611: Electromagnetic Theory II

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1 Electrodynamics and Special Relativity

1.1 Introduction

In Newtonian mechanics, the fundamental laws of physics, such as the dynamics of moving objects, are valid in all inertial frames (i.e. all non-accelerating frames). If S is an inertial frame, then the set of all inertial frames comprises all frames that are in uniform motion relative to S. Suppose that two inertial frames S and S', are parallel, and that their origins coincide at at $t = 0$. If S' is moving with uniform velocity $\vec{v}$ relative to S, then a point P with position vector $\vec{r}$ with respect to S will have position vector $\vec{r}\,'$ with respect to S', where

$$\vec{r}\,' = \vec{r} - \vec{v}\, t\,. \tag{1.1}$$

Of course, it is always understood in Newtonian mechanics that time is absolute, and so the times t and t' measured by observers in the frames S and S' are the same:

$$t' = t\,. \tag{1.2}$$

The transformations (1.1) and (1.2) form part of what is called the *Galilean Group*. The full Galilean group includes also rotations of the spatial Cartesian coordinate system, so that we can define

$$\vec{r}\,' = M\cdot\vec{r} - \vec{v}\, t\,, \qquad t' = t\,, \tag{1.3}$$

where M is an orthogonal 3×3 constant matrix acting by matrix multiplication on the components of the position vector:

$$\vec{r} \leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}\,, \qquad M\cdot\vec{r} \leftrightarrow M\begin{pmatrix} x \\ y \\ z \end{pmatrix}\,, \tag{1.4}$$

where $M^T\, M = 1$.

Returning to our simplifying assumption that the two frames are parallel, i.e. that $M = \mathbb{1}$, it follows that if a particle having position vector $\vec{r}$ in S moves with velocity $\vec{u} = d\vec{r}/dt$, then its velocity $\vec{u}' = d\vec{r}\,'/dt$ as measured with respect to the frame S' is given by

$$\vec{u}' = \vec{u} - \vec{v}\,. \tag{1.5}$$

Suppose, for example, that $\vec{v}$ lies along the x axis of S; i.e. that S' is moving along the x axis of S with speed $v = |\vec{v}|$. If a beam of light were moving along the x axis of S

with speed c, then the prediction of Newtonian mechanics and the Galilean transformation would therefore be that in the frame S', the speed c' of the light beam would be

$$c' = c - v\,. \tag{1.6}$$

Of course, as is well known, this contradicts experiment. As far as we can tell, with experiments of ever-increasing accuracy, the true state of affairs is that the speed of the light beam is the same in all inertial frames. Thus the predictions of Newtonian mechanics and the Galilean transformation are falsified by experiment.

Of course, it should be emphasised that the discrepancies between experiment and the Galilean transformations are rather negligible if the relative speed v between the two inertial frames is of a typical "everyday" magnitude, such as the speed of a car or a plane. But if v begins to become appreciable in comparison to the speed of light, then the discrepancy becomes appreciable too.

By contrast, it turns out that Maxwell's equations of electromagnetism *do* predict a constant speed of light, independent of the choice of inertial frame. To be precise, let us begin with the free-space Maxwell's equations,

$$\begin{aligned} \vec\nabla\cdot\vec E &= \frac{1}{\epsilon_0}\,\rho\,, \qquad \vec\nabla\times\vec B - \mu_0\epsilon_0\,\frac{\partial\vec E}{\partial t} = \mu_0\,\vec J\,,\\ \vec\nabla\cdot\vec B &= 0\,, \qquad \vec\nabla\times\vec E + \frac{\partial\vec B}{\partial t} = 0\,, \end{aligned} \tag{1.7}$$

where $\vec E$ and $\vec B$ are the electric and magnetic fields, ρ and $\vec J$ are the charge density and current density, and ϵ_0 and μ_0 are the permittivity and permeability of free space.

To see the electromagnetic wave solutions, we can consider a region of space where there are no sources, i.e. where $\rho = 0$ and $\vec J = 0$. Then we shall have

$$\vec\nabla\times(\vec\nabla\times\vec E) = -\frac{\partial}{\partial t}\vec\nabla\times\vec B = -\mu_0\epsilon_0\,\frac{\partial^2\vec E}{\partial t^2}\,. \tag{1.8}$$

But using the vector identity $\vec\nabla\times(\vec\nabla\times\vec E) = \vec\nabla(\vec\nabla\cdot\vec E) - \nabla^2\vec E$, it follows from $\vec\nabla\cdot\vec E = 0$ that the electric field satisfies the wave equation

$$\nabla^2\vec E - \mu_0\epsilon_0\,\frac{\partial^2\vec E}{\partial t^2} = 0\,. \tag{1.9}$$

This admits plane-wave solutions of the form

$$\vec E = \vec E_0\, e^{\mathrm{i}(\vec k\cdot\vec r - \omega t)}\,, \tag{1.10}$$

where $\vec E_0$ and $\vec k$ are constant vectors, and ω is also a constant, where

$$k^2 = \mu_0\epsilon_0\,\omega^2\,. \tag{1.11}$$

Here k means $|\vec{k}|$, the magnitude of the wave-vector $\vec{k}$. Thus we see that the waves travel at speed c given by

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}\,. \tag{1.12}$$

Putting in the numbers, this gives $c \approx 3 \times 10^8$ metres per second, i.e. the familiar speed of light.

A similar calculation shows that the magnetic field $\vec{B}$ also satisfies an identical wave equation, and in fact $\vec{B}$ and $\vec{E}$ are related by

$$\vec{B} = \frac{1}{\omega}\,\vec{k} \times \vec{E}\,. \tag{1.13}$$

The situation, then, is that if the Maxwell equations (1.7) hold in a given frame of reference, then they predict that the speed of light will be $c \approx 3 \times 10^8$ metres per second in that frame. Therefore, if we assume that the Maxwell equations hold in *all* inertial frames, then they predict that the speed of light will have that same value in all inertial frames. Since this prediction is in agreement with experiment, we can reasonably expect that the Maxwell equations will indeed hold in all inertial frames. Since the prediction contradicts the implications of the Galilean transformations, it follows that the Maxwell equations are *not* invariant under Galilean transformations. This is just as well, since the Galilean transformations are wrong!

In fact, as we shall see, the transformations that correctly describe the relation between observations in different inertial frames in uniform motion are the *Lorentz Transformations* of Special Relativity. Furthermore, even though the Maxwell equations were written down in the pre-relativity days of the nineteenth century, they are in fact perfectly invariant[1] under the Lorentz transformations. No further modification is required in order to incorporate Maxwell's theory of electromagnetism into special relativity.

However, the Maxwell equations as they stand, written in the form given in equation (1.7), do not look *manifestly* covariant with respect to Lorentz transformations. This is because they are written in the language of 3-vectors. To make the Lorentz transformations look nice and simple, we should instead express them in terms of 4-vectors, where the extra component is associated with the time direction.

In order to give a nice elegant treatment of the Lorentz transformation properties of the Maxwell equations, we should first therefore reformulate special relativity in terms of 4-vectors and 4-tensors. Since there are many different conventions on offer in the marketplace,

[1]Strictly, as will be explained later, we should say *covariant* rather than *invariant.*

we shall begin with a review of special relativity in the notation that we shall be using in this course.

1.2 The Lorentz Transformation

The derivation of the Lorentz transformation follows from Einstein's two postulates:

- The laws of physics are the same for all inertial observers.
- The speed of light is the same for all inertial observers.

To derive the Lorentz transformation, let us suppose that we have two inertial frames S and S', whose origins coincide at time zero, that is to say, at $t = 0$ in the frame S, and at $t' = 0$ in the frame S'. If a flash of light is emitted at the origin at time zero, then it will spread out over a spherical wavefront given by

$$x^2 + y^2 + z^2 - c^2 t^2 = 0 \tag{1.14}$$

in the frame S, and by

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0 \tag{1.15}$$

in the frame S'. Note that, following the second of Einstein's postulates, we have used the *same* speed of light c for both inertial frames. Our goal is to derive the relation between the coordinates (x, y, z, t) and (x', y', z', t') in the two inertial frames.

Consider for simplicity the case where S' is parallel to S, and moves along the x axis with velocity v. Clearly we must have

$$y' = y\,, \qquad z' = z\,. \tag{1.16}$$

Furthermore, the transformation between (x, t) and (x', t') must be a *linear* one, since otherwise it would not be translation-invariant or time-translation invariant. Thus we may say that

$$x' = Ax + Bt\,, \qquad t' = Cx + Dt\,, \tag{1.17}$$

for constants A, B , C and D to be determined.

Now, if $x' = 0$, this must, by definition, correspond to the equation $x = vt$ in the frame S, and so from the first equation in (1.17) we have $B = -Av$. For convenience we will change the name of the constant A to γ, and thus we have

$$x' = \gamma(x - vt)\,. \tag{1.18}$$

By the same token, if we consider taking $x = 0$ then this will correspond to $x' = -vt'$ in the frame S'. It follows that

$$x = \gamma(x' + vt') \,. \tag{1.19}$$

Note that it must be the *same* constant γ in both these equations, since the two really just correspond to reversing the direction of the x axis, and the physics must be the same for the two cases.

Now we bring in the postulate that the speed of light is the same in the two frames, so if we have $x = ct$ then this must imply $x' = ct'$. Solving the resulting two equations

$$ct' = \gamma(c - v)t \,, \qquad ct = \gamma(c + v)t' \tag{1.20}$$

for γ, we obtain

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \,. \tag{1.21}$$

Solving $x^2 - c^2t^2 = x'^2 - c^2t'^2$ for t', after using (1.18), we find $t'^2 = \gamma^2(t - vx/c^2)^2$ and hence

$$t' = \gamma(t - \frac{v}{c^2}x) \,. \tag{1.22}$$

(We must choose the positive square root since it must reduce to $t' = +t$ at zero relative velocity, v.) Thus we arrive at the Lorentz transformation

$$x' = \gamma(x - vt) \,, \qquad y' = y \,, \qquad z' = z \,, \qquad t' = \gamma(t - \frac{v}{c^2}x) \,, \tag{1.23}$$

where γ is given by (1.21), for the special case where S' is moving along the x direction with velocity v.

At this point, for notational convenience, we shall introduce the simplification of working in a system of units in which the speed of light is set equal to 1. We can do this because the speed of light is the same for all inertial observers, and so we may as well choose to measure length in terms of the time it takes for light *in vacuo* to traverse the distance. In fact, the metre is nowadays *defined* to be the distance travelled by light *in vacuo* in 1/299,792,458 of a second. By making the small change of taking the *light-second* as the basic unit of length, rather than the $1/299{,}792{,}458'th$ of a light-second, we end up with a system of units in which $c = 1$. In these units, the Lorentz transformation (1.23) becomes

$$x' = \gamma(x - vt) \,, \qquad y' = y \,, \qquad z' = z \,, \qquad t' = \gamma(t - vx) \,, \tag{1.24}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \,. \tag{1.25}$$

It will be convenient to generalise the Lorentz transformation (1.24) to the case where the frame S' is moving with (constant) velocity $\vec{v}$ in arbitrary direction, rather than specifically along the x axis. It is rather straightforward to do this. We know that there is a complete rotational symmetry in the three-dimensional space parameterised by the (x, y, z) coordinate system. Therefore, if we can first rewrite the special case described by (1.24) in terms of 3-vectors, where the 3-vector velocity $\vec{v}$ happens to be simply $\vec{v} = (v, 0, 0)$, then generalisation will be immediate. It is easy to check that with $\vec{v}$ taken to be $(v, 0, 0)$, the Lorentz transformation (1.24) can be written as

$$\vec{r}\,' = \vec{r} + \frac{\gamma - 1}{v^2} (\vec{v} \cdot \vec{r})\, \vec{v} - \gamma \vec{v}\, t\,, \qquad t' = \gamma(t - \vec{v} \cdot \vec{r})\,, \tag{1.26}$$

with $\gamma = (1 - v^2)^{-1/2}$ and $v \equiv |\vec{v}|$, and with $\vec{r} = (x, y, z)$. Since these equations are manifestly covariant under 3-dimensional spatial rotations (i.e. they are written entirely in a 3-vector notation), it must be that they are the correct form of the Lorentz transformations for an arbitrary direction for the velocity 3-vector $\vec{v}$.

The Lorentz transformations (1.26) are what are called the *pure boosts.* It is easy to check that they have the property of preserving the spherical light-front condition, in the sense that points on the expanding spherical shell given by $r^2 = t^2$ of a light-pulse emitted at the origin at $t = 0$ in the frame S will also satisfy the equivalent condition $r'^2 = t'^2$ in the primed reference frame S'. (Note that $r^2 = x^2 + y^2 + z^2$.) In fact, a stronger statement is true: The Lorentz transformation (1.26) satisfies the equation

$$x^2 + y^2 + z^2 - t^2 = x'^2 + y'^2 + z'^2 - t'^2\,. \tag{1.27}$$

1.3 An interlude on 3-vectors and suffix notation

Before describing the 4-dimensional spacetime approach to special relativity, it may be helpful to give a brief review of some analogous properties of 3-dimensional Euclidean space, and Cartesian vector analysis.

Consider a 3-vector $\vec{A}$, with x, y and z components denoted by A_1, A_2 and A_3 respectively. Thus we may write

$$\vec{A} = (A_1, A_2, A_3)\,. \tag{1.28}$$

It is convenient then to denote the set of components by A_i, for $i = 1, 2, 3$.

The scalar product between two vectors $\vec{A}$ and $\vec{B}$ is given by

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \sum_{i=1}^{3} A_i B_i\,. \tag{1.29}$$

This expression can be written more succinctly using the *Einstein Summation Convention.* The idea is that when writing valid expressions using vectors, or more generally tensors, on every occasion that a sumation of the form $\sum_{i=1}^{3}$ is performed, the summand is an expression in which the summation index i occurs exactly twice. Furthermore, there will be no occasion when an index occurs exactly twice in a given term and a sum over i is not performed. Therefore, we can abbreviate the writing by simply omitting the explicit summation symbol, since we know as soon as we see an index repeated exactly twice in a term of an equation that it *must* be accompanied by a summation symbol. Thus we can abbreviate (1.29) and just write the scalar product as

$$\vec{A}\cdot\vec{B} = A_i B_i\,. \tag{1.30}$$

The index i here is called a "dummy suffix." It is just like a local summation variable in a computer program; it doesn't matter if it is called i, or j or anything else, as long as it doesn't clash with any other index that is already in use.

The next concept to introduce is the *Kronecker delta* tensor δ_{ij}. This is defined by

$$\delta_{ij} = 1 \quad \text{if} \quad i = j\,, \qquad \delta_{ij} = 0 \quad \text{if} \quad i \ne j\,, \tag{1.31}$$

Thus

$$\delta_{11} = \delta_{22} = \delta_{33} = 1\,, \qquad \delta_{12} = \delta_{13} = \cdots = 0\,. \tag{1.32}$$

Note that δ_{ij} is a symmetric tensor: $\delta_{ij} = \delta_{ji}$. The Kronecker delta clearly has the replacement property

$$A_i = \delta_{ij} A_j\,, \tag{1.33}$$

since by (1.31) the only non-zero term in the summation over j is the term when $j = i$.

Now consider the vector product $\vec{A}\times\vec{B}$. We have

$$\vec{A}\times\vec{B} = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1)\,. \tag{1.34}$$

To write this using index notation, we first define the 3-index totally-antisymmetric tensor ϵ_{ijk}. Total antisymmetry means that the tensor changes sign if any pair of indices is swapped. For example

$$\epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}\,. \tag{1.35}$$

Given this total antisymmetry, we actually only need to specify the value of one non-zero component in order to pin down the definition completely. We shall define $\epsilon_{123} = +1$. From the total antisymmetry, it then follows that

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1\,, \qquad \epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1\,, \tag{1.36}$$

with all other components vanishing.

It is now evident that in index notation, the i'th component of the vector product $\vec{A}\times\vec{B}$ can be written as

$$(\vec{A}\times\vec{B})_i = \epsilon_{ijk} A_j B_k\,. \tag{1.37}$$

For example, the $i=1$ component (the x component) is given by

$$(\vec{A}\times\vec{B})_1 = \epsilon_{1jk} A_j B_k = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2\,, \tag{1.38}$$

in agreement with the x-component given in (1.34).

Now, let us consider the vector triple product $\vec{A}\times(\vec{B}\times\vec{C})$. The i component is therefore given by

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk} A_j (\vec{B}\times\vec{C})_k = \epsilon_{ijk}\epsilon_{k\ell m} A_j B_\ell C_m\,. \tag{1.39}$$

For convenience, we may cycle the indices on the second ϵ tensor around and write this as

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk}\epsilon_{\ell mk} A_j B_\ell C_m\,. \tag{1.40}$$

There is an extremely useful identity, which can be proved simply by considering all possible values of the free indices i, j, ℓ, m:

$$\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\,. \tag{1.41}$$

Using this in (1.40), we have

$$\begin{aligned}
[\vec{A}\times(\vec{B}\times\vec{C})]_i &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) A_j B_\ell C_m\,,\\
&= \delta_{i\ell}\delta_{jm} A_j B_\ell C_m - \delta_{im}\delta_{j\ell} A_j B_\ell C_m\,,\\
&= B_i A_j C_j - C_i A_j B_j\,,\\
&= (\vec{A}\cdot\vec{C})\, B_i - (\vec{A}\cdot\vec{B})\, C_i\,.
\end{aligned} \tag{1.42}$$

In other words, we have proven that

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{A}\cdot\vec{B})\,\vec{C}\,. \tag{1.43}$$

The above are just a few simple examples of the use of index notation in order to write 3-vector and 3-tensor expressions in Cartesian 3-tensor analysis. It is a very useful notation when one needs to deal with complicated expressions. As we shall now see, there is a very natural generalisation to the case of vector and tensor analysis in 4-dimensional Minkowski spacetime.

1.4 4-vectors and 4-tensors

The Lorentz transformations given in (1.26) are linear in the space and time coordinates. They can be written more succinctly if we first define the set of four spacetime coordinates denoted by x^μ, where μ is an index, or label, that ranges over the values 0, 1, 2 and 3. The case $\mu = 0$ corresponds to the time coordinate t, while $\mu = 1$, 2 and 3 corresponds to the space coordinates x, y and z respectively. Thus we have[2]

$$(x^0, x^1, x^2, x^3) = (t, x, y, z)\,. \tag{1.44}$$

Of course, once the abstract index label μ is replaced, as here, by the specific index values 0, 1, 2 and 3, one has to be very careful when reading a formula to distinguish between, for example, x^2 meaning the symbol x carrying the spacetime index $\mu = 2$, and x^2 meaning the square of x. It should generally be obvious from the context which is meant.

