{m+2} u - 2(m+1)x \partial^{m+1} u + [\lambda - m(m+1)] \partial^m u = 0. \quad (4.100)$$

Comparing this equation with the equation (4.97) for  $w$ , we see that  $\partial^m u$  satisfies the same equation as does  $w$ , and so by setting  $w = \partial^m u$ , we have constructed a solution of (4.97) in terms of a solution  $u$  of the Legendre equation (4.98). Thus we have shown that  $y = (1 - x^2)^{m/2} \partial^m u$  satisfies the associated Legendre equation (4.96). The upshot, therefore, is that if we define

$$P_\ell^m(x) \equiv (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \quad (4.101)$$

where  $P_\ell(x)$  is a Legendre polynomial, then  $P_\ell^m(x)$  will be a solution of the Associated Legendre Equation with  $\lambda = \ell(\ell + 1)$ :

$$\frac{d}{dx} \left( (1 - x^2) \frac{dP_\ell^m}{dx} \right) + \left( \ell(\ell + 1) - \frac{m^2}{1 - x^2} \right) P_\ell^m = 0. \quad (4.102)$$

(The  $(-1)^m$  in eqn (4.101) is included to fit with standard conventions for the definition of  $P_\ell^m(x)$ .) Since  $P_\ell(x)$  is regular everywhere including  $x = \pm 1$ , it is clear that  $P_\ell^m(x)$ , viewed as a function of  $\theta$ , will be too. (Recall that  $x = \cos \theta$ , so  $(1 - x^2)^{m/2} = (\sin \theta)^m$ .) It is understood here that (for now) we are taking the integer  $m$  to be non-negative. It is clear that we should restrict  $m$  to be less than or equal to  $\ell$ , since if  $m$  exceeds  $\ell$  then the  $m$ -fold derivative of the  $\ell$ 'th Legendre polynomial (which itself is of degree  $\ell$ ) will give zero.

Recall next that we have Rodrigues' formula (4.38), which gives us an expression for  $P_\ell(x)$ . Substituting this into (4.101), we get the *Generalised Rodrigues Formula*

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} (1 - x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell. \quad (4.103)$$

A nice little miracle now occurs: this formula makes sense for negative values of  $m$  too, provided that  $m \geq -\ell$ . Thus we have a construction of Associated Legendre Functions for all integers  $m$  in the interval  $-\ell \leq m \leq \ell$ .

Looking at the Associated Legendre Equation (4.102), we note that the equation itself is invariant under sending

$$m \longrightarrow -m, \quad (4.104)$$

since  $m$  appears only as  $m^2$  in the equation. This means that if we take a solution with a given  $m$ , then sending  $m$  to  $-m$  gives us another solution. What is more, only one solution at fixed  $\ell$  and  $m^2$  can be regular at  $x = \pm 1$ , since the second solution will have logarithmic singularities there (just like we saw for the Legendre functions). Since  $P_\ell^m(x)$  and  $P_\ell^{-m}(x)$  given by 4.103 are both regular at  $x = \pm 1$  (and therefore neither of them can involve the second solution with logarithmic singularities at  $x = \pm 1$ ), it follows that they must be

linearly dependent; i.e.  $P_\ell^{-m}(x)$  must be some constant multiple of  $P_\ell^m(x)$ :

$$P_\ell^{-m}(x) = k P_\ell^m(x). \quad (4.105)$$

It is easy to determine what the constant  $k$  is, by using (4.103). From (4.105) we get

$$\partial^{\ell-m}(x^2-1)^\ell = k(1-x^2)^m \partial^{\ell+m}(x^2-1)^\ell. \quad (4.106)$$

It is good enough just to look at the highest power of  $x$ , since we have already argued that (4.105) must hold, and so all we need to do is to calculate what  $k$  is.<sup>32</sup> Thus we get

$$\frac{(2\ell)!}{(\ell+m)!} x^{\ell+m} = k(-1)^m x^{2m} \frac{(2\ell)!}{(\ell-m)!} x^{\ell-m} \quad (4.107)$$

at the leading order in  $x$ , which fixes  $k$  and hence establishes that

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x). \quad (4.108)$$

Using this result we can now very easily work out the normalisation integral for the associated Legendre functions  $P_\ell^m(x)$ . The relevant integral we shall need to evaluate is

$$\int_{-1}^1 dx P_\ell^m(x) P_n^m(x). \quad (4.109)$$

(It will become clear later why we have set the upper indices  $m$  equal here.) Using the same method as we used earlier for the Legendre polynomials,<sup>33</sup> it is easy to show that (4.109) vanishes unless  $\ell = n$ . For  $\ell = n$ , we can make use of (4.108) to write the required integral as

$$C_{\ell m} \equiv \int_{-1}^1 dx [P_\ell^m(x)]^2 = (-1)^m \frac{(\ell+m)!}{(\ell-m)!} \int_{-1}^1 dx P_\ell^m(x) P_\ell^{-m}(x). \quad (4.110)$$

Our task is to calculate the constants  $C_{\ell m}$ .

Using (4.101), and knowing from (4.103) that (4.101) actually makes perfectly good sense for the negative values of  $m$  as well as the positive ones,<sup>34</sup> we shall have

$$\begin{aligned} \int_{-1}^1 dx P_\ell^m(x) P_\ell^{-m}(x) &= \int_{-1}^1 dx (-1)^m (1-x^2)^{m/2} \partial^m P_\ell(x) (-1)^m (1-x^2)^{-m/2} \partial^{-m} P_\ell(x) \\ &= \int_{-1}^1 dx \partial^m P_\ell(x) \partial^{-m} P_\ell(x). \end{aligned} \quad (4.111)$$

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<sup>32</sup>One could, more adventurously, give another proof that  $P_\ell^{-m}(x)$  and  $P_\ell^m(x)$  are linearly dependent by checking *all* powers of  $x$ . We leave this as an exercise for the reader.

<sup>33</sup>i.e. plug  $P_\ell^m$  into the associated Legendre equation, multiply by  $P_n^m$ , integrate  $\int_{-1}^1 dx$  and then subtract the same with the quantities  $\ell$  and  $n$  exchanged.

<sup>34</sup>Remember here that the Legendre polynomial  $P_\ell(x)$  is being written here in terms of Rodrigues' formula (4.38), so  $\partial^{-m} P_\ell(x)$  actually means  $1/(2^\ell \ell!) \partial^{\ell-m} (x^2-1)^\ell$ . Thus, since we always require  $m \geq -\ell$ , this means that  $\partial^{-m} P_\ell(x)$  always involves a non-negative number of derivatives applied to  $(x^2-1)^\ell$ .



We now integrate by parts  $m$  times, to push the  $\partial^m$  derivatives onto  $\partial^{-m}P_\ell(x)$ , noting that the boundary terms will cancel for each integration by parts. (Bear in mind the precise definition of  $\partial^{-m}P_\ell(x)$ , as explained in footnote 34.) Thus we have

$$\int_{-1}^1 dx P_\ell^m(x) P_\ell^{-m}(x) = (-1)^m \int_{-1}^1 dx (P_\ell(x))^2 = \frac{2(-1)^m}{2\ell+1}, \quad (4.112)$$

where we have used the previous result (4.61) in getting to the final result. Looking back to (4.110), we have therefore established that

$$\int_{-1}^1 dx P_\ell^m(x) P_{\ell'}^m(x) = \frac{2}{(2\ell+1)} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}. \quad (4.113)$$

Recalling that in the separation of variables discussed in section 4 we had considered factorised solutions of the form  $\phi(r, \theta, \varphi) = r^{-1} R(r) Y(\theta, \varphi)$ , and that  $Y(\theta, \varphi)$  was itself factorised in the form  $\Theta(\theta)\Phi(\varphi)$  with  $\Phi(\varphi) \sim e^{im\varphi}$  and  $\Theta(\theta)$  satisfying the associated Legendre equation, we see that  $Y(\theta, \varphi)$  will be of the general form  $P_\ell^m(\cos\theta)e^{im\varphi}$ . To be precise, we shall define

$$Y_{\ell m}(\theta, \varphi) \equiv \sqrt{\frac{(2\ell+1)}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}, \quad \ell \geq 0, \quad -\ell \leq m \leq \ell. \quad (4.114)$$

The *Spherical Harmonics*  $Y_{\ell m}(\theta, \varphi)$  satisfy

$$-\nabla_{(\theta, \varphi)}^2 Y_{\ell m}(\theta, \varphi) = \ell(\ell+1) Y_{\ell m}(\theta, \varphi). \quad (4.115)$$

These spherical harmonics form the complete set of regular solutions of  $\nabla_{(\theta, \varphi)}^2 Y = -\lambda Y$  on the unit sphere. Note from (4.108) that we have

$$Y_{\ell, -m}(\theta, \varphi) = (-1)^m \bar{Y}_{\ell m}(\theta, \varphi), \quad (4.116)$$

where the bar denotes complex conjugation.