The invariant quadratic form appearing on the left-hand side of (1.27) can now be written in a nice way, if we first introduce the 2-index quantity $\eta_{\mu\nu}$, defined to be given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{1.45}$$

What this means is that the rows of the matrix on the right are labelled by the index μ and the columns are labelled by the index ν. In other words, (1.45) is saying that the only non-vanishing components of $\eta_{\mu\nu}$ are given by

$$\eta_{00} = -1\,, \qquad \eta_{11} = \eta_{22} = \eta_{33} = 1\,, \tag{1.46}$$

with $\eta_{\mu\nu} = 0$ if $\mu \ne \nu$. Note that $\eta_{\mu\nu}$ is symmetric:

$$\eta_{\mu\nu} = \eta_{\nu\mu}\,. \tag{1.47}$$

Using $\eta_{\mu\nu}$, the quadratic form on the left-hand side of (1.27) can be rewritten as

$$x^2 + y^2 + z^2 - t^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu}\, x^\mu x^\nu\,. \tag{1.48}$$

At this point, it is very convenient to introduce the *Einstein Summation Convention.* This makes the writing of expressions such as (1.48) much less cumbersome. The summation convention works as follows:

[2] The choice to put the index label μ as a superscript, rather than a subscript, is purely conventional. But, unlike the situation with many arbitrary conventions, in this case the coordinate index is placed upstairs in *all* modern literature.

In an expression such as (1.48), if an index appears *exactly twice* in a term, then it will be understood that the index is summed over the natural index range (0, 1, 2, 3 in our present case), and the explicit summation symbol will be omitted. An index that occurs twice in a term, thus is understood to be summed over, is called a *Dummy Index.*

Since in (1.48) both μ and ν occur exactly twice, we can rewrite the expression, using the Einstein summation convention, as simply

$$x^2 + y^2 + z^2 - t^2 = \eta_{\mu\nu}\, x^\mu x^\nu\,. \tag{1.49}$$

On might at first think there would be a great potential for ambiguity, but this is not the case. The point is that in any valid vectorial (or, more generally, tensorial) expression, the *only* time that a particular index can ever occur exactly twice in a term is when it is summed over. Thus, there is no ambiguity resulting from agreeing to omit the explicit summation symbol, since it is logically inevitable that a summation is intended.[3]

Now let us return to the Lorentz transformations. The pure boosts written in (1.26), being linear in the space and time coordinates, can be written in the form

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,, \tag{1.50}$$

where $\Lambda^\mu{}_\nu$ are constants, and the Einstein summation convention is operative for the dummy index ν. By comparing (1.50) carefully with (1.26), we can see that the components $\Lambda^\mu{}_\nu$ are given by

$$\begin{aligned} \Lambda^0{}_0 &= \gamma\,, \qquad \Lambda^0{}_i = -\gamma v_i\,,\\ \Lambda^i{}_0 &= -\gamma\, v_i\,, \qquad \Lambda^i{}_j = \delta_{ij} + \frac{\gamma-1}{v^2} v_i v_j\,, \end{aligned} \tag{1.51}$$

where δ_{ij} is The Kronecker delta symbol,

$$\delta_{ij} = 1 \quad \text{if} \quad i=j\,, \qquad \delta_{ij} = 0 \quad \text{if} \quad i \ne j\,. \tag{1.52}$$

A couple of points need to be explained here. Firstly, we are introducing Latin indices here, namely the i and j indices, which range only over the three spatial index values, $i = 1$, 2 and 3. Thus the 4-index μ can be viewed as $\mu = (0, i)$, where $i = 1$, 2 and 3. This piece of notation is useful because the three spatial index values always occur on a completely

[3] As a side remark, it should be noted that in a valid vectorial or tensorial expression, a specific index can **NEVER** appear more than twice in a given term. If you have written down a term where a given index occurs 3, 4 or more times then there is no need to look further at it; it is **WRONG**. Thus, for example, it is totally meaningless to write $\eta_{\mu\mu}\, x^\mu x^\mu$. If you ever find such an expression in a calculation then you must stop, and go back to find the place where an error was made.

symmetric footing, whereas the time index value $\mu = 0$ is a bit different. This can be seen, for example, in the definition of $\eta_{\mu\nu}$ in (1.45) or (1.46).

The second point is that when we consider spatial indices (for example when μ takes the values $i = 1$, 2 or 3), it actually makes no difference whether we write the index i upstairs or downstairs. Sometimes, as in (1.51), it will be convenient to be rather relaxed about whether we put spatial indices upstairs or downstairs. By contrast, when the index takes the value 0, it is very important to be careful about whether it is upstairs or downstairs. The reason why we can be cavalier about the Latin indices, but not the Greek, will become clearer as we proceed.

We already saw that the Lorentz boost transformations (1.26), re-expressed in terms of $\Lambda^\mu{}_\nu$ in (1.51), have the property that $\eta_{\mu\nu}\, x^\mu x^\nu = \eta_{\mu\nu}\, x'^\mu x'^\nu$. Thus from (1.50) we have

$$\eta_{\mu\nu}\, x^\mu x^\nu = \eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, x^\rho x^\sigma\,. \tag{1.53}$$

(Note that we have been careful to choose two different dummy indices for the two implicit summations over ρ and σ!) On the left-hand side, we can replace the dummy indices μ and ν by ρ and σ, and thus write

$$\eta_{\rho\sigma}\, x^\rho x^\sigma = \eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, x^\rho x^\sigma\,. \tag{1.54}$$

This can be grouped together as

$$(\eta_{\rho\sigma} - \eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma) x^\rho x^\sigma = 0\,, \tag{1.55}$$

and, since it is true for any x^μ, we must have that

$$\eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,. \tag{1.56}$$

(This can also be verified directly from (1.51).) The full set of Λ's that satisfy (1.56) are the Lorentz Transformations. The Lorentz Boosts, given by (1.51), are examples, but they are just a subset of the full set of Lorentz transformations that satisfy (1.56). Essentially, the additional Lorentz transformations consist of rotations of the three-dimensional spatial coordinates. Thus, one can really say that the Lorentz boosts (1.51) are the "interesting" Lorentz transformations, i.e. the ones that rotate space and time into one another. The remainder are just rotations of our familiar old 3-dimensional Euclidean space.

The coordinates $x^\mu = (x^0, x^i)$ live in a four-dimensional *spacetime*, known as *Minkowski Spacetime*. This is the four-dimensional analogue of the three-dimensional *Euclidean Space* described by the Cartesian coordinates $x^i = (x, y, z)$. The quantity $\eta_{\mu\nu}$ is called the

Minkowski Metric, and for reasons that we shall see presently, it is called a *tensor*. It is called a metric because it provides the rule for measuring distances in the four-dimensional Minkowski spacetime. The distance, or to be more precise, the *interval*, between two infinitesimally-separated points (x^0, x^1, x^2, x^3) and $(x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ in spacetime is written as ds, and is given by

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \,. \tag{1.57}$$

Clearly, this is the Minkowskian generalisation of the three-dimensional distance ds_E between neighbouring points (x, y, z) and $(x + dx, y + dy, z + dz)$ in Euclidean space, which, by Pythagoras' theorem, is given by

$$ds_E^2 = dx^2 + dy^2 + dz^2 = \delta_{ij}\, dx^i dx^j \,. \tag{1.58}$$

The Euclidean metric (1.58) is invariant under arbitrary constant rotations of the (x, y, z) coordinate system. (This is clearly true because the distance between the neighbouring points must obviously be independent of how the axes of the Cartesian coordinate system are oriented.) By the same token, the Minkowski metric (1.57) is invariant under arbitrary Lorentz transformations. In other words, as can be seen to follow immediately from (1.56), the spacetime interval $ds'^2 = \eta_{\mu\nu}\, dx'^\mu dx'^\nu$ calculated in the primed frame is identical to the interval ds^2 calculated in the unprimed frame

$$\begin{aligned} ds'^2 &= \eta_{\mu\nu}\, dx'^\mu dx'^\nu = \eta_{\mu\nu}\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, dx^\rho dx^\sigma \,, \\ &= \eta_{\rho\sigma}\, dx^\rho dx^\sigma = ds^2 \,. \end{aligned} \tag{1.59}$$

For this reason, we do not need to distinguish between ds^2 and ds'^2, since it is the same in all inertial frames. It is what is called a *Lorentz Scalar*.

The Lorentz transformation rule of the coordinate differential dx^μ, i.e.

$$dx'^\mu = \Lambda^\mu{}_\nu\, dx^\nu \,, \tag{1.60}$$

can be taken as the prototype for more general 4-vectors. Thus, we may define any set of four quantities U^μ, for $\mu = 0$, 1, 2 and 3, to be the components of a Lorentz 4-vector (often, we shall just abbreviate this to simply a 4-vector) if they transform, under Lorentz transformations, according to the rule

$$U'^\mu = \Lambda^\mu{}_\nu\, U^\nu \,. \tag{1.61}$$

The Minkowski metric $\eta_{\mu\nu}$ may be thought of as a 4×4 matrix, whose rows are labelled by μ and columns labelled by ν, as in (1.45). Clearly, the inverse of this matrix takes the same form as the matrix itself. We denote the components of the inverse matrix by $\eta^{\mu\nu}$. This is called, not surprisingly, the *inverse Minkowksi metric.* Clearly it satisfies the relation

$$\eta_{\mu\nu}\,\eta^{\nu\rho} = \delta^\rho_\mu\,, \tag{1.62}$$

where the 4-dimensional Kronecker delta is defined to equal 1 if $\mu = \rho$, and to equal 0 if $\mu \neq \rho$. Note that like $\eta_{\mu\nu}$, the inverse $\eta^{\mu\nu}$ is symmetric also: $\eta^{\mu\nu} = \eta^{\nu\mu}$.

The Minkowksi metric and its inverse may be used to lower or raise the indices on other quantities. Thus, for example, if U^μ are the components of a 4-vector, then we may define

$$U_\mu = \eta_{\mu\nu}\,U^\nu\,. \tag{1.63}$$

This is another type of 4-vector. Two distinguish the two, we call a 4-vector with an upstairs index a *contravariant* 4-vector, while one with a downstairs index is called a *covariant* 4-vector. Note that if we raise the lowered index in (1.63) again using $\eta^{\mu\nu}$, then we get back to the starting point:

$$\eta^{\mu\nu}\,U_\nu = \eta^{\mu\nu}\,\eta_{\nu\rho}\,U^\rho = \delta^\mu_\rho\,U^\rho = U^\mu\,. \tag{1.64}$$

It is for this reason that we can use the same symbol U for the covariant 4-vector $U_\mu = \eta_{\mu\nu}\,U^\nu$ as we used for the contravariant 4-vector U^μ.

In a similar fashion, we may define the quantities ${\Lambda_\mu}^\nu$ by

$${\Lambda_\mu}^\nu = \eta_{\mu\rho}\,\eta^{\nu\sigma}\,{\Lambda^\rho}_\sigma\,. \tag{1.65}$$

It is then clear that (1.56) can be restated as

$${\Lambda^\mu}_\nu\,{\Lambda_\mu}^\rho = \delta^\rho_\nu\,. \tag{1.66}$$

We can also then invert the Lorentz transformation $x'^\mu = {\Lambda^\mu}_\nu\,x^\nu$ to give

$$x^\mu = {\Lambda_\nu}^\mu\,x'^\nu\,. \tag{1.67}$$

It now follows from (1.61) that the components of the covariant 4-vector U_μ defined by (1.63) transform under Lorentz transformations according to the rule

$$U'_\mu = {\Lambda_\mu}^\nu\,U_\nu\,. \tag{1.68}$$

Any set of 4 quantities U_μ which transform in this way under Lorentz transformations will be called a covariant 4-vector.

Using (1.67), we can see that the gradient operator $\partial/\partial x^\mu$ transforms as a covariant 4-vector. Using the chain rule for partial differentiation we have

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\,\frac{\partial}{\partial x^\nu}\,. \tag{1.69}$$

But from (1.67) we have (after a relabelling of indices) that

$$\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\mu{}^\nu\,, \tag{1.70}$$

and hence (1.69) gives

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu\,\frac{\partial}{\partial x^\nu}\,. \tag{1.71}$$

As can be seen from (1.68), this is precisely the transformation rule for a a covariant 4-vector. The gradient operator arises sufficiently often that it is useful to use a special symbol to denote it. We therefore define

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,. \tag{1.72}$$

Thus the Lorentz transformation rule (1.71) is now written as

$$\partial'_\mu = \Lambda_\mu{}^\nu\,\partial_\nu\,. \tag{1.73}$$

1.5 Lorentz tensors

Having seen how contravariant and covariant 4-vectors transform under Lorentz transformations (as given in (1.61) and (1.68) respectively), we can now define the transformation rules for more general objects called tensors. These objects carry multiple indices, and each one transforms with a Λ factor, of either the (1.61) type if the index is upstairs, or of the (1.68) type if the index is downstairs. Thus, for example, a tensor $T_{\mu\nu}$ transforms under Lorentz transformations according to the rule

$$T'_{\mu\nu} = \Lambda_\mu{}^\rho\,\Lambda_\nu{}^\sigma\,T_{\rho\sigma}\,. \tag{1.74}$$

More generally, a tensor $T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}$ will transform according to the rule

$$T'^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_m}{}_{\rho_m}\,\Lambda_{\nu_1}{}^{\sigma_1}\cdots\Lambda_{\nu_n}{}^{\sigma_n}\,T^{\rho_1\cdots\rho_m}{}_{\sigma_1\cdots\sigma_n}\,. \tag{1.75}$$

Note that scalars are just special cases of tensors with no indices, while vectors are special cases with just one index.

It is easy to see that products of tensors give rise again to tensors. For example, if U^μ and V^μ are two contravariant vectors then $T^{\mu\nu} \equiv U^\mu V^\nu$ is a tensor, since, using the known transformation rules for U and V we have

$$\begin{aligned} T'^{\mu\nu} &= U'^{\mu}V'^{\nu} = \Lambda^{\mu}{}_{\rho}\, U^{\rho}\, \Lambda^{\nu}{}_{\sigma}\, V^{\sigma}\,, \\ &= \Lambda^{\mu}{}_{\rho}\, \Lambda^{\nu}{}_{\sigma}\, T^{\rho\sigma}\,. \end{aligned} \tag{1.76}$$

Note that the gradient operator ∂_μ can also be used to map a tensor into another tensor. For example, if U_μ is a vector field (i.e. a vector that changes from place to place in spacetime) then $S_{\mu\nu} \equiv \partial_\mu U_\nu$ is a tensor field.

We make also define the operation of *Contraction*, which reduces a tensor to one with a smaller number of indices. A contraction is performed by setting an upstairs index on a tensor equal to a downstairs index. The Einstein summation convention then automatically comes into play, and the result is that one has an object with one fewer upstairs indices and one fewer downstairs indices. Furthermore, a simple calculation shows that the new object is itself a tensor. Consider, for example, a tensor $T^\mu{}_\nu$. This, of course, transforms as

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\rho}\, \Lambda_{\nu}{}^{\sigma}\, T^{\rho}{}_{\sigma} \tag{1.77}$$

under Lorentz transformations. If we form the contraction and define $\phi \equiv T^\mu{}_\mu$, then we see that under Lorentz transformations we shall have

$$\begin{aligned} \phi' &\equiv T'^{\mu}{}_{\mu} = \Lambda^{\mu}{}_{\rho}\, \Lambda_{\mu}{}^{\sigma}\, T^{\rho}{}_{\sigma}\,, \\ &= \delta^{\sigma}_{\rho}\, T^{\rho}{}_{\sigma} = \phi\,. \end{aligned} \tag{1.78}$$

Since $\phi' = \phi$, it follows, by definition, that ϕ is a scalar.

An essentially identical calculation shows that for a tensor with a arbitrary numbers of upstairs and downstairs indices, if one makes an index contraction of one upstairs with one downstairs index, the result is a tensor with the corresponding reduced numbers of indices. Of course multiple contractions work on the same way.

The Minkowski metric $\eta_{\mu\nu}$ is itself a tensor, but of a rather special type, known as an *invariant* tensor. This is because, unlike a generic 2-index tensor, the Minkowski metric is identical in all Lorentz frames. This can be seen from (1.56), which can be rewritten as the statement

$$\eta'_{\mu\nu} \equiv \Lambda_{\mu}{}^{\rho}\, \Lambda_{\nu}{}^{\sigma}\, \eta_{\rho\sigma} = \eta_{\mu\nu}\,. \tag{1.79}$$

The same is also true for the inverse metric $\eta^{\mu\nu}$.

We already saw that the gradient operator $\partial_\mu \equiv \partial/\partial X^\mu$ transforms as a covariant vector. If we define, in the standard way, $\partial^\mu \equiv \eta^{\mu\nu}\, \partial_\nu$, then it is evident from what we have seen above that the operator

$$\Box \equiv \partial^\mu \partial_\mu = \eta^{\mu\nu}\, \partial_\mu \partial_\nu \tag{1.80}$$

transforms as a scalar under Lorentz transformations. This is a very important operator, which is otherwise known as the wave operator, or d'Alembertian:

$$\Box = -\partial_0 \partial_0 + \partial_i \partial_i = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{1.81}$$

It is worth commenting further at this stage about a remark that was made earlier. Notice that in (1.81) we have been cavalier about the location of the Latin indices, which of course range only over the three spatial directions $i = 1$, 2 and 3. We can get away with this because the metric that is used to raise or lower the Latin indices is just the Minkowski metric restricted to the index values 1, 2 and 3. But since we have

$$\eta_{00} = -1\,, \qquad \eta_{ij} = \delta_{ij}\,, \qquad \eta_{0i} = \eta_{i0} = 0\,, \tag{1.82}$$

this means that Latin indices are lowered and raised using the Kronecker delta δ_{ij} and its inverse δ^{ij}. But these are just the components of the unit matrix, and so raising or lowering Latin indices has no effect. It is because of the minus sign associated with the η_{00} component of the Minkowski metric that we have to pay careful attention to the process of raising and lowering Greek indices. Thus, we can get away with writing $\partial_i \partial_i$, but we cannot write $\partial_\mu \partial_\mu$.

1.6 Proper time and 4-velocity

We defined the Lorentz-invariant interval ds between infinitesimally-separated spacetime events by

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{1.83}$$

This is the Minkowskian generalisation of the spatial interval in Euclidean space. Note that ds^2 can be positive, negative or zero. These cases correspond to what are called spacelike, timelike or null separations, respectively.

On occasion, it is useful to define the negative of ds^2, and write

$$d\tau^2 = -ds^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2\,. \tag{1.84}$$

This is called the *Proper Time* interval, and τ is the proper time. Since ds is a Lorentz scalar, it is obvious that $d\tau$ is a scalar too.

We know that dx^μ transforms as a contravariant 4-vector. Since $d\tau$ is a scalar, it follows that

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \tag{1.85}$$

is a contravariant 4-vector also. If we think of a particle following a path, or *worldline* in spacetime parameterised by the proper time τ, i.e. it follows the path $x^\mu = x^\mu(\tau)$, then U^μ defined in (1.85) is called the *4-velocity* of the particle.