As we shall see now, the spherical harmonics satisfy the orthogonality properties

$$\int d\Omega \bar{Y}_{\ell' m'}(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \delta_{\ell\ell'} \delta_{mm'}, \quad (4.117)$$

where

$$d\Omega \equiv \sin\theta d\theta d\varphi \quad (4.118)$$

is the area element on the unit sphere, and  $\int d\Omega X$  means

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta X. \quad (4.119)$$

Thus (4.117) says that the integral on the left-hand side is zero unless  $\ell' = \ell$  and  $m' = m$ . The calculations goes as follows. Using the definition in eqn (4.114) for the spherical harmonics, we have

$$\int d\Omega \bar{Y}_{\ell'm'}(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{(2\ell'+1)}{4\pi}} \sqrt{\frac{(\ell'-m')!}{(\ell'+m')!}} \times \int_0^\pi d\theta \sin \theta P_\ell^m(\cos \theta) P_{\ell'}^{m'}(\cos \theta) \int_0^{2\pi} d\varphi e^{i(m-m')\varphi}. \quad (4.120)$$

Since

$$\int_0^{2\pi} d\varphi e^{i(m-m')\varphi} = 2\pi \delta_{mm'}, \quad (4.121)$$

(that is, this integral is zero if the integers  $m$  and  $m'$  are unequal, and it equals  $2\pi$  if  $m$  and  $m'$  are equal), it follows that eqn (4.120) becomes

$$\int d\Omega \bar{Y}_{\ell'm'}(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{(2\ell'+1)}{4\pi}} \sqrt{\frac{(\ell'-m)!}{(\ell'+m)!}} \times \int_0^\pi d\theta \sin \theta P_\ell^m(\cos \theta) P_{\ell'}^m(\cos \theta) 2\pi \delta_{mm'}. \quad (4.122)$$

Then, using the result in eqn (4.113) for the orthogonality of the associated Legendre functions, this gives the final result in (4.117).

Note that the integration over  $\varphi$ , which we performed first, gave the result that the answer would be non-zero only when  $m' = m$ , which is why we were able to replace  $P_{\ell'}^{m'}(\cos \theta)$  by  $P_\ell^m(\cos \theta)$  in eqn (4.122), and then we could make use of the orthogonality result that we already obtained in eqn (4.113). If we had tried performing the integrations in eqn (4.120) in the other order, first integrating over  $\theta$ , we would have had a lot more work to do, because we would then have needed to know the results for the integrals

$$\int_0^\pi d\theta \sin \theta P_\ell^m(\cos \theta) P_{\ell'}^{m'}(\cos \theta) \quad (4.123)$$

where  $m$  and  $m'$  are unequal. This would in fact have been a huge waste of time and effort, because having laboured to evaluate all the integrals (4.123), all the hard-earned results for the cases when  $m \neq m'$  would get thrown away at the second stage, when the integral over  $\varphi$  was performed.

It is instructive to look at the first few spherical harmonics explicitly. From (4.114), and

using (4.103) to give the expressions for the  $P_\ell^m$ , we find

$$\begin{aligned}
Y_{0,0}(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}}, \\
Y_{1,1}(\theta, \varphi) &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi}, \\
Y_{1,0}(\theta, \varphi) &= \sqrt{\frac{3}{4\pi}} \cos \theta, \\
Y_{1,-1}(\theta, \varphi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi}, \\
Y_{2,2}(\theta, \varphi) &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\varphi}, \\
Y_{2,1}(\theta, \varphi) &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\varphi}, \\
Y_{2,0}(\theta, \varphi) &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1), \\
Y_{2,-1}(\theta, \varphi) &= \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{-i\varphi}, \\
Y_{2,-2}(\theta, \varphi) &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{-2i\varphi}.
\end{aligned} \tag{4.124}$$

It is also instructive to rewrite the spherical harmonics in terms of Cartesian, rather than spherical polar, coordinates. Recall that the two coordinate systems are related by

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta. \tag{4.125}$$

We can write the expressions for  $x$  and  $y$  more succinctly in a single complex equation,

$$x + iy = r \sin \theta e^{i\varphi}, \tag{4.126}$$

since we have the well-known result that  $e^{i\varphi} = \cos \varphi + i \sin \varphi$ . Thus for the spherical

harmonics listed in (4.124) we have

$$\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, \\
Y_{1,1} &= -\sqrt{\frac{3}{8\pi}} \frac{x + iy}{r}, \\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}} \frac{z}{r}, \\
Y_{1,-1} &= \sqrt{\frac{3}{8\pi}} \frac{x - iy}{r}, \\
Y_{2,2} &= \sqrt{\frac{15}{32\pi}} \frac{(x + iy)^2}{r^2}, \\
Y_{2,1} &= -\sqrt{\frac{15}{8\pi}} \frac{z(x + iy)}{r^2}, \\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}} \left( \frac{3z^2}{r^2} - 1 \right) = \sqrt{\frac{5}{16\pi}} \frac{2z^2 - x^2 - y^2}{r^2}, \\
Y_{2,-1} &= \sqrt{\frac{15}{8\pi}} \frac{z(x - iy)}{r^2}, \\
Y_{2,-2} &= \sqrt{\frac{15}{32\pi}} \frac{(x - iy)^2}{r^2}.
\end{aligned} \tag{4.127}$$

What we are seeing here is that for each value of  $\ell$ , we are getting a set of functions, labelled by  $m$  with  $-\ell \leq m \leq \ell$ , that are all of the form of polynomials  $\mathcal{Q}_\ell$  of degree  $\ell$  in the Cartesian coordinates  $(x, y, z)$ , divided by  $r^\ell$ :

$$Y_{\ell m} \sim \frac{\mathcal{Q}_\ell}{r^\ell}. \tag{4.128}$$

The larger  $\ell$  is, the larger the number of possible such polynomials. Looking at  $\ell = 1$ , we have in total three  $Y_{1,m}$  functions, which could be reorganised, by taking appropriate linear combinations, as

$$\frac{x}{r}, \quad \frac{y}{r}, \quad \frac{z}{r}. \tag{4.129}$$

Thus once we know one of them, the other two just correspond to rotating the coordinate system through 90 degrees about one or another axis. The same is true of all the higher harmonics too. The spherical harmonics thus have built into them the “knowledge” of the rotational symmetries of the sphere. Our procedure for deriving the spherical harmonics was completely non-transparent, in the sense that no explicit use of the rotational symmetries of the sphere was made in the derivation. But at the end of the day, we see that the harmonics we have obtained do indeed exhibit these symmetries. In the language of group theory, one says that the spherical harmonics  $Y_{\ell m}$  fall into representations of the rotation group. One of the rather remarkable “miracles” that we encountered during our derivation, namely that

the solutions to the associated Legendre equation could be constructed from solutions of the ordinary Legendre equation, ultimately has its explanation in the fact that the harmonics  $Y_{\ell m}$  with  $m \neq 0$  are simply related to the  $m = 0$  harmonic  $Y_{\ell 0}$  by symmetry rotations of the sphere.

#### 4.8 General solution of Laplace's equation without azimuthal symmetry

We have now established that the well-behaved factorised solutions of Laplace's equation in spherical polar coordinates are of the form

$$\phi(r, \theta, \varphi) = r^\ell Y_{\ell m}(\theta, \varphi) \quad \text{or} \quad \phi(r, \theta, \varphi) = r^{-\ell-1} Y_{\ell m}(\theta, \varphi), \quad (4.130)$$

where  $\ell$  is any non-negative integer and  $m$  is any integer in the interval  $-\ell \leq m \leq \ell$ . Therefore, the most general solution of Laplace's equation in spherical polar coordinates<sup>35</sup> can be written as

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} (A_{\ell m} r^\ell + B_{\ell m} r^{-\ell-1}) Y_{\ell m}(\theta, \varphi). \quad (4.131)$$

The constants  $A_{\ell m}$  and  $B_{\ell m}$ , which depend on both  $\ell$  and  $m$ , are as yet arbitrary. Their values are determined by boundary conditions, as in the previous potential-theory examples that we have looked at. Because we are now allowing the azimuthal separation constant  $m$  to be non-zero, the class of solutions described by (4.131) now includes those that are dependent on the azimuthal angle  $\varphi$ .