It is useful to see how the 4-velocity is related to the usual notion of 3-velocity of a particle. By definition, the 3-velocity $\vec{u}$ is a 3-vector with components u^i given by

$$u^i = \frac{dx^i}{dt}\,. \tag{1.86}$$

From (1.84), it follows that

$$d\tau^2 = dt^2[1 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2)] = dt^2(1 - u^2)\,, \tag{1.87}$$

where $u = |\vec{u}|$, or in other words, $u = \sqrt{u^i u^i}$. In view of the definition of the γ factor in (1.25), it is natural to define

$$\gamma \equiv \frac{1}{\sqrt{1 - u^2}}\,. \tag{1.88}$$

Thus we have $d\tau = dt/\gamma$, and so from (1.85) the 4-velocity can be written as

$$U^\mu = \frac{dt}{d\tau}\,\frac{dx^\mu}{dt} = \gamma\,\frac{dx^\mu}{dt}\,. \tag{1.89}$$

Since $dx^0/dt = 1$ and $dx^i/dt = u^i$, we therefore have that

$$U^0 = \gamma\,, \qquad U^i = \gamma\, u^i\,. \tag{1.90}$$

Note that $U^\mu U_\mu = -1$, since, from (1.84), we have

$$U^\mu U_\mu = \eta_{\mu\nu} U^\mu U^\nu = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{(d\tau)^2} = \frac{-(d\tau)^2}{(d\tau)^2} = -1\,. \tag{1.91}$$

We shall sometimes find it convenient to rewrite (1.90) as

$$U^\mu = (\gamma, \gamma\, u^i) \qquad \text{or} \qquad U^\mu = (\gamma, \gamma\,\vec{u})\,. \tag{1.92}$$

Having set up the 4-vector formalism, it is now completely straightforward write down how velocities transform under Lorentz transformations. We know that the 4-velocity U^μ will transform according to (1.61), and this is identical to the way that the coordinates x^μ transform:

$$U'^\mu = \Lambda^\mu{}_\nu\, U^\nu\,, \qquad x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,. \tag{1.93}$$

Therefore, if we want to know how the 3-velocity transforms, we need only write down the Lorentz transformations for (t, x, y, z), and then replace (t, x, y, z) by (U^0, U^1, U^2, U^3). Finally, using (1.92) to express (U^0, U^1, U^2, U^3) in terms of $\vec{u}$ will give the result.

Consider, for simplicity, the case where S' is moving along the x axis with velocity v. The Lorentz transformation for U^μ can therefore be read off from (1.24) and (1.25):

$$\begin{aligned}
U'^0 &= \gamma_v\,(U^0 - v U^1)\,,\\
U'^1 &= \gamma_v\,(U^1 - v U^0)\,,\\
U'^2 &= U^2\,,\\
U'^3 &= U^3\,,
\end{aligned} \tag{1.94}$$

where we are now using $\gamma_v \equiv (1-v^2)^{-1/2}$ to denote the gamma factor of the Lorentz transformation, to distinguish it from the γ constructed from the 3-velocity $\vec{u}$ of the particle in the frame S, which is defined in (1.88). Thus from (1.92) we have

$$\begin{aligned}
\gamma' &= \gamma\,\gamma_v\,(1 - v u_x)\,,\\
\gamma'\,u'_x &= \gamma\,\gamma_v\,(u_x - v)\,,\\
\gamma'\,u'_y &= \gamma\,u_y\,,\\
\gamma'\,u'_z &= \gamma\,u_z\,,
\end{aligned} \tag{1.95}$$

where, of course, $\gamma' = (1-u'^2)^{-1/2}$ is the analogue of γ in the frame S'. Thus we find

$$u'_x = \frac{u_x - v}{1 - v u_x}\,,\qquad u'_y = \frac{u_y}{\gamma_v\,(1 - v u_x)}\,,\qquad u'_z = \frac{u_z}{\gamma_v\,(1 - v u_x)}\,. \tag{1.96}$$

2 Electrodynamics and Maxwell's Equations

2.1 Natural units

We saw earlier that the supposition of the universal validity of Maxwell's equations in all inertial frames, which in particular would imply that the speed of light should be the same in all frames, is consistent with experiment. It is therefore reasonable to expect that Maxwell's equations should be compatible with special relativity. However, written in their standard form (1.7), this compatibility is by no means apparent. Our next task will be to re-express the Maxwell equations, in terms of 4-tensors, in a way that makes their Lorentz covariance manifest.

We shall begin by changing units from the S.I. system in which the Maxwell equations are given in (1.7). The first step is to change to Gaussian units, by performing the rescalings

$$\begin{aligned}
&\vec{E} \longrightarrow \frac{1}{\sqrt{4\pi\epsilon_0}}\,\vec{E}\,, \qquad \vec{B} \longrightarrow \sqrt{\frac{\mu_0}{4\pi}}\,\vec{B}\,,\\
&\rho \longrightarrow \sqrt{4\pi\epsilon_0}\,\rho\,, \qquad \vec{J} \longrightarrow \sqrt{4\pi\epsilon_0}\,\vec{J}\,.
\end{aligned} \tag{2.1}$$

Bearing in mind that the speed of light is given by $c = 1/\sqrt{\mu_0\epsilon_0}$, we see that the Maxwell equations (1.7) become

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 4\pi\,\rho\,, \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\,\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\,\vec{J}\,,\\ \vec{\nabla}\cdot\vec{B} &= 0\,, \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\,\frac{\partial\vec{B}}{\partial t} = 0\,,\end{aligned} \tag{2.2}$$

Finally, we pass from Gaussian units to Natural units, by choosing our units of length and time so that $c = 1$, as we did in our discussion of special relativity. Thus, in natural units, the Maxwell equations become

$$\vec{\nabla}\cdot\vec{E} = 4\pi\,\rho\,, \qquad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 4\pi\,\vec{J}\,, \tag{2.3}$$

$$\vec{\nabla}\cdot\vec{B} = 0\,, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0\,, \tag{2.4}$$

The equations (2.3), which have sources on the right-hand side, are called the Field Equations. The equations (2.4) are called Bianchi Identities. We shall elaborate on this a little later.

2.2 Gauge potentials and gauge invariance

We already remarked that the two Maxwell equations (2.4) are know as Bianchi identities. They are not field equations, since there are no sources; rather, they impose constraints on the electric and magnetic fields. The first equation in (2.4), i.e. $\vec{\nabla}\cdot\vec{B} = 0$, can be solved by writing

$$\vec{B} = \vec{\nabla}\times\vec{A}\,, \tag{2.5}$$

where $\vec{A}$ is the magnetic 3-vector potential. Note that (2.5) *identically* solves $\vec{\nabla}\cdot\vec{B} = 0$, because of the vector identity that **div curl** $\equiv 0$. Substituting (2.5) into the second equation in (2.4), we obtain

$$\vec{\nabla}\times\Big(\vec{E} + \frac{\partial\vec{A}}{\partial t}\Big) = 0\,. \tag{2.6}$$

This can be solved, again *identically*, by writing

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi\,, \tag{2.7}$$

where ϕ is the electric scalar potential. Thus we can *solve* the Bianchi identities (2.4) by writing $\vec{E}$ and $\vec{B}$ in terms of scalar and 3-vector potentials ϕ and $\vec{A}$:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\,, \qquad \vec{B} = \vec{\nabla}\times\vec{A}\,. \tag{2.8}$$

Although we have now "disposed of" the two Maxwell equations in (2.4), it has been achieved at a price, in that there is a redundancy in the choice of gauge potentials ϕ and $\vec{A}$. First, we may note that that $\vec{B}$ in (2.8) is unchanged if we make the replacement

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\lambda \,, \tag{2.9}$$

where λ is an arbitrary function of position and time. The expression for $\vec{E}$ will also be invariant, if we simultaneously make the replacement

$$\phi \longrightarrow \phi - \frac{\partial\lambda}{\partial t} \,. \tag{2.10}$$

To summarise, if a given set of electric and magnetic fields $\vec{E}$ and $\vec{B}$ are described by a scalar potential ϕ and 3-vector potential $\vec{A}$ according to (2.8), then the identical physical situation (i.e. identical electric and magnetic fields) is equally well described by a new pair of scalar and 3-vector potentials, related to the original pair by the *Gauge Transformations* given in (2.9) and (2.10), where λ is an arbitrary function of position and time.

We can in fact use the gauge invariance to our advantage, by making a convenient and simplifying *gauge choice* for the scalar and 3-vector potentials. We have one arbitrary function (i.e. $\lambda(t,\vec{r})$) at our disposal, and so this allows us to impose one functional relation on the potentials ϕ and $\vec{A}$. For our present purposes, the most useful gauge choice is to use this freedom to impose the *Lorenz gauge condition*,[4]

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{\partial t} = 0 \,. \tag{2.11}$$

Substituting (2.8) into the remaining Maxwell equations (i.e. (2.3), and using the Lorenz gauge condition (2.11), we therefore find

$$\begin{aligned} \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \,, \\ \nabla^2\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} &= -4\pi\vec{J} \,. \end{aligned} \tag{2.12}$$

The important thing, which we shall make use of shortly, is that in each case we have on the left-hand side the d'Alembertian operator $\square = \partial^\mu\partial_\mu$, which we discussed earlier.

[4]Note that, contrary to the belief of many physicists, this gauge choice was introduced by the Danish physicist Ludvig Lorenz, and not the Dutch physicist Hendrik Lorentz who is responsible for the Lorentz transformation. Adding to the confusion is that unlike many other gauge choices that one encounters, the Lorenz gauge condition is, as we shall see later, Lorentz invariant.

2.3 Maxwell's equations in 4-tensor notation

The next step is to write the Maxwell equations in terms of four-dimensional quantities. Since the 3-vectors describing the electric and magnetic fields have three components each, there is clearly no way in which they can be "assembled" into 4-vectors. However, we may note that in four dimensional a two-index antisymmetric tensor has $(4 \times 3)/2 = 6$ independent components. Since this is equal to $3 + 3$, it suggests that perhaps we should be grouping the electric and magnetic fields together into a single 2-index antisymmetric tensor. This is in fact exactly what is needed. Thus we introduce a tensor $F_{\mu\nu}$, satisfying

$$F_{\mu\nu} = -F_{\nu\mu}\,. \tag{2.13}$$

It turns out that we should define its components in terms of $\vec{E}$ and $\vec{B}$ as follows:

$$F_{0i} = -E_i\,, \qquad F_{i0} = E_i\,, \qquad F_{ij} = \epsilon_{ijk}\,B_k\,. \tag{2.14}$$

Here ϵ_{ijk} is the usual totally-antisymmetric tensor of 3-dimensional vector calculus. It is equal to $+1$ if (ijk) is an even permutation of (123), to $= -1$ if it is an odd permutation, and to zero if it is no permautation (i.e. if two or more of the indices (ijk) are equal). In other words, we have

$$\begin{aligned} &F_{23} = B_1\,, \qquad F_{31} = B_2\,, \qquad F_{12} = B_3\,, \\ &F_{32} = -B_1\,, \qquad F_{13} = -B_2\,, \qquad F_{21} = -B_3\,. \end{aligned} \tag{2.15}$$

Viewing $F_{\mu\nu}$ as a matrix with rows labelled by μ and columns labelled by ν, we shall have

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}\,. \tag{2.16}$$

We also need to combine the charge density ρ and the 3-vector current density $\vec{J}$ into a four-dimensional quantity. This is easy; we just define a 4-vector J^μ, whose spatial components J^i are just the usual 3-vector current components, and whose time component J^0 is equal to the charge density ρ:

$$J^0 = \rho\,, \qquad J^i = J^i\,. \tag{2.17}$$

A word of caution is in order here. Although we have defined objects $F_{\mu\nu}$ and J^μ that have the appearance of a 4-tensor and a 4-vector, we are only entitled to call them such if we have verified that they transform in the proper way under Lorentz transformations. In fact they do, and we shall justify this a little later.

For now, we shall proceed to see how the Maxwell equations look when expressed in terms of $F_{\mu\nu}$ and J^μ. The answer is that they become

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -4\pi J^\nu\,, && (2.18)\\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= 0\,. && (2.19) \end{aligned}$$

Two very nice things have happened. First of all, the original four Maxwell equations (2.3) and (2.4) have become just two four-dimensional equations; (2.18) is the field equation, and (2.19) is the Bianchi identity. Secondly, the equations are manifestly Lorentz covariant; i.e. they transform tensorially under Lorentz transformations. This means that they keep exactly the same form in all Lorentz frames. If we start with (2.18) and (2.19) in the unprimed frame S, then we know that in the frame S', related to S by the Lorentz transformation (1.50), the equations will look identical, except that they will now have primes on all the quantities.

We should first verify that indeed (2.18) and (2.19) are equivalent to the Maxwell equations (2.3) and (2.4). Consider first (2.18). This equation is vector-valued, since it has the free index ν. Therefore, to reduce it down to three-dimensional equations, we have two cases to consider, namely $\nu = 0$ or $\nu = j$. For $\nu = 0$ we have

$$\partial_i F^{i0} = -4\pi J^0\,, \qquad (2.20)$$

which therefore corresponds (see (2.14) and (2.17)) to

$$-\partial_i E_i = -4\pi\rho\,, \qquad \text{i.e.} \qquad \vec\nabla\cdot\vec E = 4\pi\rho\,. \qquad (2.21)$$

For $\nu = j$, we shall have

$$\partial_0 F^{0j} + \partial_i F^{ij} = -4\pi J^j\,, \qquad (2.22)$$

which gives

$$\partial_0 E_j + \epsilon_{ijk}\partial_i B_k = -4\pi J^j\,. \qquad (2.23)$$

This is just[5]

$$-\frac{\partial\vec E}{\partial t} + \vec\nabla\times\vec B = 4\pi\vec J\,. \qquad (2.24)$$

Thus (2.18) is equivalent to the two Maxwell field equations in (2.3).

[5]Recall that the i'th component of $\vec\nabla\times\vec V$ is given by $(\vec\nabla\times\vec V)_i = \epsilon_{ijk}\partial_j V_k$ for any 3-vector $\vec V$.

Turning now to (2.19), it follows from the antisymmetry (2.13) of $F_{\mu\nu}$ that the left-hand side is totally antisymmetric in $(\mu\nu\rho)$ (i.e. it changes sign under any exchange of a pair of indices). Thefore there are two distinct assignments of indices, after we make the $1+3$ decomposition $\mu = (0, i)$ etc. Either one of the indices is a 0 with the other two Latin, or else all three are Latin. Consider first $(\mu, \nu, \rho) = (0, i, j)$:

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0\,, \tag{2.25}$$

which, from (2.14), means

$$\epsilon_{ijk}\,\frac{\partial B_k}{\partial t} + \partial_i E_j - \partial_j E_i = 0\,. \tag{2.26}$$

Since this is antisymmetric in ij there is no loss of generality involved in contracting with $\epsilon_{ij\ell}$, which gives[6]

$$2\frac{\partial B_\ell}{\partial t} + 2\epsilon_{ij\ell}\,\partial_i E_j = 0\,. \tag{2.27}$$

This is just the statement that

$$\vec\nabla \times \vec E + \frac{\partial \vec B}{\partial t} = 0\,, \tag{2.28}$$

which is the second of the Maxwell equations in (2.4).

The other distinct possibility for assigning decomposed indices in (2.19) is to take $(\mu, \nu, \rho) = (i, j, k)$, giving

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0\,. \tag{2.29}$$

Since this is totally antisymmetric in (i, j, k), no generality is lost by contracting it with ϵ_{ijk}, giving

$$3\epsilon_{ijk}\,\partial_i F_{jk} = 0\,. \tag{2.30}$$

From (2.14), this implies

$$3\epsilon_{ijk}\epsilon_{jk\ell}\partial_i B_\ell = 0\,, \qquad \text{and hence} \qquad 6\partial_i B_i = 0\,. \tag{2.31}$$

This has just reproduced the first Maxwell equation in (2.4), i.e. $\vec\nabla \cdot \vec B = 0$.

We have now demonstrated that the equations (2.18) and (2.19) are equivalent to the four Maxwell equations (2.3) and (2.4). Since (2.18) and (2.19) are written in a four-dimensional notation, it is highly suggestive that they are indeed Lorentz covariant. However, we should be a little more careful, in order to be sure about this point. Not every set of objects V^μ

[6]Recall that $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$, and hence $\epsilon_{ijm}\epsilon_{kjm} = 2\delta_{ik}$. These identities are easily proven by considering the possible assignments of indices and explicitly verifying that the two sides of the identities agree.

can be viewed as a Lorentz 4-vector, after all. The test is whether they transform properly, as in (1.61), under Lorentz transformations.

We may begin by considering the quantities $J^\mu = (\rho, J^i)$. Note first that by applying ∂_ν to the Maxwell field equation (2.18), we get identically zero on the left-hand side, since partial derivatives commute and $F^{\mu\nu}$ is antisymmetric. Thus from the left-hand side we get

$$\partial_\mu J^\mu = 0\,. \tag{2.32}$$

This is the equation of charge conservation. Decomposed into the $3+1$ language, it takes the familiar form

$$\frac{\partial\rho}{\partial t} + \vec\nabla\cdot\vec J = 0\,. \tag{2.33}$$

By integrating over a closed 3-volume V and using the divergence theorem on the second term, we learn that the rate of change of charge inside V is balanced by the flow of charge through its boundary S:

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_S \vec J\cdot d\vec S\,. \tag{2.34}$$

Now we are in a position to show that $J^\mu = (\rho, \vec J)$ is indeed a 4-vector. Considering $J^0 = \rho$ first, we may note that

$$dQ \equiv \rho dx dy dz \tag{2.35}$$

is clearly Lorentz invariant, since it is an electric charge. Clearly, for example, all Lorentz observers will agree on the number of electrons in a given closed spatial region, and so they will agree on the amount of charge. Another quantity that is Lorentz invariant is

$$dv = dt dx dy dz\,, \tag{2.36}$$

the volume of an infinitesinal region in spacetime. This can be seen from the fact that the Jacobian $\mathcal{J}$ of the transformation from dv to $dv' = dt'dx'dy'dz'$ is given by

$$\mathcal{J} = \det\Big(\frac{\partial x'^\mu}{\partial x^\nu}\Big) = \det(\Lambda^\mu{}_\nu)\,. \tag{2.37}$$

Now the defining property (1.56) of the Lorentz transformation can be written in a matrix notation as

$$\Lambda^T\,\eta\,\Lambda = \eta\,, \tag{2.38}$$

and hence taking the determinant, we get $(\det\Lambda)^2 = 1$ and hence

$$\det\Lambda = \pm 1\,. \tag{2.39}$$

Assuming that we restrict attention to Lorentz transformations without reflections, then they will be connected to the identity (we can take the boost velocity $\vec{v}$ to zero and/or the rotation angle to zero and continuously approach the identity transformation), and so $\det \Lambda = 1$. Thus it follows from (2.37) that for Lorentz transformations without reflections, the 4-volume element $dtdxdydz$ is Lorentz invariant.

Comparing $dQ = \rho dxdydz$ and $dv = dtdxdydz$, both of which we have argued are Lorentz invariant, we can conclude that ρ must transform in the same way as dt under Lorentz transformations. In other words, ρ must transform like the 0 component of a 4-vector. Thus writing, as we did, that $J^0 = \rho$, is justified.

In the same way, we may consider the spatial components J^i of the putative 4-vector J^μ. Considering J^1, for example, we know that $J^1 dydz$ is the current flowing through the area element $dydz$. Therefore in time dt, there will have been a flow of charge $J^1 dtdydz$. Being a charge, this must be Lorentz invariant, and so it follows from the known Lorentz invariance of $dv = dtdxdydz$ that J^1 must transform the same way as dx under Lorentz transformations. Thus J^1 does indeed transform like the 1 component of a 4-vector. Similar arguments apply to J^2 and J^3. (It is important in this argument that, because of the charge-conservation equation (2.32) or (2.34), the flow of charges we are discussing when considering the J^i components are the *same* charges we discussed when considering the J^0 component.)