In a boundary-value problem where the potential  $\phi(r, \theta, \varphi)$  is specified on a spherical boundary surface at  $r = a$ , one simply uses the orthogonality conditions (4.117) in order to determine the coefficients  $A_{\ell m}$  and  $B_{\ell m}$  in the general solution (4.131). For example, suppose we are solving for  $\phi$  in the exterior region  $r > a$ , with the requirement that  $\phi \rightarrow 0$  as  $r \rightarrow \infty$ . This means that  $A_{\ell m} = 0$ , and so the potential must take the form

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} B_{\ell m} r^{-\ell-1} Y_{\ell m}(\theta, \varphi), \quad (4.132)$$

where the coefficients  $B_{\ell m}$  will be determined by imposing boundary conditions on the surface at  $r = a$ .

Suppose we wish to impose Dirichlet boundary data on this surface. Setting  $r = a$  in

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<sup>35</sup>That is, the most general solution that is regular on the spherical surfaces at constant  $r$ .

eqn (4.132), multiplying by  $\bar{Y}_{\ell',m'}(\theta, \varphi)$  and integrating, we shall have

$$\begin{aligned}
\int \phi(a, \theta, \varphi) \bar{Y}_{\ell',m'}(\theta, \varphi) d\Omega &= \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} B_{\ell m} a^{-\ell-1} \int Y_{\ell m}(\theta, \varphi) \bar{Y}_{\ell',m'}(\theta, \varphi) d\Omega \\
&= \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} B_{\ell m} a^{-\ell-1} \delta_{\ell,\ell'} \delta_{m,m'} \\
&= B_{\ell',m'} a^{-\ell'-1},
\end{aligned} \tag{4.133}$$

and so we have

$$B_{\ell m} = a^{\ell+1} \int \phi(a, \theta, \varphi) \bar{Y}_{\ell m}(\theta, \varphi) d\Omega. \tag{4.134}$$

Plugging these  $B_{\ell m}$  coefficients back into eqn (4.132) then gives the result for the potential everywhere in the region  $r \geq a$ .

It is sometimes useful to consider an expansion of a potential in spherical harmonics even when the problem is not specified in terms of boundary values on a spherical surface. Typically, expansions in spherical harmonics can be useful whenever one wants to express a potential in the form of a large- $r$  expansion, or a small- $r$  expansion. Consider, for example, our discussion of the Green function for an infinite planar surface, in section 2.4.1. Equation (2.43) gives the expression for  $\phi(x, y, z)$  everywhere in the half-space  $z > 0$  in terms of its boundary value on the plane  $z = 0$ . Suppose we take the boundary potential to be equal to  $V$  (a constant) within the square of side 2 whose vertices are located at  $x = \pm 1$ ,  $y = \pm 1$ , and take the potential to be zero everywhere outside this square in the  $z = 0$  plane. Thus, the potential for  $z > 0$  is given by

$$\phi(x, y, z) = \frac{Vz}{2\pi} \int_{-1}^1 dx' \int_{-1}^1 dy' \frac{1}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}. \tag{4.135}$$

Although this can be integrated explicitly, the result is rather complicated. Instead, let us write  $x$ ,  $y$  and  $z$  in terms of spherical polar coordinates, expand the integrand in inverse powers of  $r$ , and integrate term by term. (Note that we shall stick with the Cartesian coordinates  $x'$  and  $y'$  for the integration variables. This is much more convenient, since the region of integration (the square) is much more easily parameterised using Cartesian coordinates.) After some algebra, which is easily done using *Mathematica*, we find

$$\begin{aligned}
\phi(r, \theta, \varphi) &= \frac{2cV}{\pi r^2} + \frac{(3c - 5c^3)V}{\pi r^4} + \frac{7c(15 - 70c^2 + 63c^4 - 9s^4 \cos 4\varphi)V}{24\pi r^6} \\
&\quad + \frac{3c(35 - 315c^2 + 693c^4 - 429c^6 + 11s^4(13c^2 - 3) \cos 4\varphi)V}{16\pi r^8} + \dots,
\end{aligned} \tag{4.136}$$

where we have defined  $c = \cos \theta$  and  $s = \sin \theta$ .

Of course, since  $\phi$  satisfies Laplace's equation it must necessarily be possible to express it in a large- $r$  expansion in spherical harmonics, of the form

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} B_{\ell m} r^{-\ell-1} Y_{\ell m}(\theta, \varphi). \quad (4.137)$$

From the definition of  $Y_{\ell m}(\theta, \varphi)$ , it is not hard to express (4.136) in such a series, giving

$$\begin{aligned} \phi(r, \theta, \varphi) = & \frac{4V}{\sqrt{3\pi}} \frac{1}{r^2} Y_{1,0}(\theta, \varphi) - \frac{4V}{\sqrt{7\pi}} \frac{1}{r^4} Y_{3,0}(\theta, \varphi) \\ & + \frac{V}{r^6} \left( \frac{14}{3\sqrt{11\pi}} Y_{5,0}(\theta, \varphi) - \sqrt{\frac{14}{55\pi}} (Y_{5,4}(\theta, \varphi) + Y_{5,-4}(\theta, \varphi)) \right) \\ & + \frac{V}{r^8} \left( -2\sqrt{\frac{3}{5\pi}} Y_{7,0}(\theta, \varphi) + \sqrt{\frac{22}{35\pi}} (Y_{7,4}(\theta, \varphi) + Y_{7,-4}(\theta, \varphi)) \right) + \dots \end{aligned} \quad (4.138)$$

Observe that the expression for the potential is azimuthally symmetric at the first couple of orders in inverse powers of  $1/r$ . That is, only the spherical harmonics of the form  $Y_{\ell 0}$ , with  $m = 0$ , arise in the  $1/r^2$  and  $1/r^4$  terms in the expansion. The non-azimuthal symmetry, which must of course necessarily be present since the original boundary value specification is not azimuthally-symmetric because it involves a square of non-zero potential at  $z = 0$ , sets in at orders  $1/r^6$  and above.

Observe also that the potential exhibits the discrete four-fold rotational symmetry that we should expect, given that the boundary value specification itself has a discrete four-fold rotational symmetry. In other words, the potential must exhibit the property of quarter-rotational symmetry:

$$\phi(r, \theta, \varphi) = \phi(r, \theta, \varphi + \tfrac{1}{2}\pi). \quad (4.139)$$

Note that this means that the answer *must* involve only those spherical harmonics  $Y_{\ell m}(\theta, \varphi)$  for which  $Y_{\ell m}(\theta, \varphi) = Y_{\ell m}(\theta, \varphi + \frac{1}{2}\pi)$ , and hence  $e^{im\varphi} = e^{im(\varphi + \frac{1}{2}\pi)}$ , which implies  $e^{\frac{1}{2}im\pi} = 1$  and hence

$$m = 4n, \quad n = \text{integer}. \quad (4.140)$$

Since the spherical harmonic  $Y_{\ell m}(\theta, \varphi)$  must have  $|m| \leq \ell$ , this means that it would be *impossible* for any azimuthal dependence to appear before order  $1/r^5$  at the absolute earliest. In fact, because other symmetries of the problem mean that only *even* inverse powers of  $r$  occur here, the azimuthal dependence is deferred until order  $1/r^6$ .

It is worth emphasising at this point that a new feature that arises when there is no azimuthal symmetry is that the technique of off-axis extrapolation, which was so useful for azimuthally-symmetric problems, no longer works. The reason for this is easily understood.

When there is no azimuthal symmetry the general solution takes the form

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi), \quad (4.141)$$

and so there are  $2 \times (2\ell + 1)$  constants,  $A_{\ell m}$  and  $B_{\ell m}$ , for each value of  $\ell$  in the expansion. But, as we saw previously, on the  $z$  axis there are only 2 terms in the expansion for  $\phi(z)$ , for each value of  $\ell$ :

$$\phi(z) = \sum_{\ell \geq 0} (a_{\ell} z^{\ell} + b_{\ell} z^{-\ell-1}). \quad (4.142)$$

Thus the information available in an on-axis expansion is insufficient to determine all the coefficients in the general expansion (4.141). Explicitly, if we take the expression (4.141) and restrict it to the  $z$  axis (consider just the positive  $z$  axis, for simplicity), we have to set  $\theta = 0$ , whereupon  $r = z$ . Now the associated Legendre function  $P_{\ell}^m(\cos \theta)$  has a factor  $(\sin \theta)^{|m|}$ , which vanishes at  $\theta = 0$  (or  $\theta = \pi$ ) when  $m \neq 0$ , and so restricting to the positive  $z$  axis (4.141) becomes

$$\phi(z) = \phi(z, 0, \varphi) = \sum_{\ell \geq 0} \left( A_{\ell 0} z^{\ell} + B_{\ell 0} z^{-\ell-1} \right) \sqrt{\frac{2\ell + 1}{4\pi}}. \quad (4.143)$$

(See eqn (4.114).) Thus one can determine the expansion coefficients  $A_{\ell 0}$  and  $B_{\ell 0}$  in (4.141) by matching with the solution on the  $z$  axis, but the coefficients  $A_{\ell m}$  and  $B_{\ell m}$  for which  $m \neq 0$  cannot be determined by this means.

## 4.9 Another look at the generating function

We now return to the generating function for the Legendre polynomials, defined in (4.41). There is a nice physical interpretation of this construction, which we shall now describe.

Consider the problem of a point charge of unit strength, sitting on the  $z$  axis at a point  $z = r'$ . We know, since it is an axially-symmetric situation, that the potential at an arbitrary point must be expressible in the form (4.69)

$$\phi(r, \theta) = \sum_{\ell \geq 0} (A_{\ell} r^{\ell} + B_{\ell} r^{-\ell-1}) P_{\ell}(\cos \theta). \quad (4.144)$$

To determine the coefficients, we must first make a choice between considering the expansion either in the region where  $r > r'$ , or in the region where  $r < r'$ .

For  $r > r'$ , the solution should be an infinite series in inverse powers of  $r$ , so that it dies off at infinity. Thus for  $r > r'$  we must have  $A_{\ell} = 0$ . On the other hand, for  $r < r'$  the solution should be an infinite series in positive powers of  $r$ , so that it remains regular at  $r = 0$ . For  $r < r'$ , therefore, we must have  $B_{\ell} = 0$ .



The non-zero coefficients  $B_\ell$  or  $A_\ell$  in the two regions can be determined by the method we discussed earlier, of first finding the potential when  $\vec{r}$  is on the  $z$  axis. This is easy; we shall have

$$\begin{aligned} z > r' : \quad \phi &= \frac{1}{z - r'} = \frac{1}{z} \left(1 - \frac{r'}{z}\right)^{-1} = \sum_{\ell \geq 0} \frac{r'^\ell}{z^{\ell+1}}, \\ z < r' : \quad \phi &= \frac{1}{r' - z} = \frac{1}{r'} \left(1 - \frac{z}{r'}\right)^{-1} = \sum_{\ell \geq 0} \frac{z^\ell}{r'^{\ell+1}}, \end{aligned} \quad (4.145)$$

The general off-axis solution, where  $\vec{r}$  is arbitrary, is therefore given by

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos \theta), \quad r > r', \quad (4.146)$$

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos \theta), \quad r < r'. \quad (4.147)$$

These expressions give the potential at a point  $(r, \theta, \varphi)$  due to a unit point charge at the point  $z = r'$  on the  $z$  axis. The answer is, of course, independent of the azimuthal angle  $\varphi$  because the point charge is on the  $z$  axis.

There is a point that is perhaps worth emphasising here, and it applies also in all the many examples we have encountered where we have separated the discussion of expansions in Legendre polynomials or spherical harmonics into a “small- $r$  region” and a “large- $r$  region.” The  $r > r'$  expansion in eqn (4.146) is the complete expression for the potential in the  $r > r'$  region. Furthermore, it applies *only* in the region  $r > r'$ ; it is not valid in the region  $r < r'$ . Likewise, eqn (4.147) is the complete expression that is valid in the small- $r$  region  $r < r'$ . It is not valid in the region  $r > r'$ . Thus, in a given region we use *either* the expression (4.146) (if  $r > r'$ ), or else we use the expression (4.147) (if  $r < r'$ ). People occasionally make the mistake of thinking that the complete answer for the potential is obtained by *adding* the expressions (4.146) and (4.147) together. This is simply wrong, for the reasons we just stated.<sup>36</sup>

Note that it is sometimes convenient to write the two expressions in (4.146) and (4.147)

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<sup>36</sup>A possible psychological explanation for why this mistake is sometimes made might be that having essentially calculated some coefficients  $B_\ell = r'^\ell$  in the large- $r$  expansion (4.146), and some coefficients  $A_\ell = r'^{-\ell-1}$  in the small- $r$  expansion (4.147), they then think back to the form of the general expansion and want to stick both  $A_\ell = r'^{-\ell-1}$  and  $B_\ell = r'^\ell$  into (4.144) at the same time. The error lies in failing to “clear all the variables” in between doing one particular calculation and doing the other. The correct statement in this example is that, in terms of the general expansion (4.144), the coefficients  $A_\ell$  and  $B_\ell$  are different in



"Just a darn minute! — Yesterday  
you said that X equals **two**!"

Figure 1: “Always clear the variables between calculations.”

in the form of a single equation

$$\phi(r, \theta) = \sum_{\ell \geq 0} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} P_{\ell}(\cos \theta), \quad (4.149)$$

where  $r_{<}$  means whichever of  $r$  and  $r'$  is the smaller, and  $r_{>}$  means whichever of  $r$  and  $r'$  is the larger.

We can relate these results to the generating function, by observing that we can in fact write down the solution to this problem in closed form. The potential  $\phi(r, \theta)$  will just be the inverse of the distance from the point  $(r, \theta)$  to the point  $z = r'$  on the  $z$  axis where the unit charge is located. Using the cosine rule, this distance is  $(r^2 - 2r r' \cos \theta + r'^2)^{1/2}$ , and so

$$\phi(r, \theta) = \frac{1}{(r^2 - 2r r' \cos \theta + r'^2)^{1/2}}. \quad (4.150)$$

But when  $r > r'$ , we have just obtained the expansion (4.146) for the potential, and so we

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the two regions, with

$$\begin{aligned} r > r' : \quad & A_{\ell} = 0, \quad B_{\ell} = r'^{\ell}, \\ r < r' : \quad & A_{\ell} = r'^{-\ell-1}, \quad B_{\ell} = 0. \end{aligned} \quad (4.148)$$

One has to “clear the variables” when switching from one region to the other.

have

$$\phi(r, \theta) = \frac{1}{r[1 - 2(r'/r) \cos \theta + (r'/r)^2]^{1/2}} = \sum_{\ell \geq 0} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos \theta). \quad (4.151)$$

Letting  $t = r'/r$ , we therefore have

$$(1 - 2t \cos \theta + t^2)^{-1/2} = \sum_{\ell \geq 0} t^\ell P_\ell(\cos \theta), \quad (4.152)$$

which is nothing but the generating function formula (4.41) with  $x$  set equal to  $\cos \theta$ .

It is straightforward to repeat the above exercise for the region where  $r < r'$ . This time, we pull out a factor of  $r'$  in the denominator of (4.150), and define  $t = r/r'$ . Equating to (4.147) then again gives rise to the generating function formula (4.41).

The discussions above show how the generating function (4.41) admits a very simple physical interpretation as giving the expansion, in terms of Legendre polynomials, of the potential due to a unit point charge on the  $z$  axis.

We may also generalise the discussion, to the case where the unit charge is placed at a general position  $(r', \theta', \varphi')$  that need not lie on the  $z$  axis. The expression for the potential at  $(r, \theta, \varphi)$  is therefore no longer azimuthally symmetric, and so it must be expanded in the general form (4.131), in terms of spherical harmonics.