We have now established that $J^\mu = (\rho, J^i)$ is indeed a Lorentz 4-vector, where ρ is the charge density and J^i the 3-vector current density.

At this point, we recall that by choosing the Lorenz gauge (2.11), we were able to reduce the Maxwell field equations (2.3) to (2.12). Furthermore, we can write these equations together as

$$\square A^\mu = -4\pi\, J^\mu\,, \tag{2.40}$$

where

$$A^\mu = (\phi, \vec{A})\,, \tag{2.41}$$

where the d'Alembertian, or wave operator, $\square = \partial^\mu \partial_\mu = \partial_i \partial_i - \partial_0^2$ was introduced in (1.81). We saw that it is manifestly a Lorentz *scalar* operator, since it is built from the contraction of indices on the two Lorentz-vector gradient operators. Since we have already established that J^μ is a 4-vector, it therefore follows that A^μ is a 4-vector. Note, *en passant*, that the Lorenz gauge condition (2.11) that we imposed earlier translates, in the four-dimensional language, into

$$\partial_\mu A^\mu = 0\,, \tag{2.42}$$

which is nicely Lorentz invariant.

The final step is to note that our definition (2.14) is precisely consistent with (2.41) and (2.8), if we write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{2.43}$$

First, we note from (2.41) that because of the $\eta_{00} = -1$ needed when lowering the 0 index, we shall have

$$A_\mu = (-\phi, \vec{A}) \,. \tag{2.44}$$

Therefore we find

$$\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = \frac{\partial A_i}{\partial t} + \partial_i \phi = -E_i \,, \\ F_{ij} &= \partial_i A_j - \partial_j A_i = \epsilon_{ijk} (\vec{\nabla} \times \vec{A})_k = \epsilon_{ijk} B_k \,. \end{aligned} \tag{2.45}$$

In summary, we have shown that J^μ is a 4-vector, and hence, using (2.40), that A^μ is a 4-vector. Then, it is manifest from (2.43) that $F_{\mu\nu}$ is a 4-tensor. Hence, we have established that the Maxwell equations, written in the form (2.18) and (2.19), are indeed expressed in terms of 4-tensors and 4-vectors, and so the manifest Lorentz covariance of the Maxwell equations is established.

Finally, it is worth remarking that in the 4-tensor description, the way in which the gauge invariance arises is very straightforward. First, it is manifest that the Bianchi identity (2.19) is solved identically by writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{2.46}$$

for some 4-vector A_μ. This is because (2.19) is totally antisymmetric in $\mu\nu\rho$, and so, when (2.46) is substituted into it, one gets identically zero since partial derivatives commute. (Try making the substitution and verify this explicitly. The vanishing because of the commutativity of partial derivatives is essentially the same as the reason why **curl grad** $\equiv 0$ and **div curl** $\equiv 0$.) It is also clear from (2.46) that $F_{\mu\nu}$ will be unchanged if we make the replacement

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda \,, \tag{2.47}$$

where λ is an arbitrary function of position and time. Again, the reason is that partial derivatives commute. Comparing (2.47) with (2.44), we see that (2.47) implies

$$\phi \longrightarrow \phi - \frac{\partial \lambda}{\partial t} \,, \qquad A_i \longrightarrow A_i + \partial_i \lambda \,, \tag{2.48}$$

and so we have reproduced the gauge transformations (2.9) and (2.10).

It should have become clear by now that all the familiar features of the Maxwell equations are equivalently described in the spacetime formulation in terms of 4-vectors and 4-tensors. The only difference is that everything is described much more simply and elegantly in the four-dimensional language.

2.4 Lorentz transformation of $\vec{E}$ and $\vec{B}$

Although for many purposes the four-dimensional decsription of the Maxwell equations is the most convenient, it is sometimes useful to revert to the original description in terms of $\vec{E}$ and $\vec{B}$. For example, we may easily derive the Lorentz transformation properties of $\vec{E}$ and $\vec{B}$, making use of the four-dimensional formulation. In terms of $F_{\mu\nu}$, there is no work needed to write down its behaviour under Lorentz transformations. Raising the indices for convenience, we shall have

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, F^{\rho\sigma}\,. \tag{2.49}$$

From this, and the fact (see (2.14) that $F^{0i} = E_i$, $F^{ij} = \epsilon_{ijk} B_k$, we can then immediately read of the Lorentz transformations for $\vec{E}$ and $\vec{B}$.

From the expressions (1.51) for the most general Lorentz boost transformation, we may first calculate $\vec{E}\,'$, calculated from

$$\begin{aligned}
E_i' &= F'^{0i} = \Lambda^0{}_\rho\, \Lambda^i{}_\sigma\, F^{\rho\sigma}\,,\\
&= \Lambda^0{}_0\, \Lambda^i{}_k\, F^{0k} + \Lambda^0{}_k\, \Lambda^i{}_0\, F^{k0} + \Lambda^0{}_k\, \Lambda^i{}_\ell\, F^{k\ell}\,,\\
&= \gamma\Big(\delta_{ik} + \frac{\gamma-1}{v^2}\, v_i v_k\Big) E_k - \gamma^2 v_i v_k E_k - \gamma\, v_k \Big(\delta_{i\ell} + \frac{\gamma-1}{v^2} v_i v_\ell\Big)\, \epsilon_{k\ell m}\, B_m\,,\\
&= \gamma E_i + \gamma \epsilon_{ijk}\, v_j B_k - \frac{\gamma-1}{v^2} v_i v_k E_k\,.
\end{aligned} \tag{2.50}$$

(Note that because $F^{\mu\nu}$ is antisymmetric, there is no F^{00} term on the right-hand side on the second line.) Thus, in terms of 3-vector notation, the Lorentz boost transformation of the electric field is given by

$$\vec{E}' = \gamma(\vec{E} + \vec{v}\times\vec{B}) - \frac{\gamma-1}{v^2}\,(\vec{v}\cdot\vec{E})\,\vec{v}\,. \tag{2.51}$$

An analogous calculation shows that the Lorentz boost transformation of the magnetic field is given by

$$\vec{B}' = \gamma(\vec{B} - \vec{v}\times\vec{E}) - \frac{\gamma-1}{v^2}\,(\vec{v}\cdot\vec{B})\,\vec{v}\,. \tag{2.52}$$

Suppose, for example, that in the frame S there is just a magnetic field $\vec{B}$, while $\vec{E} = 0$. An observer in a frame S' moving with uniform velocity $\vec{v}$ relative to S will therefore observe not only a magnetic field, given by

$$\vec{B}' = \gamma \vec{B} - \frac{\gamma - 1}{v^2} (\vec{v} \cdot \vec{B}) \, \vec{v} \,, \tag{2.53}$$

but also an electric field, given by

$$\vec{E}' = \gamma \vec{v} \times \vec{B} \,. \tag{2.54}$$

This, of course, is the principle of the dynamo.[7]

It is instructive to write out the Lorentz transformations explicitly in the case when the boost is along the x direction, $\vec{v} = (v, 0, 0)$. Equations (2.51) and (2.52) become

$$\begin{aligned} E'_x &= E_x \,, \qquad & E'_y &= \gamma(E_y - v B_z) \,, \qquad & E'_z &= \gamma(E_z + v B_y) \,, \\ B'_x &= B_x \,, \qquad & B'_y &= \gamma(B_y + v E_z) \,, \qquad & B'_z &= \gamma(B_z - v E_y) \,. \end{aligned} \tag{2.55}$$

2.5 The Lorentz force

Consider a point particle following the path, or worldline, $x^i = x^i(t)$. It has 3-velocity $u^i = dx^i/dt$, and, as we saw earlier, 4-velocity

$$U^\mu = (\gamma, \gamma \, \vec{u}) \,, \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - u^2}} \,. \tag{2.56}$$

Multiplying by the rest mass m of the particle gives another 4-vector, namely the 4-momentum

$$p^\mu = m U^\mu = (m\gamma, m\gamma \, \vec{u}) \,. \tag{2.57}$$

The quantity $p^0 = m\gamma$ is called the relativistic energy E, and $p^i = m\gamma \, u^i$ is called the relativistic 3-momentum. Note that since $U^\mu U_\mu = -1$, we shall have

$$p^\mu p_\mu = -m^2 \,. \tag{2.58}$$

We now define the relativistic 4-force f^μ acting on the particle to be

$$f^\mu = \frac{dp^\mu}{d\tau} \,, \tag{2.59}$$

where τ is the proper time. Clearly f^μ is indeed a 4-vector, since it is the 4-vector dp^μ divided by the scalar $d\tau$.

Using (2.57), we can write the 4-force as

$$f^\mu = \Big(m\gamma^3 \vec{u} \cdot \frac{d\vec{u}}{d\tau}, m\gamma^3 \vec{u} \cdot \frac{d\vec{u}}{d\tau} \, \vec{u} + m\gamma \frac{d\vec{u}}{d\tau} \Big) \,. \tag{2.60}$$

[7] In a practical dynamo the rotor is moving with a velocity $\vec{v}$ which is much less than the speed of light, i.e. $|\vec{v}| << 1$ in natural units. This means that the gamma factor $\gamma = (1 - v^2)^{-1/2}$ is approximately equal to unity in such cases.

It follows that if we move to the instantaneous rest frame of the particle, i.e. the frame in which $\vec{u} = 0$ at the particular moment we are considering, then f^μ reduces to

$$f^\mu\Big|_{\text{rest frame}} = (0, \vec{F})\,, \tag{2.61}$$

where

$$\vec{F} = m\frac{d\vec{u}}{dt} \tag{2.62}$$

is the Newtonian force measured in the rest frame of the particle.[8] Thus, we should interpret the 4-force physically as describing the Newtonian 3-force when measured in the instantaneous rest frame of the accelerating particle.

If we now suppose that the particle has electric charge e, and that it is moving under the influence of an electromagnetic field $F_{\mu\nu}$, then its motion is given by the *Lorentz force equation*

$$f^\mu = eF^{\mu\nu}\, U_\nu\,. \tag{2.63}$$

One can more or less justify this equation on the grounds of "what else could it be?", since we know that there must exist a relativistic equation (i.e. a Lorentz covariant equation) that describes the motion. In fact it is easy to see that (2.63) is correct. We calculate the spatial components:

$$\begin{aligned} f^i &= eF^{i\nu}\, U_\nu = eF^{i0}\, U_0 + eF^{ij}\, U_j\,, \\ &= e(-E_i)(-\gamma) + e\epsilon_{ijk}\, B_k \gamma\, u_j\,, \end{aligned} \tag{2.64}$$

and thus

$$\vec{f} = e\gamma\,(\vec{E} + \vec{u}\times\vec{B})\,. \tag{2.65}$$

But $f^\mu = dp^\mu/d\tau$, and so $\vec{f} = d\vec{p}/d\tau = \gamma\, d\vec{p}/dt$ (recall from section 1.6 that $d\tau = dt/\gamma$) and so we have

$$\frac{d\vec{p}}{dt} = e\,(\vec{E} + \vec{u}\times\vec{B})\,, \tag{2.66}$$

where $d\vec{p}/dt$ is the rate of change of relativistic 3-momentum. This is indeed the standard expression for the motion of a charged particle under the Lorentz force.

2.6 Action principle for charged particles

In this section, we shall show how the equations of motion for a charged particle moving in an electromagnetic field can be derived from an action principle. To begin, we shall

[8]Note that we can replace the proper time τ by the coordinate time t in the instantaneous rest frame, since the two are the same.

consider an uncharged particle of mass m, with no forces acting on it. It will, of course, move in a straight line. It turns out that its equation of motion can be derived from the Lorentz-invariant action

$$S = -m \int_{\tau_1}^{\tau_2} d\tau\,, \tag{2.67}$$

where τ is the proper time along the trajectory $x^\mu(\tau)$ of the particle, starting at proper time $\tau = \tau_1$ and ending at $\tau = \tau_2$. The action principle then states that if we consider all possible paths between the initial and final spacetime points on the path, then the actual path followed by the particle will be such that the action S is stationary. In other words, if we consider small variations of the path around the actual path, then to first order in the variations we shall have $\delta S = 0$.

To see how this works, we note that $d\tau^2 = dt^2 - dx^i dx^i = dt^2(1 - v_i v_i) = dt^2(1 - v^2)$, where $v_i = dx^i/dt$ is the 3-velocity of the particle. Thus

$$S = -m \int_{t_1}^{t_2} (1 - v^2)^{1/2}\, dt = -m \int_{t_1}^{t_2} (1 - \dot{x}^i \dot{x}^i)^{1/2}\, dt\,. \tag{2.68}$$

In other words, the Lagrangian L, for which $S = \int_{t_1}^{t_2} L dt$, is given by

$$L = -m(1 - \dot{x}^i \dot{x}^i)^{1/2}\,. \tag{2.69}$$

As a check, if we expand (2.69) for small velocities (i.e. small compared with the speed of light, so $|\dot{x}^i| << 1$), we shall have

$$L = -m + \tfrac{1}{2} m v^2 + \cdots\,. \tag{2.70}$$

Since the Lagrangian is given by $L = T - V$ we see that T is just the usual kinetic energy $\frac{1}{2} m v^2$ for a non-relativistic particle of mass m, while the potential energy is just m. Of course if we were not using units where the speed of light were unity, this energy would be mc^2. Since it is just a constant, it does not affect the equations of motion that will follow from the action principle.

Now let us consider small variations $\delta x^i(t)$ around the path $x^i(t)$ followed by the particle. The action will vary according to

$$\delta S = m \int_{t_1}^{t_2} (1 - \dot{x}^j \dot{x}^j)^{-1/2}\, \dot{x}^i \delta \dot{x}^i dt\,. \tag{2.71}$$

Integrating by parts then gives

$$\delta S = -m \int_{t_1}^{t_2} \frac{d}{dt}\Big((1 - \dot{x}^j \dot{x}^j)^{-1/2}\, \dot{x}^i\Big) \delta x^i dt + m\Big[(1 - \dot{x}^j \dot{x}^j)^{-1/2}\, \dot{x}^i \delta x^i\Big]_{t_1}^{t_2}\,. \tag{2.72}$$

As usual in an action principle, we restrict to variations of the path that vanish at the endpoints, so $\delta x^i(t_1) = \delta x^i(t_2) = 0$ and the boundary term can be dropped. The variation δx^i is allowed to be otherwise arbitrary in the time interval $t_1 < t < t_2$, and so we conclude from the requirement of stationary action $\delta S = 0$ that

$$\frac{d}{dt}\Big((1-\dot{x}^j\dot{x}^j)^{-1/2}\,\dot{x}^i\Big) = 0\,. \tag{2.73}$$

Now, recalling that we define $\gamma = (1-v^2)^{-1/2}$, we see that

$$\frac{d(m\gamma\vec{v})}{dt} = 0\,, \tag{2.74}$$

or, in other words,

$$\frac{d\vec{p}}{dt} = 0\,, \tag{2.75}$$

where $\vec{p} = m\gamma\vec{v}$ is the relativistic 3-momentum. We have, of course, derived the equation for straight-line motion in the absence of any forces acting.

Now we extend the discussion to the case of a particle of mass m and charge e, moving under the influence of an electromagnetic field $F_{\mu\nu}$. This field will be written in terms of a 4-vector potential:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{2.76}$$

The action will now be the sum of the free-particle action (2.68) above plus a term describing the interaction of the particle with the electromagnetic field. The total action turns out to be

$$S = \int_{\tau_1}^{\tau_2} (-m d\tau + eA_\mu dx^\mu)\,. \tag{2.77}$$

Note that it is again Lorentz invariant.

From (2.44) we have $A_\mu = (-\phi, \vec{A})$, and so

$$A_\mu dx^\mu = A_\mu \frac{dx^\mu}{dt}\,dt = (A_0 + A_i\dot{x}^i)dt = (-\phi + A_i\dot{x}^i)dt\,. \tag{2.78}$$

Thus we have $S = \int_{t_1}^{t_2} L dt$ with the Lagrangian L given by

$$L = -m(1-\dot{x}^j\dot{x}^j)^{1/2} - e\phi + eA_i\dot{x}^i\,, \tag{2.79}$$

where potentials ϕ and A_i depend on t and x. The first-order variation of the action under a variation δx^i in the path gives

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2}\Big[m(1-\dot{x}^j\dot{x}^j)^{-1/2}\,\dot{x}^i\delta\dot{x}^i - e\partial_i\phi\,\delta x^i + eA_i\delta\dot{x}^i + e\partial_j A_i\dot{x}^i\delta x^j\Big]dt\,,\\ &= \int_{t_1}^{t_2}\Big[-\frac{d}{dt}(m\gamma\dot{x}^i) - e\partial_i\phi - e\frac{dA_i}{dt} + e\partial_i A_j\,\dot{x}^j\Big]\delta x^i dt\,. \end{aligned} \tag{2.80}$$

(We have dropped the boundary terms immediately, since δx^i is again assumed to vanish at the endpoints.) Thus the principle of stationary action $\delta S = 0$ implies

$$\frac{d(m\gamma \dot{x}^i)}{dt} = -e\partial_i \phi - \frac{dA_i}{dt} + e\partial_i A_j\, \dot{x}^j\,. \tag{2.81}$$

Now, the total time derivative dA_i/dt has two contributions, and we may write it as

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \partial_j A_i\, \frac{dx^j}{dt} = \frac{\partial A_i}{\partial t} + \partial_j A_i\, \dot{x}^j\,. \tag{2.82}$$

This arises because first of all, A_i can depend explicitly on the time coordinate; this contribution is $\partial A_i/\partial t$. Additionally, A_i depends on the spatial coordinates x^i, and along the path followed by the particle, x^i depends on t because the path is $x^i = x^i(t)$. This accounts for the second term.

Putting all this together, we have

$$\begin{aligned} \frac{d(m\gamma \dot{x}^i)}{dt} &= e\Big(-\partial_i\phi - \frac{\partial A_i}{\partial t}\Big) + e(\partial_i A_j - \partial_j A_i)\, \dot{x}^j\,, \\ &= e(E_i + \epsilon_{ijk}\, \dot{x}^j B_k)\,. \end{aligned} \tag{2.83}$$

In other words, we have

$$\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v}\times\vec{B})\,, \tag{2.84}$$

which is the Lorentz force equation (2.66).

It is worth noting that although we gave a "three-dimensional" derivation of the equations of motion following from the action (2.77), we can also instead directly derive the four-dimensional equation $dp^\mu/d\tau = eF^{\mu\nu}U_\nu$. To begin, we write the proper time interval as $d\tau = (-\eta_{\rho\sigma}dx^\rho dx^\sigma)^{1/2}$, and so its variation under a variation of the path $x^\mu(\tau)$ gives

$$\begin{aligned} \delta(d\tau) &= -(-\eta_{\rho\sigma}dx^\rho dx^\sigma)^{-1/2}\, \eta_{\mu\nu}\, dx^\mu d\delta x^\nu\,, \\ &= -\eta_{\mu\nu}\, \frac{dx^\mu}{d\tau}\, d\delta x^\nu\,, \\ &= -U_\mu\, d\delta x^\mu\,, \end{aligned} \tag{2.85}$$

where U_μ is the 4-velocity. Thus the variation of the action (2.77) gives

$$\begin{aligned} \delta S &= \int_{\tau_1}^{\tau_2} (mU_\mu d\delta x^\mu + eA_\mu\, d\delta x^\mu + e\partial_\nu A_\mu\, \delta x^\nu dx^\mu)\,, \\ &= \int_{\tau_1}^{\tau_2} (-mdU_\mu\, \delta x^\mu - edA_\mu\, \delta x^\mu + e\partial_\mu A_\nu\, \delta x^\mu dx^\nu)\,, \\ &= \int_{\tau_1}^{\tau_2} (-m\frac{dU_\mu}{d\tau} - e\frac{dA_\mu}{d\tau} + e\partial_\mu A_\nu\, \frac{dx^\nu}{d\tau})\delta x^\mu d\tau\,. \end{aligned} \tag{2.86}$$

Now we have

$$\frac{dA_\mu}{d\tau} = \partial_\nu A_\mu \frac{dx^\nu}{d\tau} = \partial_\nu A_\mu\, U^\nu\,, \tag{2.87}$$

and so

$$\delta S = \int_{\tau_1}^{\tau_2} \Big(- m\frac{dU_\mu}{d\tau} - e\partial_\nu A_\mu\, U^\nu + e\partial_\mu A_\nu\, U^\nu \Big)\delta x^\mu d\tau\,. \tag{2.88}$$

Requiring $\delta S = 0$ for all variations (that vanish at the endpoints) we therefore obtain the equation of motion

$$\begin{aligned} m\frac{dU_\mu}{d\tau} &= e(\partial_\mu A_\nu - \partial_\nu A_\mu)\, U^\nu\,, \\ &= eF_{\mu\nu}\, U^\nu\,. \end{aligned} \tag{2.89}$$

Thus we have reproduced the Lorentz force equation (2.66).

2.7 Gauge invariance of the action

In writing down the relativistic action (2.77) for a charged particle we had to make use of the 4-vector potential A_μ. This is itself not physically observable, since, as we noted earlier, A_μ and $A'_\mu = A_\mu + \partial\lambda$ describe the same physics, where λ is any arbitrary function in spacetime, since A_μ and A'_μ give rise to the same electromagnetic field $F_{\mu\nu}$. One might worry, therefore, that the action itself would be gauge dependent, and therefore might not properly describe the required physical situation. However, all is in fact well. This already can be seen from the fact that, as we demonstrated, the variational principle for the action (2.77) does in fact produce the correct gauge-invariant Lorentz force equation (2.66).

It is instructive also to examine the effects of a gauge transformation directly at the level of the action. If we make the gaug transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu\lambda$, we see from (2.77) that the action S transforms to S' given by

$$\begin{aligned} S' &= \int_{\tau_1}^{\tau_2} (-m d\tau + eA_\mu dx^\mu + e\partial_\mu\lambda dx^\mu)\,, \\ &= S + e\int_{\tau_1}^{\tau_2} \partial_\mu\lambda dx^\mu = e\int_{\tau_1}^{\tau_2} d\lambda\,, \end{aligned} \tag{2.90}$$

and so

$$S' = S + e[\lambda(\tau_2) - \lambda(\tau_1)]\,. \tag{2.91}$$

Thus provided we restrict ourselves to gauge transformations that vanish at the endpoints, the action will be gauge invariant, $S' = S$.

2.8 Canonical momentum, and Hamiltonian

Given any Lagrangian $L(x^i, \dot x^i, t)$ one defines the canonical momentum π_i as

$$\pi_i = \frac{\partial L}{\partial \dot x^i}\,. \tag{2.92}$$

The relativistic Lagrangian for the charged particle is given by (2.79), and so we have

$$\pi_i = m(1 - \dot{x}^j \dot{x}^j)^{-1/2}\, \dot{x}^i + eA_i\,, \tag{2.93}$$

or, in other words,

$$\begin{aligned} \pi_i &= m\gamma\, \dot{x}^i + eA_i\,, && (2.94) \\ &= p_i + eA_i\,, && (2.95) \end{aligned}$$

where p_i as usual is the standard mechanical relativistic 3-momentum of the particle.

As usual, the Hamiltonian for the system is given by

$$H = \pi_i\, \dot{x}^i - L\,, \tag{2.96}$$

and so we find

$$H = m\gamma \dot{x}^i \dot{x}^i + \frac{m}{\gamma} + e\phi\,. \tag{2.97}$$

Now, $\dot{x}^i = v_i$ and $m\gamma v^2 + m/\gamma = m\gamma(v^2 + (1 - v^2)) = m\gamma$, so we have

$$H = m\gamma + e\phi\,. \tag{2.98}$$

The Hamiltonian is to be viewed as a function of the coordinates x^i and the canonical momenta π_i. To express γ in terms of π_i, we note from (2.94) that $m\gamma\dot{x}^i = \pi_i - eA_i$, and so squaring, we get $m^2\gamma^2 v^2 = m^2 v^2/(1 - v^2) = (\pi_i - eA_i)^2$. Solving for v^2, and hence for γ, we find that $m^2\gamma^2 = (\pi_i - eA_i)^2 + m^2$, and so finally, from (2.98), we arrive at the Hamiltonian

$$H = \sqrt{(\pi_i - eA_i)^2 + m^2} + e\phi\,. \tag{2.99}$$

Note that Hamilton's equations, which will necessarily give rise to the same Lorentz force equations of motion we encountered previously, are given by

$$\frac{\partial H}{\partial \pi_i} = \dot{x}^i\,, \qquad \frac{\partial H}{\partial x^i} = -\dot{\pi}_i\,. \tag{2.100}$$

As a check of the correctness of the Hamiltonian (2.99) we may examine it in the non-relativistic limit when $(\pi_i - eA_i)^2$ is much less than m^2. We then extract an m^2 factor from inside the square root in $\sqrt{(\pi_i - eA_i)^2 + m^2}$ and expand to get

$$\begin{aligned} H &= m\sqrt{1 + (\pi_i - eA_i)^2/m^2} + e\phi\,, \\ &= m + \frac{1}{2m}\,(\pi_i - eA_i)^2 + e\phi + \cdots\,. && (2.101) \end{aligned}$$

The first term is the rest-mass energy, which is just a constant, and the remaining terms presented explicitly in (2.101) give the standard non-relativistic Hamiltonian for a charged particle

$$H_{\text{non-rel.}} = \frac{1}{2m}(\pi_i - eA_i)^2 + e\phi\,. \tag{2.102}$$

This should be familiar from quantum mechanics, when one writes down the Schrödinger equation for the wave function for a charged particle in an electromagnetic field.

3 Particle Motion in Static Electromagnetic Fields

In this chapter, we discuss the motion of a charged particle in static (i.e. time-independent) electromagnetic fields.

3.1 Description in terms of potentials

If we are describing static electric and magnetic fields, $\vec{E} = \vec{E}(\vec{r})$ and $\vec{B} = \vec{B}(\vec{r})$, it is natural (and always possible) to describe them in terms of scalar and 3-vector potentials that are also static, $\phi = \phi(\vec{r})$, $\vec{A} = \vec{A}(\vec{r})$. Thus we write

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi(\vec{r})\,, \\ \vec{B} &= \vec{\nabla}\times\vec{A}(\vec{r})\,. \end{aligned} \tag{3.1}$$

We can still perform gauge transformations, as given in (2.9) and (2.10). The most general gauge transformation that preserves the time-independence of the potentials is therefore given by taking the parameter λ to be of the form

$$\lambda(\vec{r},t) = \lambda(\vec{r}) + k\,t\,, \tag{3.2}$$

where k is an arbitrary constant. This implies that ϕ and $\vec{A}$ will transform according to

$$\phi \longrightarrow \phi - k\,, \qquad \vec{A} \longrightarrow \vec{A} + \vec{\nabla}\lambda(\vec{r})\,. \tag{3.3}$$

Note, in particular, that the electrostatic potential ϕ can just be shifted by an arbitrary constant. This is the familiar freedom that one typically uses to set $\phi = 0$ at infinity.

Recall that the Hamiltonian for a particle of mass m and charge e in an electromagnetic field is given by (2.98)

$$H = m\gamma + e\phi\,, \tag{3.4}$$

where $\gamma = (1-v^2)^{-1/2}$. In the present situation with static fields, the Hamiltonian does not depend explicitly on time, i.e. $\partial H/\partial t = 0$. In this circumstance, it follows that the energy $\mathcal{E}$ is conserved, and is given simply by H:

$$\mathcal{E} = H = m\gamma + e\phi\,. \tag{3.5}$$

The time-independence of $\mathcal{E}$ can be seen from Hamilton's equations (2.100):

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x^i}\,\dot{x}^i + \frac{\partial H}{\partial \pi_i}\,\dot{\pi}_i\,, \\ &= 0 - \dot{\pi}_i\,\dot{x}^i + \dot{x}^i\,\dot{\pi}_i = 0\,. \end{aligned} \tag{3.6}$$

We may think of the first term in $\mathcal{E}$ as being the *mechanical* term,

$$\mathcal{E}_{\rm mech} = m\gamma\,, \tag{3.7}$$

since this is just the total energy of a particle of rest mass m moving with velocity $\vec{v}$. The second term, $e\phi$, is the contribution to the total energy from the electric field. Note that the magnetic field, described by the 3-vector potential $\vec{A}$, does not contribute to the conserved energy. This is because the magnetic field $\vec{B}$ does no work on the charge:

Recall that the Lorentz force equation can be written as

$$\frac{d(m\gamma v^i)}{dt} = e(E_i + \epsilon_{ijk}\,v^j B_k)\,. \tag{3.8}$$

Multiplying by v^i we therefore have

$$m\gamma v^i \frac{dv^i}{dt} + m v^i v^i \frac{d\gamma}{dt} = e v^i E_i\,. \tag{3.9}$$

Now $\gamma = (1-v^2)^{-1/2}$, so

$$\frac{d\gamma}{dt} = (1-v^2)^{-3/2} v^i \frac{dv^i}{dt} = \gamma^3 v^i \frac{dv^i}{dt}\,, \tag{3.10}$$

and so (3.9) gives

$$m\frac{d\gamma}{dt} = e v^i E_i\,. \tag{3.11}$$

Since $\mathcal{E}_{\rm mech} = m\gamma$, and m is a constant, we therefore have

$$\frac{d\mathcal{E}_{\rm mech}}{dt} = e\vec{v}\cdot\vec{E}\,. \tag{3.12}$$

Thus, the mechanical energy of the particle is changed only by the electric field, and not by the magnetic field.

Note that another derivation of the constancy of $\mathcal{E} = m\gamma + e\phi$ is as follows:

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \frac{d(m\gamma)}{dt} + e\frac{d\phi}{dt} \\ &= \frac{d\mathcal{E}_{\rm mech}}{dt} + e\partial_i\phi\,\frac{dx^i}{dt}\,, \\ &= e\vec{v}\cdot\vec{E} - e\vec{v}\cdot\vec{E} = 0\,. \end{aligned} \tag{3.13}$$

3.2 Particle motion in static uniform $\vec{E}$ and $\vec{B}$ fields

Let us consider the case where a charged particle is moving in static (i.e. time-independent) uniform $\vec{E}$ and $\vec{B}$ fields. In other words, $\vec{E}$ and $\vec{B}$ are *constant* vectors, independent of time and of position. In this situation, it is easy to write down explicit expressions for the corresponding scalar and 3-vector potentials. For the scalar potential, we can take

$$\phi = -\vec{E}\cdot\vec{r} = -E_i x^i\,. \tag{3.14}$$

Clearly this gives the correct electric field, since

$$-\partial_i\phi = \partial_i(E_j x^j) = E_j\partial_i x^j = E_j\delta_{ij} = E_i\,. \tag{3.15}$$

(It is, of course, essential that E_j is *constant* for this calculation to be valid.)

Turning now to the uniform $\vec{B}$ field, it is easily seen that this can be written as $\vec{B} = \vec{\nabla}\times\vec{A}$, with the 3-vector potential given by

$$\vec{A} = \tfrac{1}{2}\vec{B}\times\vec{r}\,. \tag{3.16}$$

It is easiest to check this using index notation. We have

$$\begin{aligned}
(\vec{\nabla}\times\vec{A})_i &= \epsilon_{ijk}\,\partial_j A_k = \epsilon_{ijk}\partial_j(\tfrac{1}{2}\epsilon_{k\ell m}B_\ell x^m)\,,\\
&= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{\ell mk}\,B_\ell\partial_j x^m = \tfrac{1}{2}\epsilon_{ijk}\epsilon_{\ell jk}\,B_\ell\,,\\
&= \delta_{i\ell}\,B_\ell = B_i\,.
\end{aligned} \tag{3.17}$$

Of course the potentials we have written above are not unique, since we can still perform gauge transformations. If we restrict attention to transformations that maintain the time-independence of ϕ and $\vec{A}$, then for ϕ the only remaining freedom is to add an arbitrary constant to ϕ. For the 3-vector potential, we can still add $\vec{\nabla}\lambda(\vec{r})$ to $\vec{A}$, where $\lambda(\vec{r})$ is an arbitrary function of position. It is sometimes helpful, for calculational reasons, to do this. Suppose, for example, that the uniform $\vec{B}$ field lies along the z axis: $\vec{B} = (0,0,B)$. From (3.16), we may therefore write the 3-vector potential

$$\vec{A} = (-\tfrac{1}{2}By, \tfrac{1}{2}Bx, 0)\,. \tag{3.18}$$

Another choice is to take $\vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{r})$, with $\lambda = -\frac{1}{2}Bxy$. This gives

$$\vec{A}' = (-By, 0, 0)\,. \tag{3.19}$$

One easily verifies that indeed $\vec{\nabla}\times\vec{A}' = (0,0,B)$.

3.2.1 Motion in a static uniform electric field

From the Lorentz force equation, we shall have

$$\frac{d\vec{p}}{dt} = e\vec{E}\,, \tag{3.20}$$

where $\vec{p} = m\gamma\vec{v}$ is the relativistic 3-momentum. Without loss of generality, we may take the electric field to lie along the x axis, and so we will have

$$\frac{dp_x}{dt} = eE\,, \qquad \frac{dp_y}{dt} = 0\,, \qquad \frac{dp_z}{dt} = 0\,. \tag{3.21}$$

Since there is a rotational symmetry in the (y, z) plane, we can, without loss of generality, choose to take $p_z = 0$, since the motion in the (yz) plane is evidently, from (3.21), simply linear. Thus we may take the solution to (3.21) to be

$$p_x = eEt\,, \qquad p_y = \bar{p}\,, \qquad p_z = 0\,, \tag{3.22}$$

where $\bar{p}$ is a constant. We have also chosen the origin for the time coordinate t such that $p_x = 0$ at $t = 0$.

Recalling that the 4-momentum is given by $p^\mu = (m\gamma, \vec{p}) = (\mathcal{E}_{\rm mech}, \vec{p})$, and that $p^\mu p_\mu = m^2 U^\mu U_\mu = -m^2$, we see that

$$\mathcal{E}_{\rm mech} = \sqrt{m^2 + p_x^2 + p_y^2} = \sqrt{m^2 + \bar{p}^2 + (eEt)^2}\,, \tag{3.23}$$

and hence we may write

$$\mathcal{E}_{\rm mech} = \sqrt{\mathcal{E}_0^2 + (eEt)^2}\,, \tag{3.24}$$

where $\mathcal{E}_0^2 = m^2 + \bar{p}^2$ is the square of the mechanical energy at time $t = 0$.

We have $\vec{p} = m\gamma\,\vec{v} = \mathcal{E}_{\rm mech}\,\vec{v}$, and so

$$\frac{dx}{dt} = \frac{p_x}{\mathcal{E}_{\rm mech}} = \frac{eEt}{\sqrt{\mathcal{E}_0^2 + (eEt)^2}}\,, \tag{3.25}$$

which can be integrated to give

$$x = \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + (eEt)^2}\,. \tag{3.26}$$

(The constant of integration has been absorbed into a choice of origin for the x coordinate.) Note from (3.25) that the x-component of the 3-velocity asymptotically approaches 1 as t goes to infinity. Thus the particle is accelerated closer and closer to the speed of light, but never reaches it.

We also have

$$\frac{dy}{dt} = \frac{p_y}{\mathcal{E}_{\rm mech}} = \frac{\bar p}{\sqrt{\mathcal{E}_0^2 + (eEt)^2}}\,. \tag{3.27}$$

This can be integrated by changing variable from t to u, defined by

$$eEt = \mathcal{E}_0\, \sinh u\,. \tag{3.28}$$

This gives $y = \bar p\, u/(eE)$, and hence

$$y = \frac{\bar p}{eE}\,{\rm arcsinh}\Big(\frac{eEt}{\mathcal{E}_0}\Big)\,. \tag{3.29}$$

(Again, the constant of integration has been absorbed into the choice of origin for y.)

The solutions (3.26) and (3.29) for x and y as functions of t can be combined to give x as a function of y, leading to

$$x = \frac{\mathcal{E}_0}{eE}\, \cosh\Big(\frac{eEy}{\bar p}\Big)\,. \tag{3.30}$$

This is a catenary.

In the non-relativistic limit when $|v| << 1$, we have $\bar p \approx m\bar v$ and then, expanding (3.30) we find the standard "Newtonian" parabolic motion

$$x \approx \hbox{constant} + \frac{eE}{2m\bar v^2}\, y^2\,. \tag{3.31}$$

3.2.2 Motion in a static uniform magnetic field

From the Lorentz force equation we shall have

$$\frac{d\vec p}{dt} = e\vec v\times\vec B\,. \tag{3.32}$$

Recalling (3.11), we see that in the absence of an electric field we shall have $\gamma = $ constant, and hence $d\vec p/dt = d(m\gamma\vec v)/dt = m\gamma\, d\vec v/dt$, leading to

$$\frac{d\vec v}{dt} = \frac{e}{m\gamma}\,\vec v\times\vec B = \frac{e}{\mathcal{E}}\,\vec v\times\vec B\,, \tag{3.33}$$

since $\mathcal{E} = m\gamma + e\phi = m\gamma$ (a constant) here.