Of course we could expand the potential simply using Legendre polynomials if we introduce the angle  $\gamma$  between  $\vec{r}$  and  $\vec{r}'$ , so that we may write

$$\phi(r, \theta, \varphi) = (r^2 + r'^2 - 2rr' \cos \gamma)^{-1/2}. \quad (4.153)$$

Using the generating function formula (4.41) we may therefore write

$$\begin{aligned} \frac{1}{|\vec{r} - \vec{r}'|} &= \sum_{\ell \geq 0} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos \gamma), & r > r', \\ \frac{1}{|\vec{r} - \vec{r}'|} &= \sum_{\ell \geq 0} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos \gamma), & r < r'. \end{aligned} \quad (4.154)$$

However, as noted earlier the expression for  $\gamma$  in terms of  $\theta, \varphi, \theta'$  and  $\varphi'$  is actually rather complicated, namely  $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')$  (see (2.65)).

It will be much more useful to have an expression for  $|\vec{r} - \vec{r}'|^{-1}$  that is written directly as a series in spherical harmonics. To do this, we first note from (4.131) that in the region  $r > r'$ , and viewing  $|\vec{r} - \vec{r}'|^{-1}$  as the potential at  $\vec{r}$  due to a unit point charge located at  $\vec{r}'$ , we shall have an expansion of the form

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{1}{r^{\ell+1}} B_{\ell m}(r', \theta', \varphi') Y_{\ell m}(\theta, \varphi), \quad (4.155)$$

with the  $A_{\ell m}$  coefficients being zero, where we have indicated explicitly that the expansion coefficients  $B_{\ell m}$  will depend upon the chosen location  $(r', \theta', \varphi')$  of the unit charge. On the other hand, in the region  $r < r'$  we shall have an expansion of the form

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} r^{\ell} A_{\ell m}(r', \theta', \varphi') Y_{\ell m}(\theta, \varphi), \quad (4.156)$$

with the  $B_{\ell m}$  coefficients being zero in this case.

How do we determine the expansion coefficients  $A_{\ell m}(r', \theta', \varphi')$  and  $B_{\ell m}(r', \theta', \varphi')$ ? The first thing to notice is that the function  $|\vec{r} - \vec{r}'|^{-1}$  itself is symmetrical under the exchange of  $\vec{r}$  and  $\vec{r}'$ . This means that the way in which  $r', \theta'$  and  $\varphi'$  enter into the expansions (4.155) and (4.156) should be symmetrically related to the way in which  $r, \theta$  and  $\varphi$  enter. In saying this, due allowance must be made for the fact that if  $r > r'$ , then, of course  $r' < r$ , so the way in which  $r'$  appears in (4.155) should be very like the way that  $r$  appears in (4.156), and *vice versa*. In fact, roughly speaking, we can expect in symmetry grounds that  $B_{\ell m}(r', \theta', \varphi')$  should be something like  $r'^{\ell}$  multiplied by one of the  $Y_{\ell m'}(\theta', \varphi')$  spherical harmonics, and that  $A_{\ell m}(r', \theta', \varphi')$  should be something like  $r'^{-\ell-1}$  multiplied by some  $Y_{\ell m'}(\theta', \varphi')$  spherical harmonic.

At this point, we shall write down the answer, and then prove that it is correct. The result is that

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi), \quad r > r', \quad (4.157)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r^{\ell}}{r'^{\ell+1}} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi), \quad r < r', \quad (4.158)$$

These expressions can also be written as the single equation

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi). \quad (4.159)$$

where, as before,  $r_{<}$  means the lesser of  $r$  and  $r'$ , while  $r_{>}$  means the greater of  $r$  and  $r'$ .

To prove that (4.159) is true, we shall follow a strategy that makes use of the uniqueness theorem. We know that  $|\vec{r} - \vec{r}'|^{-1}$  obeys the Poisson equation

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta^3(\vec{r} - \vec{r}'), \quad (4.160)$$

and that  $|\vec{r} - \vec{r}'|^{-1}$  goes to zero as  $\vec{r}$  goes to infinity. Invoking the uniqueness theorem, we know that  $|\vec{r} - \vec{r}'|^{-1}$  is the unique function that obeys this equation with this boundary

condition. Our strategy for proving (4.159) will be to show that the expression on the right-hand side also obeys this same Poisson equation, with the same delta-function source, and that it goes to zero at infinity. By the uniqueness theorem, it will therefore follow that the right-hand side of (4.159) must in fact be equal to the function  $|\vec{r} - \vec{r}'|^{-1}$ .

We shall first need to establish a couple of preliminary results. First, we note that since the spherical harmonics form, by construction, a complete set of functions on the unit sphere, we can expand an arbitrary smooth function  $f(\theta, \varphi)$  defined on the sphere in the form of a generalised Fourier expansion

$$f(\theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} b_{\ell m} Y_{\ell m}(\theta, \varphi). \quad (4.161)$$

Using the orthonormality relation (4.117) for the spherical harmonics, we find

$$b_{\ell m} = \int d\Omega \bar{Y}_{\ell m}(\theta, \varphi) f(\theta, \varphi), \quad (4.162)$$

and hence substituting this back into (4.161), after first changing variables from  $(\theta, \varphi)$  to  $(\theta', \varphi')$  in (4.162) to avoid a clash with the  $(\theta, \varphi)$  variables in (4.161), we obtain

$$f(\theta, \varphi) = \int d\Omega' f(\theta', \varphi') \left( \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi) \right). \quad (4.163)$$

The quantity enclosed in the large parentheses is evidently behaving just like a 2-dimensional Dirac delta function, and we shall have

$$\sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi) = \frac{\delta(\theta - \theta') \delta(\varphi - \varphi')}{\sin \theta'}. \quad (4.164)$$

(The  $\sin \theta'$  in the denominator is needed to cancel the  $\sin \theta'$  factor in the area element  $d\Omega' = \sin \theta' d\theta' d\varphi'$ .) The expression (4.164) is called the *Completeness Relation* for the spherical harmonics. It signifies the fact that the spherical harmonics comprise a complete set of functions on the sphere, in terms of which one can always expand any smooth function, as in (4.161).

Secondly, it is useful to introduce the *Heaviside function*  $\vartheta(x)$ , often called the step function or theta-function, whose definition is

$$\begin{aligned} \vartheta(x) &= 0, & x < 0, \\ \vartheta(x) &= 1, & x > 0, \end{aligned} \quad (4.165)$$

The discontinuous jump as  $x$  passes through zero implies that there is an infinite spike in the derivative of  $\vartheta(x)$  at  $x = 0$ , and in fact

$$\vartheta'(x) = \delta(x). \quad (4.166)$$

This can be verified by integrating  $\vartheta'(x)$  over the interval  $x_1 \leq x \leq x_2$ :

$$\int_{x_1}^{x_2} \vartheta'(x) dx = \left[ \vartheta(x) \right]_{x_1}^{x_2}. \quad (4.167)$$

From the definition (4.165), we see that this is equal to 1 if  $x = 0$  lies in the integration range, but it is instead 0 if  $x = 0$  lies outside the integration range. The integral of  $\delta(x)$  has exactly the same features.

Using the Heaviside function, the two expressions (4.157) and (4.158) for  $|\vec{r} - \vec{r}'|^{-1}$  can be combined into the single formula:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \left( \vartheta(r' - r) \frac{r^\ell}{r'^{\ell+1}} + \vartheta(r - r') \frac{r'^\ell}{r^{\ell+1}} \right) \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi). \quad (4.168)$$

We are now in a position to verify that (4.168) is indeed correct, by verifying that it does indeed satisfy the Poisson equation (4.160). As discussed above, this, together with the fact that the right-hand side of (4.168) obviously satisfies the proper boundary condition that it goes to zero as  $r$  goes to infinity, pins down the Green function  $|\vec{r} - \vec{r}'|^{-1}$  uniquely.

Viewed as a function of  $r$ ,  $\theta$  and  $\varphi$ , the right-hand side of (4.168) *by construction* is annihilated by the Laplacian in the bulk, i.e. away from the crossover at  $r = r'$ . This is obvious from the fact that in each region  $r > r'$  and  $r < r'$ , it is a particular instance of an infinite series of the form (4.131), which by construction is a solution of Laplace's equation. Therefore, when we act with  $\nabla^2$  on the right-hand side of (4.168), all the terms involving  $\theta$  and  $\varphi$  derivatives, together with those from the  $r$  derivatives that *do not act on the Heaviside functions*, will add to zero. We therefore need only to retain those terms in which the  $r$  derivatives act on the Heaviside functions.