Without loss of generality we may choose the uniform $\vec B$ field to lie along the z axis: $\vec B = (0,0,B)$. Defining

$$\omega \equiv \frac{eB}{\mathcal{E}} = \frac{eB}{m\gamma}\,, \tag{3.34}$$

we then find

$$\frac{dv_x}{dt} = \omega\, v_y\,,\qquad \frac{dv_y}{dt} = -\omega\, v_x\,,\qquad \frac{dv_z}{dt} = 0\,. \tag{3.35}$$

From this, it follows that

$$\frac{d(v_x + \mathrm{i}\, v_y)}{dt} = -\mathrm{i}\, \omega\, (v_x + \mathrm{i}\, v_y)\,, \tag{3.36}$$

and so the first two equations in (3.35) can be integrated to give

$$v_x + \mathrm{i}\, v_y = v_0\, e^{-\mathrm{i}\,(\omega t + \alpha)}\,, \tag{3.37}$$

where v_0 is a real constant, and α is a constant (real) phase. Thus after further integrations we obtain

$$x = x_0 + r_0 \sin(\omega t + \alpha)\,, \qquad y = y_0 + r_0 \cos(\omega t + \alpha)\,, \qquad z = z_0 + \bar{v}_z t\,, \tag{3.38}$$

for constants r_0, x_0, y_0, z_0 and $\bar{v}_z$, with

$$r_0 = \frac{v_0}{\omega} = \frac{m\gamma v_0}{eB} = \frac{\bar{p}}{eB}\,, \tag{3.39}$$

where $\bar{p}$ is the relativistic 3-momentum in the (x, y) plane. The particle therefore follows a helical path, of radius r_0.

3.2.3 Adiabatic invariant

In any conservative system with a periodic motion, it can be shown that the quantity

$$I \equiv \oint \pi_i dx^i\,, \tag{3.40}$$

integrated over a complete cycle of the coordinates x^i is conserved under slow (adiabatic) changes of the external parameters. Specifically, if there is an external parameter a, then dI/dt is of order $\mathcal{O}(\dot{a}^2, \ddot{a})$, but there is no *linear* dependence on the first derivative $\dot{a}$.

In our previous discussion, of a charged particle moving under the influence of a uniform magnetic field $\vec{B}$ that lies along the z direction, we may consider the invariant I that one obtains by integrating around its closed path in the (x, y) plane. We shall have

$$I \equiv \oint \pi_i dx^i = \oint (p_i + eA_i) dx^i\,, \tag{3.41}$$

and

$$\oint p_i dx^i = 2\pi r_0 \bar{p} = 2\pi r_0^2 eB\,. \tag{3.42}$$

We shall also have

$$e \oint A_i dx^i = e \int_S \vec{B} \cdot d\vec{S} = -e\pi r_0^2 B\,. \tag{3.43}$$

Hence we find

$$I = 2\pi r_0^2 eB - \pi r_0^2 eB\,, \tag{3.44}$$

and so

$$I = \pi r_0^2 eB = \frac{\pi \bar{p}^2}{eB}\,. \tag{3.45}$$

The statement is that since I is an adiabatic invariant, it will remain essentially unchanged if B, which we can view here as the external parameter, is slowly changed. Thus we may say that

$$r_0 \propto B^{-1/2}\,, \qquad \text{or} \qquad \bar{p} \propto B^{1/2}\,. \tag{3.46}$$

Note that since $\pi r_0^2 = A$, the area of the loop, it follows from (3.45) that

$$I = e\Phi\,, \tag{3.47}$$

where $\Phi = AB$ is the magnetic flux threading the loop. Thus if we make a slow change to the magnetic field, then the radius of the particle's orbit adjusts itself so that the magnetic flux through the loop remains constant.

As an application, we may consider a charged particle moving in a static magnetic field that changes gradually with position. We have already seen that $\mathcal{E}_{\text{mech}}$ is constant in a pure magnetic field. Since we have

$$p^\mu p_\mu = -\mathcal{E}_{\text{mech}}^2 + \vec{p}^{\,2} = -m^2\,, \tag{3.48}$$

it follows that $|\vec{p}|$ is also a constant. In our discussion of the particle motion in the magnetic field, we defined $\bar{p}$ to be the component of transverse 3-momentum; i.e. the component in the (x, y) plane. Thus we shall have

$$\vec{p}^{\,2} = \bar{p}^2 + p_L^2\,, \tag{3.49}$$

where p_L denotes the longitudinal component of 3-momentum. It follows that

$$p_L^2 = \vec{p}^{\,2} - \bar{p}^2 = \vec{p}^{\,2} - \frac{eIB}{\pi}\,. \tag{3.50}$$

Since $\vec{p}^{\,2}$ is a constant, it follows that as the particle penetrates into a region where the magnetic field increases, the longitudinal momentum p_L (i.e. the momentum in the direction of its forward motion) gets smaller and smaller. If the B field becomes large enough, the forward motion will be brought to zero, and the particle will be repelled out of the region of high magnetic field.

3.2.4 Motion in uniform $\vec{E}$ and $\vec{B}$ fields

Having considered the case of particle motion in a uniform $\vec{E}$ field, and in a uniform $\vec{B}$ field, we may also consider the situation of motion in uniform $\vec{E}$ and $\vec{B}$ fields together. To discuss this in detail is quite involved, and we shall not pursue it extensively here. Instead, consider the situation where we take

$$\vec{B} = (0, 0, B)\,, \qquad \vec{E} = (0, E_y, E_z)\,, \tag{3.51}$$

(there is no loss of generality in choosing axes so that this is the case), and we make the simplifying assumption that the motion is non-relativistic, i.e. $|\vec{v}| << 1$. The equations of motion will therefore be

$$m\frac{d\vec{v}}{dt} = e(\vec{E} + \vec{v}\times\vec{B})\,, \tag{3.52}$$

and so

$$m\ddot{x} = eB\dot{y}\,, \qquad m\ddot{y} = eE_y - eB\dot{x}\,, \qquad m\ddot{z} = eE_z\,. \tag{3.53}$$

We can immediately solve for z, finding

$$z = \frac{e}{2m}\,E_z\,t^2 + \bar{v}t\,, \tag{3.54}$$

where we have chosen the z origin so that $z = 0$ at $t = 0$. The x and y equations can be combined into

$$\frac{d}{dt}(\dot{x} + \mathrm{i}\,\dot{y}) + \mathrm{i}\,\omega(\dot{x} + \mathrm{i}\,\dot{y}) = \frac{\mathrm{i}\,e}{m}\,E_y\,, \tag{3.55}$$

where $\omega = eB/m$. Thus we find

$$\dot{x} + \mathrm{i}\,\dot{y} = ae^{-\mathrm{i}\,\omega t} + \frac{e}{m\omega}\,E_y = ae^{-\mathrm{i}\,\omega t} + \frac{E_y}{B}\,. \tag{3.56}$$

Choosing the origin of time so that a is real, we have

$$\dot{x} = a\cos\omega t + \frac{E_y}{B}\,, \qquad \dot{y} = -a\sin\omega t\,. \tag{3.57}$$

Taking the time averages, we see that

$$\langle\dot{x}\rangle = \frac{E_y}{B}\,, \qquad \langle\dot{y}\rangle = 0\,. \tag{3.58}$$

The averaged velocity along the x direction is called the *drift velocity.* Notice that it is perpendicular to $\vec{E}$ and $\vec{B}$. It can be written in general as

$$\vec{v}_{\rm drift} = \frac{\vec{E}\times\vec{B}}{B^2}\,. \tag{3.59}$$

For our assumption that $|\vec{v}| << 1$ to be valid, we must have $|\vec{E}\times\vec{B}| << B^2$, i.e. $|E_y| << |B|$.

Integrating (3.57) once more, we find

$$x = \frac{a}{\omega}\,\sin\omega t + \frac{E_y}{B}\,t\,, \qquad y = \frac{a}{\omega}\,(\cos\omega t - 1)\,, \tag{3.60}$$

where the origins of x and y have been chosen so that $x = y = 0$ at $t = 0$. These equations describe the projection of the particle's motion onto the (x, y) plane. The curve is called a *trochoid*. If $|a| > E_y/B$ there will be loops in the motion, and in the special case $a = -E_y/B$ the curve becomes a *cycloid*, with cusps:

$$x = \frac{E_y}{\omega B}\,(\omega t - \sin\omega t)\,, \qquad y = \frac{E_y}{\omega B}\,(1 - \cos\omega t)\,. \tag{3.61}$$

4 Action Principle for Electrodynamics

In this section, we shall show how the Maxwell equations themselves can be derived from an action principle. We shall also introduce the notion of the *energy-momentum tensor* for the electromagnetic field. We begin with a discussion of Lorentz invariant quantities that can be built from the Maxwell field strength tensor $F_{\mu\nu}$.

4.1 Invariants of the electromagnetic field

As we shall now show, it is possible to build two independent Lorentz invariants that are quadratic in the electromagnetic field. One of these will turn out to be just what is needed in order to construct an action for electrodynamics.

4.1.1 The first invariant

The first quadratic invariant is very simple; we may write

$$I_1 \equiv F_{\mu\nu}\,F^{\mu\nu}\,. \tag{4.1}$$

Obviously this is Lorentz invariant, since it is built from the product of two Lorentz tensors, with all indices contracted. It is instructive to see what this looks like in terms of the electric and magnetic fields. From the expressions given in (2.14), we see that

$$\begin{aligned} I_1 &= F_{0i}\,F^{0i} + F_{i0}\,F^{i0} + F_{ij}\,F^{ij}\,, \\ &= 2F_{0i}\,F^{0i} + F_{ij}\,F^{ij} = -2E_iE_i + \epsilon_{ijk}\,B_k\,\epsilon_{ij\ell}\,B_\ell\,, \\ &= -2E_iE_i + 2B_iB_i\,, \end{aligned} \tag{4.2}$$

and so

$$I_1 \equiv F_{\mu\nu}\,F^{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)\,. \tag{4.3}$$

One could, of course, verify from the Lorentz transformations (2.51) and (2.52) for $\vec{E}$ and $\vec{B}$ that indeed $(\vec{B}^2 - \vec{E}^2)$ was invariant, i.e. $I_1' = I_1$ under Lorentz transformations. This would be quite an involved computation. However, the great beauty of the 4-dimensional language is that there is absolutely no work needed at all; one can see *by inspection* that $F_{\mu\nu}\, F^{\mu\nu}$ is Lorentz invariant.

4.1.2 The second invariant

The second quadratic invariant that we can write down is given by

$$I_2 \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu} F_{\rho\sigma}\,. \tag{4.4}$$

First, we need to explain the tensor $\epsilon^{\mu\nu\rho\sigma}$. This is the four-dimensional Minkowski spacetime generalisation of the totally-antisymmetric tensor ϵ_{ijk} of three-dimensional Cartesian tensor analysis. The tensor $\epsilon^{\mu\nu\rho\sigma}$ is also totally antisymmetric in all its indices. That means that it changes sign if any two indices are exchanged. For example,[9]

$$\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\nu\mu\rho\sigma} = -\epsilon^{\mu\nu\sigma\rho} = -\epsilon^{\sigma\nu\rho\mu}\,. \tag{4.5}$$

Since all the non-vanishing components of $\epsilon^{\mu\nu\rho\sigma}$ are related by the antisymmetry, we need only specify one non-vanishing component in order to define the tensor completely. We shall define

$$\epsilon^{0123} = -1\,, \qquad \text{or, equivalently} \qquad \epsilon_{0123} = +1\,. \tag{4.6}$$

Thus $\epsilon^{\mu\nu\rho\sigma}$ is -1, $+1$ or 0 according to whether $(\mu\nu\rho\sigma)$ is an even permutation of (0123), and odd permutation, or no permutation at all. We use this definition of $\epsilon^{\mu\nu\rho\sigma}$ in *all* frames. This can be done because, like the Minkowski metric $\eta_{\mu\nu}$, the tensor $\epsilon^{\mu\nu\rho\sigma}$ is an *invariant* tensor, as we shall now discuss.

Actually, to be more precise, $\epsilon^{\mu\nu\rho\sigma}$ is an invariant *pseudo-tensor*. This means that under Lorentz transformations that are connected to the identity (pure boosts and/or pure rotations), it is truly an invariant tensor. However, it reverses its sign under Lorentz transformations that involve a reflection. To see this, let us calculate what the transformation of $\epsilon^{\mu\nu\rho\sigma}$ would be if we assume it behaves as an ordinary Lorentz tensor:

$$\begin{aligned} \epsilon'^{\mu\nu\rho\sigma} &\equiv \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta\, \epsilon^{\alpha\beta\gamma\delta}\,, \\ &= (\det\Lambda)\, \epsilon^{\mu\nu\rho\sigma}\,. \end{aligned} \tag{4.7}$$

[9]Beware that in an odd dimension, such as 3, the process of "cycling" the indices on ϵ_{ijk} (for example, pushing one off the right-hand end and bringing it to the front) is an *even* permutation; $\epsilon_{kij} = \epsilon_{ijk}$. By contrast, in an even dimension, such as 4, the process of cycling is an *odd* permutation; $\epsilon^{\sigma\mu\nu\rho} = -\epsilon^{\mu\nu\rho\sigma}$. This is an elementary point, but easily overlooked if one is familiar only with three dimensions!

The last equality can easily be seen by writing out all the terms. (It is easier to play around with the analogous identity in 2 or 3 dimensions, to convince oneself of it in an example with fewer terms to write down.) Now, we already saw in section 2.3 that $\det\Lambda = \pm 1$, with $\det\Lambda = +1$ for pure boosts and/or rotations, and $\det\Lambda = -1$ if there is a reflection as well. (See the discussion leading up to equation (2.39).) Thus we see from (4.7) that $\epsilon^{\mu\nu\rho\sigma}$ behaves like an invariant tensor, taking the same values in all Lorentz frames, provided there is no reflection. (Lorentz transformations connected to the identity, i.e. where there is no reflection, are sometimes called *proper* Lorentz transformations.) In practice, we shall almost always be considering only proper Lorentz transformations, and so the distinction between a tensor and a pseudo-tensor will not concern us.

Returning now to the second quadratic invariant, (4.4), we shall have

$$\begin{aligned} I_2 &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu} F_{\rho\sigma} = \tfrac{1}{2}\times 4\times \epsilon^{0ijk} F_{0i}\, F_{jk}\,, \\ &= 2(-\epsilon_{ijk})(-E_i)\epsilon_{jk\ell} B_\ell\,, \\ &= 4E_i B_i = 4\vec{E}\cdot\vec{B}\,. \end{aligned} \tag{4.8}$$

Thus, to summarise, we have the two quadratic invariants

$$\begin{aligned} I_1 &= F_{\mu\nu}F^{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)\,, \\ I_2 &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu} F_{\rho\sigma} = 4\vec{E}\cdot\vec{B}\,. \end{aligned} \tag{4.9}$$

Since the two quantities I_1 and I_2 are (manifestly) Lorentz invariant, this means that, even though it is not directly evident in the three-dimensional language without quite a lot of work, the two quantities

$$\vec{B}^2 - \vec{E}^2\,, \qquad \text{and} \qquad \vec{E}\cdot\vec{B} \tag{4.10}$$

are Lorentz invariant; i.e. they take the same values in all Lorentz frames. This has a number of consequences. For example

1. If $\vec{E}$ and $\vec{B}$ are perpendicular in one Lorentz frame, then they are perpendicular in all Lorentz frames.

2. In particular, if there exists a Lorentz frame where the electromagnetic field is purely electric ($\vec{B}=0$), or purely magnetic ($\vec{E}=0$), then $\vec{E}$ and $\vec{B}$ are perpendicular in any other frame.

3. If $|\vec{E}| > |\vec{B}|$ in one frame, then it is true in all frames. Conversely, if $|\vec{E}| < |\vec{B}|$ in one frame, then it is true in all frames.

4. By making an appropriate Lorentz transformation, we can, *at a given point*, make $\vec{E}$ and $\vec{B}$ equal to any values we like, subject only to the conditions that we cannot alter the values of $(\vec{B}^2 - \vec{E}^2)$ and $\vec{E}\cdot\vec{B}$ at that point.

4.2 Action for electrodynamics

We have already discussed the action principle for a charged particle moving in an electromagnetic field. In that discussion, the electromagnetic field was just a specified background, which, of course, would be a solution of the Maxwell equations. We can also derive the Maxwell equations themselves from an action principle, as we shall now show.

We begin by introducing the notion of *Lagrangian density*. This is a quantity that is integrated over a three-dimensional spatial volume (typically, all of 3-space) to give the Lagrangian:

$$L = \int \mathcal{L} d^3x\,. \tag{4.11}$$

Then, the Lagrangian is integrated over a time interval $t_1 \le t \le t_2$ to give the action,

$$S = \int_{t_1}^{t_2} L dt = \int \mathcal{L} d^4x\,. \tag{4.12}$$

Consider first the vacuum Maxwell equations without sources,

$$\partial_\mu F^{\mu\nu} = 0\,, \qquad \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,. \tag{4.13}$$

We immediately solve the second equation (the Bianchi identity) by writing $F_{\mu\nu}$ in terms of a potential:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{4.14}$$

Since the Maxwell field equations are linear in the fields, it is natural to expect that the action should be quadratic. In fact, it turns out that the first invariant we considered above provides the appropriate Lagrangian density. We take

$$\mathcal{L} = -\frac{1}{16\pi}\, F_{\mu\nu} F^{\mu\nu}\,, \tag{4.15}$$

and so the action will be

$$S = -\frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} d^4x\,. \tag{4.16}$$

We can now derive the source-free Maxwell equations by requiring that this action be stationary with respect to variations of the gauge field A_μ. It must be emphasised that we treat A_μ as the fundamental field here.

The derivation goes as follows. We shall have

$$
\begin{aligned}
\delta S &= -\frac{1}{16\pi}\int(\delta F_{\mu\nu}F^{\mu\nu}+F_{\mu\nu}\delta F^{\mu\nu})d^4x = -\frac{1}{8\pi}\int \delta F_{\mu\nu}F^{\mu\nu}\,d^4x\,,\\
&= -\frac{1}{8\pi},\int F^{\mu\nu}\,(\partial_\mu\delta A_\nu-\partial_\nu\delta A_\mu)d^4x = -\frac{1}{4\pi}\int F^{\mu\nu}\,\partial_\mu\delta A_\nu d^4x\,,\\
&= -\frac{1}{4\pi}\int \partial_\mu(F^{\mu\nu}\,\delta A_\nu)d^4x + \frac{1}{4\pi}\int(\partial_\mu F^{\mu\nu})\,\delta A_\nu d^4x\,,\\
&= -\frac{1}{4\pi}\int_\Sigma F^{\mu\nu}\,\delta A_\nu d\Sigma_\mu + \frac{1}{4\pi}\int(\partial_\mu F^{\mu\nu})\,\delta A_\nu d^4x\,,\\
&= \frac{1}{4\pi}\int(\partial_\mu F^{\mu\nu})\,\delta A_\nu d^4x\,.
\end{aligned}
\tag{4.17}
$$

Note that in the final steps, we have used the 4-dimensional analogue of the divergence theorem to turn the 4-volume integral of the divergence of a vector into a 3-volume integral over the bouding surface Σ. The next step is to say that this integral vanishes, because we restrict attention to variations δA_μ that vanish on Σ. Finally, we argue that if δS is to vanish for all possible variations δA_μ (that vanish on Σ), it must be that

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{4.18}$$

Thus we have derived the source-free Maxwell field equation. Of course the Bianchi identity has already been taken care of by writing $F_{\mu\nu}$ in terms of the 4-vector potential A_μ.

The action (4.16), whose variation gave the Maxwell field equation, is written in what is called *second-order formalism*; that is, the action is expressed in terms of the 4-vector-potential A_μ as the fundamental field, with $F_{\mu\nu}$ just being a short-hand notation for $\partial_\mu A_\nu - \partial_\nu A_\mu$. It is sometimes convenient to use instead the *first-order formalism*, in which one treats A_μ and $F_{\mu\nu}$ as *independent* fields. In this formalism, the equation of motion coming from demanding that S be stationary under variations of $F_{\mu\nu}$ will *derive* the equation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. To do this, we need a different action as our starting point, namely

$$S_{\text{f.o.}} = \frac{1}{4\pi}\int(\tfrac14 F^{\mu\nu}F_{\mu\nu} - F^{\mu\nu}\,\partial_\mu A_\nu)d^4x\,. \tag{4.19}$$

First, consider the variation of $F^{\mu\nu}$, *now treated as an independent fundamental field.* This gives

$$
\begin{aligned}
\delta S_{\text{f.o.}} &= \frac{1}{4\pi}\int(\tfrac12 F_{\mu\nu}\delta F^{\mu\nu} - \delta F^{\mu\nu}\partial_\mu A_\nu)d^4x\,,\\
&= \frac{1}{4\pi}\int[\tfrac12 F_{\mu\nu}\delta F^{\mu\nu} - \tfrac12\delta F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)]d^4x\,,
\end{aligned}
\tag{4.20}
$$

where, in getting to the second line, we have used the fact that $F^{\mu\nu}$ is antisymmetric. The reason for doing this is that when we vary $F^{\mu\nu}$ we can take $\delta F^{\mu\nu}$ to be arbitary, but it must still be *antisymmetric.* Thus it is helpful to force an explicit antisymmetrisation on the $\partial_\mu A_\nu$ that multiplies it, since the symmetric part automatically gives zero when contracted onto the antisymmetric $\delta F^{\mu\nu}$. Requiring $\delta S_{\rm f.o.} = 0$ for arbitrary $\delta F^{\mu\nu}$ then implies the integrand must vanish. This gives, as promised, the equation of motion

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{4.21}$$

Vraying $S_{\rm f.o.}$ in (4.19) instead with respect to A_μ, we get

$$\begin{aligned} \delta S_{\rm f.o.} &= -\frac{1}{4\pi} \int F^{\mu\nu}\, \partial_\mu \delta A_\nu\, d^4x \,, \\ &= \frac{1}{4\pi} \int (\partial_\mu F^{\mu\nu})\, \delta A_\nu\, d^4x \,, \end{aligned} \tag{4.22}$$

and hence reuiring that the variation of $S_{\rm f.o.}$ with respect to A_μ vanish gives the Maxwell field equation

$$\partial_\mu F^{\mu\nu} = 0 \tag{4.23}$$

again. Note that in this calculation, we immediately dropped the boundary term coming from the integration by parts, for the usual reason that we only allow variations that vanish on the boundary.

In practice, we shall usually use the previous, second-order, formalism.

4.3 Inclusion of sources

In general, the Maxwell field equation reads

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu \,. \tag{4.24}$$

So far, we have seen that by varying the second-order action (4.16) with respect to A_μ, we obtain

$$\delta S = \frac{1}{4\pi} \int \partial_\mu F^{\mu\nu}\, \delta A_\nu\, d^4x \,. \tag{4.25}$$

To derive the Maxwell field equation with a source current J^μ, we can simply add a term to the action, to give

$$S = \int \Big(-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \Big) d^4x \,. \tag{4.26}$$

Treating J^μ as *independent* of A_μ, we therefore find

$$\delta S = \int \Big(\frac{1}{4\pi} \partial_\mu F^{\mu\nu} + J^\nu \Big) \delta A_\nu\, d^4x \,, \tag{4.27}$$

and so requiring $\delta S = 0$ gives the Maxwell field equation (4.24) with the source on the right-hand side.

The form of the source current J^μ depends, of course, on the details of the situation one is considering. One might simply have a situation where J^μ is an externally-supplied source field. Alternatively, the source J^μ might itself be given dynamically in terms of some charged matter fields, or in terms of a set of moving point charges. Let us consider this possibility in more detail.

If there is a single point charge q at the location $\vec r_0$, then it will be described by the charge density

$$\rho = q\,\delta^3(\vec r - \vec r_0)\,, \tag{4.28}$$

where the three-dimensional delta-function $\delta^3(\vec r)$, with $\vec r = (x,y,z)$, means

$$\delta^3(\vec r) = \delta(x)\delta(y)\delta(z)\,. \tag{4.29}$$

If the charge is moving, so that its location at time t is at $\vec r = \vec r_0(t)$, then of course we shall have

$$\rho = q\,\delta^3(\vec r - \vec r_0(t))\,. \tag{4.30}$$

The 3-vector current will be given by

$$\vec J = q\,\delta^3(\vec r - \vec r_0(t))\,\frac{d\vec r_0}{dt}\,, \tag{4.31}$$

and so the 4-current is

$$J^\mu = (\rho, \rho\vec v)\,, \qquad \text{where} \qquad \vec v = \frac{d\vec r_0}{dt}\,, \tag{4.32}$$

and ρ is given by (4.30). We can verify that this is the correct current vector, by checking that it properly satisfies the charge-conservation equation $\partial_\mu J^\mu = \partial\rho/\partial t + \partial_i J^i = 0$. Thus we have

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= q\frac{\partial}{\partial t}\delta^3(\vec r - \vec r_0(t)) = q\frac{\partial}{\partial x_0^i}\delta^3(\vec r - \vec r_0(t))\,\frac{dx_0^i}{dt}\,,\\
&= -q\frac{\partial}{\partial x^i}\delta^3(\vec r - \vec r_0(t))\,\frac{dx_0^i}{dt} = -\partial_i\Big(\rho\,\frac{dx_0^i}{dt}\Big)\,,\\
&= -\partial_i(\rho v^i) = -\partial_i J_i\,.
\end{aligned} \tag{4.33}$$

Note that we used the chain rule for differentiation in the first line, and that in getting to the second line we used the result that $\partial/\partial x\, f(x-y) = f'(x-y) = -\partial/\partial y\, f(x-y)$ for any function f with argument $(x-y)$ (where f' denotes the derivative of f with respect to its argument). It is also useful to note that we can write (4.32) as

$$J^\mu = \rho\frac{dx_0^\mu}{dt}\,, \tag{4.34}$$

where we simply define x_0^μ with $\mu = 0$ to be t.

Note that the integral $\int J^\mu A_\mu$ for the point charge gives a contribution to that action that is precisely of the form we saw in equation (2.77):

$$\begin{aligned}\int J^\mu A_\mu d^4x &= \int q\delta^3(\vec{r}-\vec{r}_0)\frac{dx_0^\mu}{dt} A_\mu d^3x dt\,,\\ &= \int_{\text{path}} q\frac{dx^\mu}{dt}\, A_\mu\, dt = q\int_{\text{path}} A_\mu dx^\mu\,. \end{aligned} \tag{4.35}$$

Suppose now we have N charges q_a, following paths $\vec{r}_a(t)$. Then the total charge density will be given by

$$\rho = \sum_{a=1}^N q_a\, \delta^3(\vec{r}-\vec{r}_a(t))\,. \tag{4.36}$$

Since we have alluded several times to the fact that $\partial_\mu J^\mu = 0$ is the equation of charge conservation, it is appropriate to examine this in a little more detail. The total charge Q at time t_1 is given by integrating the charge density over the spatial 3-volume:

$$Q(t_1) = \int_{t=t_1} J^0 d\Sigma_0\,, \qquad \text{where} \qquad d\Sigma_0 = dx dy dz\,. \tag{4.37}$$

This can be written covariantly as

$$Q(t_1) = \int_{t=t_1} J^\mu d\Sigma_\mu\,, \tag{4.38}$$

where we define also

$$d\Sigma_1 = -dt dy dz\,, \qquad d\Sigma_2 = -dt dz dx\,, \qquad d\Sigma_3 = -dt dy dz\,. \tag{4.39}$$

Because the integral in (4.37) is *defined* to be over the 3-surface at constant t, it follows that the extra terms, for $\mu = 1, 2, 3$, in (4.38) do not contribute.

If we now calculate the charge at a later time t_2, and then take the difference between the two charges, we will obtain

$$Q(t_2) - Q(t_1) = \int_\Sigma J^\mu d\Sigma_\mu\,, \tag{4.40}$$

where Σ is the cylindrical closed spatial 3-volume bounded by the "end caps" formed by the surfaces $t = t_1$ and $t = t_2$, and by the sides at spatial infinity. We are assuming the charges are confined to a finite region, and so the current J^μ is zero on the sides of the cylinder.

By the 4-dimensional analogue of the divergence theorem we shall have

$$\int_\Sigma J^\mu d\Sigma_\mu = \int_V \partial_\mu J^\mu\, d^4x\,, \tag{4.41}$$

where V is the 4-volume bounded by Σ. Thus we have

$$Q(t_2) - Q(t_1) = \int_V \partial_\mu J^\mu d^4x = 0\,, \tag{4.42}$$

since $\partial_\mu J^\mu = 0$. Thus we see that $\partial_\mu J^\mu = 0$ implies that the total charge in an isolated finite region is independent of time.

Note that the equation of charge conservation implies the gauge invariance of the action. We have

$$S = \int \Big(-\frac{1}{16\pi}\, F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \Big) d^4x\,, \tag{4.43}$$

and so under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$, we find

$$\begin{aligned} S \quad &\longrightarrow \quad \int \Big(-\frac{1}{16\pi}\, F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \Big) d^4x + \int J^\mu \partial_\mu \lambda d^4x\,, \\ &= \quad S + \int J^\mu \partial_\mu \lambda d^4x = S - \int \lambda \partial_\mu J^\mu d^4x\,, \\ &= \quad S\,. \end{aligned} \tag{4.44}$$

4.4 Energy density and energy flux

Here, we review the calculation of energy density and energy flux in the 3-dimensional language. After that, we shall give the more elegant 4-dimensional description.

Consider the two Maxwell equations

$$\vec\nabla \times \vec B - \frac{\partial \vec E}{\partial t} = 4\pi \vec J\,, \qquad \vec\nabla \times \vec E + \frac{\partial \vec B}{\partial t} = 0\,. \tag{4.45}$$

From these, we can deduce

$$\begin{aligned} \vec E \cdot \frac{\partial \vec E}{\partial t} + \vec B \cdot \frac{\partial \vec B}{\partial t} &= \vec E \cdot (\vec\nabla \times \vec B - 4\pi \vec J) - \vec B \cdot (\vec\nabla \times \vec E)\,, \\ &= \epsilon_{ijk}(E_i \partial_j B_k - B_i \partial_j E_k) - 4\pi \vec J \cdot \vec E\,, \\ &= -\epsilon_{ijk}(B_i \partial_j E_k + E_k \partial_j B_i) - 4\pi \vec J \cdot \vec E\,, \\ &= -\partial_j(\epsilon_{jki} E_k B_i) - 4\pi \vec J \cdot \vec E\,, \\ &= -\vec\nabla \cdot (\vec E \times \vec B) - 4\pi \vec J \cdot \vec E\,. \end{aligned} \tag{4.46}$$

We then define the *Poynting vector*

$$\vec S \equiv \frac{1}{4\pi} \vec E \times \vec B\,, \tag{4.47}$$

and so

$$\tfrac12 \frac{\partial}{\partial t}(\vec E^2 + \vec B^2) = -4\pi \vec\nabla \cdot \vec S - 4\pi \vec J \cdot \vec E\,, \tag{4.48}$$

since $\vec{E}\cdot\partial\vec{E}/\partial t = \frac{1}{2}\partial/\partial t(\vec{E}^2)$, etc.

We now assume that the $\vec{E}$ and $\vec{B}$ fields are confined to some finite region of space. Integrating (4.48) over all space, we obtain

$$\begin{aligned}\int \vec{J}\cdot\vec{E} d^3x + \frac{1}{8\pi}\frac{d}{dt}\int(\vec{E}^2+\vec{B}^2)d^3x &= -\int \vec{\nabla}\cdot\vec{S} d^3x\,,\\ &= -\int_\Sigma \vec{S}\cdot d\vec{\Sigma}\,,\\ &= 0\,. \end{aligned} \tag{4.49}$$

We get zero on the right-hand side because, having used the divergence theorem to convert it to an integral over Σ, the "sphere at infinity," the integral vanishes since $\vec{E}$ and $\vec{B}$, and hence $\vec{S}$, are assumed to vanish there.

If the current $\vec{J}$ is assumed to be due to the motion of a set of charges q_a with 3-velocities $\vec{v}_a$ and rest masses m_a, we shall have from (4.31) that

$$\int \vec{J}\cdot\vec{E} d^3x = \sum_a q_a \vec{v}_a\cdot\vec{E} = \frac{d\mathcal{E}_{\rm mech}}{dt}\,, \tag{4.50}$$

where

$$\mathcal{E}_{\rm mech} = \sum_a m_a\gamma_a \tag{4.51}$$

is the total mechanical energy for the set of particles, as defined in (3.7). Note that here

$$\gamma_a \equiv (1-v_a^2)^{-1/2}\,. \tag{4.52}$$

Thus we conclude that

$$\frac{d}{dt}\Big(\mathcal{E}_{\rm mech} + \frac{1}{8\pi}\int(\vec{E}^2+\vec{B}^2)d^3x\Big) = 0\,. \tag{4.53}$$

This is the equation of total energy conservation. It says that the sum of the total mechanical energy plus the energy contained in the electromagnetic fields is a constant. Thus we interpret

$$W \equiv \frac{1}{8\pi}(\vec{E}^2+\vec{B}^2) \tag{4.54}$$

as the energy density of the electromagnetic field.

Returning now to equation (4.48), we can consider integrating it over just a finite volume V, bounded by a closed 2-surface Σ. We will have

$$\frac{d}{dt}\Big(\mathcal{E}_{\rm mech} + \int_V W d^3x\Big) = -\int_\Sigma \vec{S}\cdot d\vec{\Sigma}\,. \tag{4.55}$$

We now know that the left-hand side should be interpreted as the rate of change of total energy in the volume V and so clearly, since the total energy must be conserved, we should interpret the right-hand side as the flux of energy passing through the boundary surface Σ. Thus we see that the Poynting vector

$$S = \frac{1}{4\pi} \vec{E} \times \vec{B} \tag{4.56}$$

is to be interpreted as the energy flux across the boundary; i.e. the energy per unit area per unit time.

4.5 Energy-momentum tensor

The discussion above was presented within the 3-dimensional framework. In this section we shall give a 4-dimensional spacetime description, which involves the introduction of the energy-momentum tensor. We shall begin with a rather general introduction. In order to simplify this discussion, we shall first describe the construction of the energy-momentum tensor for a scalar field $\phi(x^\mu)$. When we then apply these ideas to electromagnetism, we shall need to make the rather simple generalisation to the case of a Lagrangian for the vector field $A_\mu(x^\nu)$.

We begin by considering a Lagrangian density $\mathcal{L}$ for the scalar field ϕ. We shall assume that this depends on ϕ, and on its first derivatives $\partial_\nu \phi$, but that it has no *explicit* dependence[10] on the spacetime coordinates x^μ:

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\nu \phi) \, . \tag{4.57}$$

The action is then given by

$$S = \int \mathcal{L}(\phi, \partial_\nu \phi) \, d^4 x \, . \tag{4.58}$$

The Euler-Lagrange equations for the scalar field then follow from requiring that the action be stationary. Thus we have[11]

$$\begin{aligned} \delta S &= \int \Big[\frac{\partial \mathcal{L}}{\partial \phi} \, \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \, \partial_\nu \delta\phi \Big] d^4 x \, , \\ &= \int \Big[\frac{\partial \mathcal{L}}{\partial \phi} \, \delta\phi - \partial_\nu \Big(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \Big) \delta\phi \Big] d^4 x + \int_\Sigma \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \delta\phi d\Sigma_\nu \, , \end{aligned}$$

[10]This is the analogue of a Lagrangian in classical mechanics that depends on the coordinates q_i and velocities $\dot{q}^i$, but which does not have explicit time dependence. Energy is conserved in a system described by such a Lagrangian.

[11]Note that $\partial\mathcal{L}/\partial\partial_\nu\phi$ means taking the partial derivative of $\mathcal{L}$ viewed as a function of ϕ and $\partial_\mu\phi$, with respect to $\partial_\nu\phi$. For example, if $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2} m^2 \phi$, then

$$\partial\mathcal{L}/\partial\partial_\nu\phi = -(\partial^\mu\phi) \, \frac{\partial(\partial_\mu\phi)}{\partial\partial_\nu\phi} = -(\partial^\mu\phi) \, \delta^\nu_\mu = -\partial^\nu\phi \, . \tag{4.59}$$

$$= \int \Big[\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi - \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\Big)\delta\phi\Big]\,d^4x\,, \tag{4.60}$$

where, in getting to the last line, we have as usual dropped the surface term integrated over the boundary cylinder Σ, since we shall insist that $\delta\phi$ vanishes on Σ. Thus the requirement that $\delta S = 0$ for all such $\delta\phi$ implies the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\Big) = 0\,. \tag{4.61}$$

Now consider the expression $\partial_\rho\mathcal{L} = \partial\mathcal{L}/\partial x_\rho$. Since we are assuming $\mathcal{L}$ has no *explicit* dependence on the spacetime coordinates, it follows that $\partial_\rho\mathcal{L}$ is given by the chain rule,

$$\partial_\rho\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\partial_\rho\phi + \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\partial_\rho\partial_\nu\phi\,. \tag{4.62}$$

Now, using the Euler-Lagrange equations (4.61), we can write this as

$$\begin{aligned} \partial_\rho\mathcal{L} &= \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\Big)\partial_\rho\phi + \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\partial_\nu\partial_\rho\phi\,,\\ &= \partial_\nu\Big[\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\,\partial_\rho\phi\Big]\,, \end{aligned} \tag{4.63}$$

and thus we have

$$\partial_\nu\Big[\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\,\partial_\rho\phi - \delta^\nu_\rho\,\mathcal{L}\Big] = 0\,. \tag{4.64}$$

We are therefore led to define the 2-index tensor

$$T_\rho{}^\nu \equiv -\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\,\partial_\rho\phi + \delta^\nu_\rho\,\mathcal{L}\,, \tag{4.65}$$

which then satisfies

$$\partial_\nu T_\rho{}^\nu = 0\,. \tag{4.66}$$

$T^{\mu\nu}$ is called an *energy-momentum tensor.*

We saw previously that the equation $\partial_\mu J^\mu = 0$ for the 4-vector current density J^μ implies that there is a conserved charge

$$Q = \int_{t=\text{const}} J^0 d\Sigma_0 = \int_{t=\text{const}} J^\mu d\Sigma_\mu\,, \tag{4.67}$$

where $d\Sigma_0 = dxdydz$, etc. By an identical argument, it follows that the equation $\partial_\nu T_\rho{}^\nu = 0$ implies that there is a conserved 4-vector:

$$P^\mu \equiv \int_{t=\text{const}} T^{\mu 0} d\Sigma_0 = \int_{t=\text{const}} T^{\mu\nu} d\Sigma_\nu\,. \tag{4.68}$$

(Of course $T^{\mu\nu} = \eta^{\mu\rho}\,T_\rho{}^\nu$.) Thus we may check

$$\begin{aligned} \frac{dP^\mu}{dt} &= \partial_0\int_{t=\text{const}} T^{\mu 0} d^3x = \int_{t=\text{const}} \partial_0 T^{\mu 0} d^3x = -\int_{t=\text{const}} \partial_i T^{\mu i} d^3x\,,\\ &= -\int_S T^{\mu i} dS_i = 0\,, \end{aligned} \tag{4.69}$$

where in the last line we have used the divergence theorem to turn the integral into a 2-dimensional integral over the boundary sphere S at infinity. This vanishes since we shall assume the fields are zero at infinity.