To save some writing, let us temporarily define

$$W_{\ell m} \equiv \frac{4\pi}{2\ell+1} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi). \quad (4.169)$$

Recalling that the radial part of  $\nabla^2 \phi$  can be written as  $(\nabla^2 \phi)_{\text{radial}} = r^{-1} \partial^2(r\phi)/\partial r^2$ , we

shall have

$$\begin{aligned}
(\nabla^2 \phi)_{\text{radial}} &= \\
&= \frac{1}{r} \frac{\partial^2}{\partial r^2} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} \left( \vartheta(r' - r) \frac{r^{\ell+1}}{r'^{\ell+1}} + \vartheta(r - r') \frac{r'^{\ell}}{r^{\ell}} \right) \\
&= \frac{1}{r} \frac{\partial}{\partial r} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} \left( \frac{d\vartheta(r' - r)}{dr} \frac{r^{\ell+1}}{r'^{\ell+1}} + (\ell + 1) \vartheta(r' - r) \frac{r^{\ell}}{r'^{\ell+1}} + \frac{d\vartheta(r - r')}{dr} \frac{r'^{\ell}}{r^{\ell}} - \ell \vartheta(r - r') \frac{r'^{\ell}}{r^{\ell+1}} \right) \\
&= \frac{1}{r} \frac{\partial}{\partial r} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} \left( -\delta(r - r') \frac{r^{\ell+1}}{r'^{\ell+1}} + (\ell + 1) \vartheta(r' - r) \frac{r^{\ell}}{r'^{\ell+1}} + \delta(r - r') \frac{r'^{\ell}}{r^{\ell}} - \ell \vartheta(r - r') \frac{r'^{\ell}}{r^{\ell+1}} \right) \\
&= \frac{1}{r} \frac{\partial}{\partial r} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} \left( (\ell + 1) \vartheta(r' - r) \frac{r^{\ell}}{r'^{\ell+1}} - \ell \vartheta(r - r') \frac{r'^{\ell}}{r^{\ell+1}} \right) \\
&= \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} \left[ \ell(\ell + 1) \left( \vartheta(r' - r) \frac{r^{\ell-1}}{r'^{\ell+1}} + \vartheta(r - r') \frac{r'^{\ell}}{r^{\ell+2}} \right) \right. \\
&\quad \left. + (\ell + 1) \frac{d\vartheta(r' - r)}{dr} \frac{r^{\ell}}{r'^{\ell+1}} - \ell \frac{d\vartheta(r - r')}{dr} \frac{r'^{\ell}}{r^{\ell+1}} \right].
\end{aligned} \tag{4.170}$$

Note that we have used the results that  $d\vartheta(r - r')/dr = \delta(r - r')$  and  $d\vartheta(r' - r)/dr = -\delta(r - r')$ , and that since  $\delta(r - r')$  is non-zero only when  $r = r'$ , then in any function of  $r$  and  $r'$  that is multiplied by  $\delta(r - r')$ , we can always set  $r' = r$ . Thus, for example,  $\delta(r - r')r'^{\ell}/r^{\ell} = \delta(r - r')$ .

The terms on the first line of the last expression here are just the usual “bulk” terms, with no derivatives on the Heaviside functions, which will cancel against the terms from the angular derivatives in  $\nabla^2$ , as we mentioned above. It is the second line of the last expression that we are after, since these are the terms where the radial derivatives have landed on the Heaviside functions. This second line then becomes

$$\begin{aligned}
&= \frac{1}{r} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} \left( -(\ell + 1) \delta(r - r') \frac{r^{\ell}}{r'^{\ell+1}} - \ell \delta(r - r') \frac{r'^{\ell}}{r^{\ell+1}} \right) \\
&= -\frac{1}{r^2} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} W_{\ell m} (2\ell + 1) \delta(r - r') \\
&= -\frac{4\pi}{r^2} \delta(r - r') \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi) \\
&= -4\pi \frac{\delta(r - r') \delta(\theta - \theta') \delta(\varphi - \varphi')}{r^2 \sin \theta} \\
&= -4\pi \delta^3(\vec{r} - \vec{r}').
\end{aligned} \tag{4.171}$$

Note that we used the completeness relation (4.164) in getting from the third to the fourth line in eqn (4.164).

To summarise, we have shown that when  $\nabla^2$  acts on the expression (4.159) it gives precisely  $-4\pi\delta^3(\vec{r}-\vec{r}')$ . Therefore, since by the uniqueness theorem there can be only one function that has this property and that also falls off at large  $r$ , we have indeed verified that the expression given by (4.159) (or equivalently, by (4.157) and (4.158)), must be equal to  $|\vec{r}-\vec{r}'|^{-1}$ .

It is interesting to see how the delta function on the right-hand side of (4.160) arises in this representation of  $|\vec{r}-\vec{r}'|^{-1}$  in (4.159). The factors in the  $\theta$  and  $\varphi$  directions come from the completeness relation (4.164), but the factor in the radial direction arises quite differently. It comes from the fact that the expressions (4.157) and (4.158) in the  $r > r'$  and  $r < r'$  regions, which are combined into one formula in (4.159) (or in other words, in (4.168)), are themselves *equal* at  $r = r'$ , but there is a discontinuity in the gradient. This in turn implies that there is a delta function in the second derivative with respect to  $r$ , and it is this that produces the radial delta function factor.

Observe also that by comparing the expansions (4.154) with those in (4.157) and (4.158), we can deduce that

$$P_\ell(\cos \gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi), \quad (4.172)$$

where, it will be recalled,  $\gamma$  is given by (2.65).

#### 4.10 Dirichlet Green function expansion

The expression (4.168) obtained in the previous section is useful in its own right, in situations where there is a charge distribution  $\rho(\vec{r})$  and one wants to calculate the resulting contribution to the potential as given by (2.13) in the case that there are no boundaries.

We can also make use of (4.168) in order to obtain an expansion for the Dirichlet Green function for a spherical boundary. Recall that we used the method of images in order to construct the Dirichlet Green function (2.60) for the exterior boundary value problem in which the potential is specified on the spherical surface  $r = a$ , and one wants to determine  $\phi(r, \theta, \varphi)$  for all  $r \geq a$ . For convenience, we give the expression (2.60) again here:

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{a/r'}{|\vec{r} - (a^2/r'^2)\vec{r}'|}. \quad (4.173)$$

It then follows straightforwardly from (4.157) and (4.158) that the expansion for  $G_D(\vec{r}, \vec{r}')$



in (4.173) will be given by

$$\begin{aligned}
G_D(\vec{r}, \vec{r}') &= \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} \left( r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}} \right) \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi), & r > r', \\
G_D(\vec{r}, \vec{r}') &= \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{1}{r'^{\ell+1}} \left( r^{\ell} - \frac{a^{2\ell+1}}{r^{\ell+1}} \right) \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi), & r < r',
\end{aligned} \tag{4.174}$$

Recall from our original discussion of the Green function that for the Dirichlet boundary-value problem, the potential inside a volume  $V$  bounded by a surface  $S$  is given by (2.29)

$$\phi(\vec{r}) = \int_V \rho(\vec{r}') G_D(\vec{r}, \vec{r}') d^3 \vec{r}' - \frac{1}{4\pi} \int_S \phi(\vec{r}') \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} dS'. \tag{4.175}$$

We therefore need to calculate the normal derivative of the upper expression in (4.10) at  $r' = a$  (approached from  $r' > a$ , since we are solving the exterior Dirichlet problem). The normal derivative should be directed *out* of the volume  $V$ , which in our case is all of space in the region  $r > a$ , and so the normal derivative that we require is  $(-\partial G_D(\vec{r}, \vec{r}')/\partial r')|_{r'=a}$ . From the upper expression in (4.10), this is therefore given by

$$-\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial r'} \Big|_{r'=a} = -\frac{4\pi}{a^2} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( \frac{a}{r} \right)^{\ell+1} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi). \tag{4.176}$$

Consider, for simplicity, the case where  $\rho = 0$  in the entire region  $r > a$ . It then follows from (4.175) and (4.176) that  $\phi(r, \theta, \varphi)$  outside the surface  $r = a$  is given in terms of the boundary values  $\phi(a, \theta, \varphi)$  on the sphere by

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( \int \phi(a, \theta', \varphi') \bar{Y}_{\ell m}(\theta', \varphi') d\Omega' \right) \left( \frac{a}{r} \right)^{\ell+1} Y_{\ell m}(\theta, \varphi). \tag{4.177}$$

This can be seen to be contained, as it must be, within the general class of solutions (4.131). It corresponds to  $A_{\ell m} = 0$  and

$$B_{\ell m} = a^{\ell+1} \int \phi(a, \theta', \varphi') \bar{Y}_{\ell m}(\theta', \varphi') d\Omega'. \tag{4.178}$$

This is indeed the same result that we obtained previously, in (4.134), by a different (but of course equivalent) method.

An analogous discussion can be given for the interior problem, where one solves for  $\phi(r, \theta, \varphi)$  for  $0 < r < a$  in terms of boundary values  $\phi(a, \theta, \varphi)$  on the sphere at  $r = a$ .

#### 4.11 Inversion symmetry revisited

Before closing the discussion of solving Laplace's equation in spherical polar coordinates, we shall return again to the topic of the inversion symmetry that was discussed in section 4.6. Although the discussion there was for solutions with azimuthal symmetry, the notion of applying inversion in order to map one solution into another can be applied also to cases where there is no azimuthal symmetry.

In a general setting, the idea of inversion can be described as follows. Suppose that  $\phi(r, \theta, \varphi)$  is any solution of Laplace's equation. We then define

$$\psi(r, \theta, \varphi) \equiv \frac{a}{r} \phi\left(\frac{a^2}{r}, \theta, \varphi\right), \quad (4.179)$$

where  $a$  is any constant. It is straightforward to see that  $\psi$  is also a solution of Laplace's equation. This can be shown either by direct substitution of (4.179) into the Laplacian, or, perhaps more simply, as follows: Since  $\phi(r, \theta, \varphi)$  solves Laplace's equation, we know from the earlier discussions that it can be expanded in spherical harmonics in the form

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi). \quad (4.180)$$

From (4.179) we therefore have

$$\begin{aligned} \psi(r, \theta, \varphi) &= \frac{a}{r} \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( A_{\ell m} a^{2\ell} r^{-\ell} + B_{\ell m} a^{-2\ell-2} r^{\ell+1} \right) Y_{\ell m}(\theta, \varphi), \\ &= \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( A_{\ell m} a^{2\ell+1} r^{-\ell-1} + B_{\ell m} a^{-2\ell-1} r^{\ell} \right) Y_{\ell m}(\theta, \varphi). \end{aligned} \quad (4.181)$$

Thus we see that

$$\psi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left( \tilde{A}_{\ell m} r^{\ell} + \tilde{B}_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi), \quad (4.182)$$

where

$$\tilde{A}_{\ell m} = \frac{B_{\ell m}}{a^{2\ell+1}}, \quad \tilde{B}_{\ell m} = A_{\ell m} a^{2\ell+1}. \quad (4.183)$$

Since we know that the right-hand side of (4.180) solves Laplace's equation for *any* values of the constants  $A_{\ell m}$  and  $B_{\ell m}$ , it follows that since  $\psi$  in (4.182) is also of this general form, it is necessarily a solution of Laplace's equation. The relations (4.183) are just the natural generalisation to the non-azimuthally symmetric case of relations such as (4.92) that we saw previously when discussing inversion. And indeed, it follows from (4.179) that

$$\psi(a, \theta, \varphi) = \phi(a, \theta, \varphi), \quad (4.184)$$

meaning that  $\phi$  and  $\psi$  are equal at the “inversion radius”  $a$ . Furthermore, the transformation (4.179) maps a solution  $\phi$  of Laplace’s equation in the region  $r > a$  into a solution  $\psi$  in the region  $r < a$ , and *vice versa*.

The upshot of the above discussion is that in a case where there exists an inversion symmetry, the mapping (4.179) will map the exterior solution into the interior solution, and *vice versa*.

Let us now specialise to the case where there is azimuthal symmetry. In such cases, as we have seen, the expression for the potential at general points  $(r, \theta)$  can be derived from the expression on the  $z$  axis, by the method of off-axis extrapolation. In these situations, one can actually restate the criterion for the existence of an inversion symmetry in a given problem as a criterion purely on the potential evaluated on the  $z$  axis.

Suppose we can solve for the potential in a particular problem on the  $z$  axis, finding  $\phi_{>}(z)$  in the region  $|z| > a$  and  $\phi_{<}(z)$  in the region  $|z| < a$ . If there is an inversion symmetry through the radius  $a$  then it would follow, from (4.179), and the discussion above, that we should have<sup>37</sup>

$$\phi_{<}(z) = \frac{a}{z} \phi_{>}\left(\frac{a^2}{z}\right). \quad (4.185)$$

If this relation holds on the  $z$  axis then the problem admits an inversion symmetry. If (4.185) does not hold, then there is no inversion symmetry.

As an example where the inversion symmetry does work, consider the problem discussed in section 2.4, where the on-axis solution was obtained for the potential outside a spherical surface at radius  $a$ , whose upper hemisphere is held at potential  $V$  and whose lower hemisphere is held at potential  $-V$ . Again, we shall simplify the discussion by considering just the region where  $z$  is positive, to avoid the trivial but tiresome complications of considering negative  $z$  as well. The potential, which we now denote by  $\phi_{>}(z)$ , is given by (2.69):

$$\phi_{>}(z) = V \left( 1 - \frac{z^2 - a^2}{z\sqrt{z^2 + a^2}} \right). \quad (4.186)$$

Using (4.185), we therefore find that

$$\begin{aligned} \phi_{<}(z) &= \frac{a}{z} V \left[ 1 - \frac{a^4/z^2 - a^2}{(a^2/z)\sqrt{a^4/z^2 + a^2}} \right], \\ &= V \left[ \frac{a}{z} - \frac{a^2 - z^2}{z\sqrt{a^2 + z^2}} \right]. \end{aligned} \quad (4.187)$$

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<sup>37</sup>For simplicity, to avoid burdening the discussion with the rather trivial but tiresome complications of considering the negative  $z$  axis as well, we shall restrict ourselves in the discussion presented here to the positive  $z$  axis. The reader can easily extend this discussion to include the negative  $z$  axis if desired.

One can readily check, using the appropriate Green function for the interior problem, that this is indeed the correct expression for the potential on the  $z$  axis in the region  $|z| < a$  that is *inside* the spherical shell.

An example where the inversion symmetry fails is the problem of finding the potential due to a thin circular conducting disk of radius  $a$  held at potential  $V$ .<sup>38</sup> This was explored at length in one of the homeworks. One can indeed easily verify in this case that the on-axis expansions for the potentials  $\phi_{>}(z)$  and  $\phi_{<}(z)$  in the two regions are *not* related by the mapping (4.185).

## 5 Separation of Variables in Cylindrical Polar Coordinates

Another common situation that arises when considering boundary-value problems in electrostatics is when there is a cylindrical symmetry, in which case cylindrical polar coordinates are typically the most convenient choice. We shall take these to be  $(\rho, \varphi, z)$ , where

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z. \quad (5.1)$$

In other words, we still use  $z$  as the coordinate along the Cartesian  $z$  axis, but in the  $(x, y)$  plane we use polar coordinates  $(\rho, \varphi)$ . A straightforward calculation shows that Laplace's equation (3.1) becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (5.2)$$

We can separate variables by writing  $\phi(\rho, \varphi, z) = R(\rho) \Phi(\varphi) Z(z)$ , which leads, after dividing out by  $\phi$ , to

$$\frac{1}{\rho R} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (5.3)$$

The first two terms depend on  $\rho$  and  $\varphi$  but not  $z$ , while the last term depends on  $z$  but not  $\rho$  and  $\varphi$ . Thus the last term must be a constant, which we shall call  $k^2$ , and then

$$\frac{1}{\rho R} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + k^2 = 0. \quad (5.4)$$

Multiplying by  $\rho^2$ , we obtain

$$\frac{\rho}{R} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) + k^2 \rho^2 + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0. \quad (5.5)$$

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<sup>38</sup>The failure of the inversion symmetry in this example is the reason why some purported solutions to this problem that one can find online are incorrect.

The first two terms depend on  $\rho$  but not  $\varphi$ , whilst the last term depends on  $\varphi$  but not  $\rho$ . We deduce that the last term is a constant, which we shall call  $-\nu^2$ . The separation process is now complete, and we have

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0, \quad \frac{d^2 \Phi}{d\varphi^2} + \nu^2 \Phi = 0, \quad (5.6)$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0, \quad (5.7)$$

where  $k^2$  and  $\nu^2$  are separation constants.

The  $Z$  equation and  $\Phi$  equation in (5.6) are easily solved, giving

$$Z(z) \sim e^{\pm kz}, \quad \Phi(\varphi) \sim e^{\pm i\nu\varphi}. \quad (5.8)$$

Usually, we shall be seeking solutions where the potential should be periodic in the polar angle  $\varphi$  in the  $(x, y)$  plane with period  $2\pi$ , so that  $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ , and hence

$$\nu = \text{integer}. \quad (5.9)$$

However, it will still be useful to think of  $\nu$  as being a more general real constant for now. The reason for this will emerge when we investigate the solutions of the radial equation (5.7).

Rescaling the radial coordinate by defining  $x = k\rho$ , and renaming  $R$  as  $y$  (these functions  $x$  and  $y$  are not to be confused with the original Cartesian coordinates  $x$  and  $y$ !), the radial equation (5.7) takes the form

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y = 0, \quad (5.10)$$

where  $y'$  means  $dy/dx$ . This is known as *Bessel's Equation*, and we can construct solutions in the form of power-series expansions, by applying Frobenius's method as we did when discussing the Legendre equation.

## 5.1 Solutions of Bessel's equation

It may be recalled that when seeking power-series solutions of Legendre's equation in the form of expansions around  $x = 0$ , it was sufficient to consider Taylor expansions in non-negative integer powers of  $x$ , since  $x = 0$  was an *ordinary point* of the Legendre equation. By contrast, the point  $x = 0$  is a *singular point* of the Bessel equation. This can be seen by dividing out (5.10) by  $x^2$  so that the  $y''$  term has unit coefficient, and then noting that the coefficients of  $y'$  and  $y$  become singular at  $x = 0$ . Technically, the nature of the behaviour

at  $x = 0$  implies that it is a *regular singular point*, and the upshot is that we should now seek solutions of the form

$$y(x) = \sum_{n \geq 0} a_n x^{n+\sigma}, \quad (5.11)$$

where  $\sigma$  is a constant. Substituting into (5.10), we obtain

$$\sum_{n \geq 0} [(n + \sigma)^2 - \nu^2] a_n x^{n+\sigma} + \sum_{n \geq 0} a_n x^{n+\sigma+2} = 0. \quad (5.12)$$

Since this must hold for all  $x$ , we can now equate to zero the coefficient of each power of  $x$ . To do this, in the first sum we make the replacement  $n \rightarrow n + 2$ , so that (5.12) is re-expressed as<sup>39</sup>

$$\begin{aligned} \sum_{n \geq 0} \left\{ [(n + \sigma + 2)^2 - \nu^2] a_{n+2} + a_n \right\} x^{n+2} \\ + (\sigma^2 - \nu^2) a_0 + [(\sigma + 1)^2 - \nu^2] a_1 x = 0, \end{aligned} \quad (5.13)$$

where we have also extracted out an overall factor of  $x^\sigma$ . Since the coefficient of each power of  $x$  in eqn (5.13) must separately vanish, we see that

$$a_{n+2} = \frac{a_n}{\nu^2 - (n + \sigma + 2)^2}, \quad (5.14)$$

for  $n \geq 0$ . In addition we have, from the two “extra” terms,

$$(\sigma^2 - \nu^2) a_0 = 0 \quad \text{and} \quad [(\sigma + 1)^2 - \nu^2] a_1 = 0. \quad (5.15)$$

We begin with the first equation in (5.15). This is called the *Indicial Equation*. Notice that we can insist, without any loss of generality, that  $a_0 \neq 0$ . The reason for this is as follows. Suppose  $a_0$  were equal to zero. The series (5.11) would then begin with the  $a_1$  term, so it would be a series whose powers of  $x$  were  $(x^{\sigma+1}, x^{\sigma+2}, x^{\sigma+3}, \dots)$ . But since at the stage when we write (5.11)  $\sigma$  is a completely arbitrary constant, not yet determined, we could as well relabel it by writing  $\sigma = \sigma' - 1$ . We would then have a series whose powers of  $x$  are  $(x^{\sigma'}, x^{\sigma'+1}, x^{\sigma'+2}, \dots)$ . But this is exactly what we would have had if the  $a_0$  term were in fact non-zero, after relabelling  $\sigma'$  as  $\sigma$ . So insisting that  $a_0$  be non-zero loses no generality at all.

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<sup>39</sup>Recall that “sending  $n \rightarrow n + 2$  in the first sum” means first setting  $n = m + 2$ , so that the summation over  $m$  runs from  $-2$  up to  $+\infty$ . Then, we write this as the sum from  $m = 0$  to  $+\infty$  together with the “extra” two terms  $m = -2$  and  $m = -1$  added on in addition. Finally, we relabel the  $m$  summation variable as  $n$ .

Proceeding, we then have the indicial equation  $\sigma^2 - \nu^2 = 0$ , i.e.

$$\sigma = \pm \nu. \quad (5.16)$$

Now we look at the second equation in (5.15). Since we already know from the indicial equation that  $\sigma^2 = \nu^2$ , we can rewrite the second equation as

$$(2\sigma + 1)a_1 = 0. \quad (5.17)$$

Thus either  $a_1 = 0$  or else  $\sigma = -\frac{1}{2}$ . But since we already know from the indicial equation that  $\sigma = \pm \nu$ , it follows that except in the very special cases where  $\nu = \pm \frac{1}{2}$ , which has to be analysed separately, we must have that  $a_1 = 0$ . Let us assume that  $\nu \neq \pm \frac{1}{2}$ , for simplicity. In fact, we shall assume for now that  $\nu$  takes a generic value, which is not equal to any integer or half integer.

Finally, in the recursion relation (5.14), we substitute the two possible values for  $\sigma$ , i.e.  $\sigma = \nu$  or  $\sigma = -\nu$ . In each of these two cases, the recursion relation then gives us expressions for all the  $a_n$  with  $n \geq 2$ , in terms of  $a_0$  (which is non-zero), and  $a_1$  (which is zero since we are assuming  $\nu \neq \pm \frac{1}{2}$ ).

We can check the radius of convergence of the series solutions, by applying the ratio test. The ratio of successive terms (bearing in mind that  $a_1 = 0$ , which means all the odd  $a_n$  are zero) is given by

$$\frac{a_{n+2} x^{n+2+\sigma}}{a_n x^{n+\sigma}} = \frac{a_{n+2} x^2}{a_n} = \frac{x^2}{\nu^2 - (n + \sigma + 2)^2}, \quad (5.18)$$

where  $\sigma = \pm \nu$ . In either case, at large  $n$  we see that the absolute value of the ratio tends to  $x^2/n^2$ , and thus the ratio becomes zero for any finite  $x$ , no matter how large. Thus the radius of convergence is infinite.

To summarise, we have obtained two solutions to the Bessel equation,  $y_1(x)$  and  $y_2(x)$ , given by

$$y_1(x) = x^\nu \sum_{n \geq 0} a_n^+ x^n, \quad y_2(x) = x^{-\nu} \sum_{n \geq 0} a_n^- x^n, \quad (5.19)$$

where  $a_0^\pm$  is arbitrary,  $a_1^\pm = 0$ , and from the recursion relation (5.14) we have

$$a_{n+2}^+ = \frac{a_n^+}{\nu^2 - (n + \nu + 2)^2}, \quad a_{n+2}^- = \frac{a_n^-}{\nu^2 - (n - \nu + 2)^2}. \quad (5.20)$$

It is straightforward to see that these two recursion relations can be solved, to give the two solutions, called  $J_\nu(x)$  and  $J_{-\nu}(x)$ :

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{p \geq 0} \frac{(-1)^p}{p! \Gamma(p + \nu + 1)} \left(\frac{x}{2}\right)^{2p}, \quad J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{p \geq 0} \frac{(-1)^p}{p! \Gamma(p - \nu + 1)} \left(\frac{x}{2}\right)^{2p}. \quad (5.21)$$