Notice that $T^{00} = -T_0{}^0$ and from (4.65) we therefore have

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\, \partial_0 \phi - \mathcal{L}\,. \tag{4.70}$$

Now for a Lagrangian $L = L(q^i, \dot{q}^i)$ we have the canonical momentum $\pi_i = \partial L / \partial \dot{q}^i$, and the Hamiltonian

$$H = \pi_i \dot{q}^i - L\,. \tag{4.71}$$

Since there is no explicit time dependence, H is conserved, and is equal to the total energy of the system. Comparing with (4.70), we can recognise that T^{00} is the energy density. From (4.68) we therefore have that

$$P^0 = \int T^{00} d^3x \tag{4.72}$$

is the total energy. Since it is manifest from its construction that P^μ is a 4-vector, and since its 0 component is the energy, it follows that P^μ is the 4-momentum.

The essential point in the discussion above is that P^μ given in (4.68) should be conserved, which requires $\partial_\nu T_\rho{}^\nu = 0$. The quantity $T_\rho{}^\nu$ we constructed is not the unique tensor with this property. We can define a new one, according to

$$T_\rho{}^\nu \longrightarrow T_\rho{}^\nu + \partial_\sigma \psi_\rho{}^{\nu\sigma}\,, \tag{4.73}$$

where $\psi_\rho{}^{\nu\sigma}$ is an arbitrary tensor that is antisymmetric in its last two indices,

$$\psi_\rho{}^{\nu\sigma} = -\psi_\rho{}^{\sigma\nu}\,. \tag{4.74}$$

We shall take $\psi_\rho{}^{\nu\sigma}$ to vanish at spatial infinity.

The antisymmetry implies, since partial derivatives commute, that

$$\partial_\nu \partial_\sigma \psi_\rho{}^{\nu\sigma} = 0\,, \tag{4.75}$$

and hence that the modified energy-momentum tensor defined by (4.73) is conserved too. Furthermore, the modification to $T_\rho{}^\nu$ does not alter P^μ, since, from (4.68), the extra term will be

$$\begin{aligned}\int_{t=\text{const}} \partial_\sigma \psi^{\mu\nu\sigma} d\Sigma_\nu &= \int_{t=\text{const}} \partial_\sigma \psi^{\mu 0\sigma} d\Sigma_0\,,\\ &= \int_{t=\text{const}} \partial_i \psi^{\mu 0 i} d^3x\,,\\ &= \int_S \psi^{\mu 0 i} dS_i = 0\,,\end{aligned} \tag{4.76}$$

where S is the sphere at spatial infinity. The modification to P^μ therefore vanishes since we are requiring that $\psi_\rho{}^{\nu\sigma}$ vanishes at spatial infinity.

The energy-momentum tensor can be pinned down uniquely by requiring that the four-dimensional angular momentum $M^{\mu\nu}$, defined by

$$M^{\mu\nu} = \int (x^\mu dP^\nu - x^\nu dP^\mu) \tag{4.77}$$

be conserved. First, let us make a remark about angular momentum in four dimensions. In three dimensions, we define the angular momentum 3-vector as $\vec{L} = \vec{r} \times \vec{p}$. In other words,

$$L_i = \epsilon_{ijk} x^j p^k = \tfrac{1}{2}\epsilon_{ijk}(x^j p^k - x^k p^j) = \tfrac{1}{2}\epsilon_{ijk} M^{jk}\,, \tag{4.78}$$

where $M^{jk} \equiv x^j p^k - x^k p^j$. Thus taking $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ in four dimensions is a plausible-looking generalisation. It should be noted that in a general dimension, angular momentum is described by a 2-index antisymmetric tensor; in other words, angular momentum is associated with a rotation in a 2-dimensional plane. It is a very special feature of three dimensions that we can use the ϵ_{ijk} tensor to map the 2-index antisymmetric tensor M^{jk} into the vector $L_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$. Put another way, a very special feature of three dimensions is that a rotation in the (x, y) plane can equivalently be described as a rotation about the orthogonal (i.e. z) axis. In higher dimensions, rotations do not occur around axes, but rather, in 2-planes. It is amusing, therefore, to try to imagine what the analogue of an axle is for a higher-dimensional car!

Getting back to our discussion of angular momentum and the energy-momentum tensor in four dimensions, we are defining

$$M^{\mu\nu} = \int (x^\mu dP^\nu - x^\nu dP^\mu) = \int (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) d\Sigma_\rho,. \tag{4.79}$$

By analogous arguments to those we used earlier, this will be conserved (i.e. $dM^{\mu\nu}/dt = 0$) if

$$\partial_\rho (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) = 0\,. \tag{4.80}$$

Distributing the derivative, we therefore have the requirement that

$$\delta^\mu_\rho T^{\nu\rho} + x^\mu \partial_\rho T^{\nu\rho} - \delta^\nu_\rho T^{\mu\rho} - x^\nu \partial_\rho T^{\mu\rho} = 0\,, \tag{4.81}$$

and hence, since $\partial_\rho T^{\mu\rho} = 0$, that $T^{\mu\nu}$ is symmetric,

$$T^{\mu\nu} = T^{\nu\mu}\,. \tag{4.82}$$

Using the freedom to add $\partial_\sigma \psi^{\mu\nu\sigma}$ to $T^{\mu\nu}$, as we discussed earlier, it is always possible to arrange for $T^{\mu\nu}$ to be symmetric. From now on, we shall assume that this is done.

We already saw that $P^\mu = \int T^{\mu 0} d^3x$ is the 4-momentum, so T^{00} is the energy density, and T^{i0} is the 3-momentum density. Let us now look at the conservation equation $\partial_\nu T^{\mu\nu} = 0$ in more detail. Taking $\mu = 0$, we have $\partial_\nu T^{0\nu} = 0$, or

$$\frac{\partial}{\partial t} T^{00} + \partial_j T^{0j} = 0\,. \tag{4.83}$$

integrating over a spatial 3-volume V with boundary S, we therefore find

$$\frac{\partial}{\partial t} \int_V T^{00} d^3x = -\int_V \partial_j T^{0j} d^3x = -\int_S T^{0j} dS_j\,. \tag{4.84}$$

The left-hand side is the rate of change of field energy in the volume V, and so we can deduce, from energy conservation, that T^{0j} is the energy flux 3-vector. But since we are now working with a *symmetric* energy-momentum tensor, we have that $T^{0j} = T^{j0}$, and we already identified T^{j0} as the 3-momentum density. Thus we have that

$$\text{energy flux} = \text{momentum density}\,. \tag{4.85}$$

From the $\mu = i$ components of $\partial_\nu T^{\mu\nu} = 0$, we have

$$\frac{\partial}{\partial t} T^{i0} + \partial_j T^{ij} = 0\,, \tag{4.86}$$

and so, integrating over the 3-volume V, we get

$$\frac{\partial}{\partial t} \int_V T^{i0} d^3x = -\int_V \partial_j T^{ij} d^3x = -\int_S T^{ij} dS_j\,. \tag{4.87}$$

The left-hand side is the rate of change of 3-momentum, and so we deduce that T^{ij} is the 3-tensor of momentum flux density. It gives the i component of 3-momentum that flows, per unit time, through the 2-surface perpendicular to the x^j axis. T^{ij} is sometimes called the 3-dimensional stress tensor.

4.6 Energy-momentum tensor for the electromagnetic field

Recall that for a scalar field ϕ, the original construction of the energy-momentum tensor $T_\rho{}^\nu$ (which we later modified by adding $\partial_\sigma \psi_\rho{}^{\nu\sigma}$ where $\psi_\rho{}^{\nu\sigma} = -\psi_\rho{}^{\sigma\nu}$) was given by

$$T_\rho{}^\nu = -\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi}\, \partial_\rho \phi + \delta_\rho^\nu\, \mathcal{L}\,. \tag{4.88}$$

If we have a set of N scalar fields ϕ_a, then it is easy to see that the analogous conserved tensor is

$$T_\rho{}^\nu = -\sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a}\, \partial_\rho \phi_a + \delta_\rho^\nu\, \mathcal{L}\,. \tag{4.89}$$

A similar calculation shows that if we consider instead a vector field A_σ, with Lagrangian density $\mathcal{L}(A_\sigma, \partial_\nu A_\sigma)$, the construction will give a conserved energy-momentum tensor

$$T_\rho{}^\nu = -\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\sigma}\, \partial_\rho A_\sigma + \delta_\rho^\nu\, \mathcal{L}\,. \tag{4.90}$$

Let us apply this to the Lagrangian density for pure electrodynamics (without sources),

$$\mathcal{L} = -\frac{1}{16\pi}\, F_{\mu\nu} F^{\mu\nu}\,. \tag{4.91}$$

We have

$$\delta\mathcal{L} = -\frac{1}{8\pi} F^{\mu\nu} \delta F_{\mu\nu} = -\frac{1}{4\pi} F^{\mu\nu} \partial_\mu \delta A_\nu\,, \tag{4.92}$$

and so

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -\frac{1}{4\pi} F^{\mu\nu}\,. \tag{4.93}$$

Thus from (4.90) we find

$$T_\rho{}^\nu = \frac{1}{4\pi} F^{\nu\sigma} \partial_\rho A_\sigma - \frac{1}{16\pi} \delta_\rho^\nu\, F_{\sigma\lambda} F^{\sigma\lambda}\,, \tag{4.94}$$

and so

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\nu\sigma} \partial^\mu A_\sigma - \frac{1}{16\pi} \eta^{\mu\nu}\, F_{\sigma\lambda} F^{\sigma\lambda}\,. \tag{4.95}$$

This expression is not symmetric in μ and ν. However, following our previous discussion, we can add a term $\partial_\sigma \psi^{\mu\nu\sigma}$ to it, where $\psi^{\mu\nu\sigma} = -\psi^{\mu\sigma\nu}$, without upsetting the conservation condition $\partial_\nu T^{\mu\nu} = 0$. Specifically, we shall choose $\psi^{\mu\nu\sigma} = -1/(4\pi) A^\mu F^{\nu\sigma}$, and so

$$\begin{aligned} \partial_\sigma \psi^{\mu\nu\sigma} &= -\frac{1}{4\pi} \partial_\sigma (A^\mu F^{\nu\sigma})\,, \\ &= -\frac{1}{4\pi} (\partial_\sigma A^\mu) F^{\nu\sigma} - \frac{1}{4\pi} A^\mu \partial_\sigma F^{\nu\sigma} = -\frac{1}{4\pi} (\partial_\sigma A^\mu) F^{\nu\sigma}\,. \end{aligned} \tag{4.96}$$

(the $\partial_\sigma F^{\nu\sigma}$ term drops as a consequence of the source-free field equation.) This leads to the new energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\nu\sigma} (\partial^\mu A_\sigma - \partial_\sigma A^\mu) - \frac{1}{16\pi} \eta^{\mu\nu}\, F_{\sigma\lambda} F^{\sigma\lambda}\,, \tag{4.97}$$

or, in other words,

$$T^{\mu\nu} = \frac{1}{4\pi} \Big(F^\mu{}_\sigma\, F^{\nu\sigma} - \tfrac{1}{4} \eta^{\mu\nu}\, F_{\sigma\lambda} F^{\sigma\lambda} \Big)\,. \tag{4.98}$$

This is indeed manifestly symmetric in μ and ν. From now on, it will be understood when we speak of the energy-momentum tensor for electrodynamics that this is the one we mean.

It is a straightforward exercise to verify directly, using the source-free Maxwell field equation and the Bianchi identity, that indeed $T^{\mu\nu}$ given by (4.98) is conserved, $\partial_\nu T^{\mu\nu} = 0$. Note that it has another simple property, namely that it is trace-free, in the sense that

$$\eta_{\mu\nu} T^{\mu\nu} = 0\,. \tag{4.99}$$

This is easily seen from (4.98), as a consequence of the fact that $\eta^{\mu\nu}\eta_{\mu\nu} = 4$ in four dimensions. The trace-free property is related to a special feature of the Maxwell equations in four dimensions, known as *conformal invariance.*

Having obtained the energy-momentum tensor (4.98) for the electromagnetic field, it is instructive to look at its components from the three-dimensional point of view. First, recall that we showed earlier that

$$F_{\sigma\lambda}F^{\sigma\lambda} = 2(\vec{B}^2 - \vec{E}^2)\,. \tag{4.100}$$

Then, we find

$$\begin{aligned} T^{00} &= \frac{1}{4\pi}(F^0{}_\sigma F^{0\sigma} - \tfrac{1}{4}\eta^{00} F_{\sigma\lambda}F^{\sigma\lambda})\,,\\ &= \frac{1}{4\pi}(F^{0i}F^{0i} + \tfrac{1}{2}\vec{B}^2 - \tfrac{1}{2}\vec{E}^2)\,,\\ &= \frac{1}{4\pi}(\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 - \tfrac{1}{2}\vec{E}^2)\,,\\ &= \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)\,. \end{aligned} \tag{4.101}$$

Thus T^{00} is equal to the energy density W that we introduced in (4.54).

Now consider T^{0i}. Since $\eta^{0i} = 0$, we have

$$\begin{aligned} T^{0i} &= \frac{1}{4\pi}F^0{}_\sigma F^{i\sigma} = \frac{1}{4\pi}F^0{}_j\, F^{ij}\,,\\ &= \frac{1}{4\pi}E_j\epsilon_{ijk}B_k = S_i\,, \end{aligned} \tag{4.102}$$

where $\vec{S} = 1/(4\pi)\vec{E}\times\vec{B}$ is the Poynting vector introduced in (4.47). Thus T^{0i} is the energy flux. As we remarked earlier, since we now have $T^{0i} = T^{i0}$, it can be equivalently interpreted as the 3-momentum density vector.

Finally, we consider the components T^{ij}. We have

$$\begin{aligned} T^{ij} &= \frac{1}{4\pi}\Big(F^i{}_\sigma F^{j\sigma} - \tfrac{1}{4}\eta^{ij}2(\vec{B}^2 - \vec{E}^2)\Big)\,,\\ &= \frac{1}{4\pi}\Big(F^i{}_0 F^{j0} + F^i{}_k F^{jk} - \tfrac{1}{2}\delta_{ij}(\vec{B}^2 - \vec{E}^2)\Big)\,,\\ &= \frac{1}{4\pi}\Big(-E_iE_j + \epsilon_{ik\ell}\epsilon_{jkm}B_\ell B_m - \tfrac{1}{2}\delta_{ij}(\vec{B}^2 - \vec{E}^2)\Big)\,,\\ &= \frac{1}{4\pi}\Big(-E_iE_j + \delta_{ij}\vec{B}^2 - B_iB_j - \tfrac{1}{2}\delta_{ij}(\vec{B}^2 - \vec{E}^2)\Big)\,,\\ &= \frac{1}{4\pi}\Big(-E_iE_j - B_iB_j + \tfrac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)\Big)\,. \end{aligned} \tag{4.103}$$

To summarise, we have

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{0j} \\ T^{i0} & \sigma_{ij} \end{pmatrix} = \begin{pmatrix} W & S_j \\ S_i & \sigma_{ij} \end{pmatrix}\,, \tag{4.104}$$

where W and $\vec{S}$ are the energy density and Poynting flux,

$$W = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2), \qquad \vec{S} = \frac{1}{4\pi}\vec{E} \times \vec{B}, \tag{4.105}$$

and

$$\sigma_{ij} \equiv \frac{1}{4\pi}(-E_i E_j - B_i B_j + \tfrac{1}{2} W \delta_{ij}). \tag{4.106}$$

Remarks

- Unless $\vec{E}$ and $\vec{B}$ are perpendicular and equal in magnitude, we can always choose a Lorentz frame where $\vec{E}$ and $\vec{B}$ are parallel *at a point*. (In the case that $\vec{E}$ and $\vec{B}$ are perpendicular (but unequal in magnitude), one or other of $\vec{E}$ or $\vec{B}$ will be zero, at the point, in the new Lorentz frame.)

  Let the direction of $\vec{E}$ and $\vec{B}$ then be along z:

$$\vec{E} = (0, 0, E), \qquad \vec{B} = (0, 0, B). \tag{4.107}$$

  Then we have $\vec{S} = 1/(4\pi)\vec{E} \times \vec{B} = 0$ and

$$\sigma_{11} = \sigma_{22} = W, \qquad \sigma_{33} = -W, \qquad \sigma_{ij} = 0 \quad \text{otherwise}, \tag{4.108}$$

  and so $T^{\mu\nu}$ is diagonal, given by

$$T^{\mu\nu} = \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & -W \end{pmatrix}, \tag{4.109}$$

  with $W = 1/(8\pi)(E^2 + B^2)$.

- If $\vec{E}$ and $\vec{B}$ are perpendicular and $|\vec{E}| = |\vec{B}|$ at a point, then at that point we can choose axes so that

$$\vec{E} = (E, 0, 0), \qquad \vec{B} = (0, B, 0) = (0, E, 0). \tag{4.110}$$

  Then we have

$$\begin{aligned} W &= \frac{1}{4\pi}E^2, \qquad \vec{S} = (0, 0, W), \\ \sigma_{11} &= \sigma_{22} = 0, \qquad \sigma_{33} = W, \qquad \sigma_{ij} = 0 \quad \text{otherwise}, \end{aligned} \tag{4.111}$$

  and therefore $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = \begin{pmatrix} W & 0 & 0 & W \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ W & 0 & 0 & W \end{pmatrix}. \tag{4.112}$$

4.7 Inclusion of massive charged particles

We now consider the energy-momentum tensor for a particle with rest masses m. We proceed by analogy with the construction of the 4-current density J^μ for charged non-interacting particles. Thus we define first a mass density, ε, for a point mass m located at $\vec{r} = \vec{r}_0(t)$. This will simply be given by a 3-dimensional delta function, with strength m, located at the instantaneous position of the mass point:

$$\varepsilon = m\delta^3(\vec{r} - \vec{r}_0(t))\,. \tag{4.113}$$

The energy density T^{00} for the particle will then be its mass density times the corresponding γ factor, where $\gamma = (1-v^2)^{-1/2}$, and $\vec{v} = d\vec{r}_0(t)/dt$ is the velocity of the particle. Since the coordinate time t and the proper time τ in the frame of the particle are related, as usual, by $dt = \gamma d\tau$, we then have

$$T^{00} = \varepsilon\,\frac{dt}{d\tau}\,. \tag{4.114}$$

The 3-momentum density will be

$$T^{0i} = \varepsilon\gamma\frac{dx^i}{dt} = \varepsilon\frac{dt}{d\tau}\frac{dx^i}{dt}\,. \tag{4.115}$$

We can therefore write

$$T^{0\nu} = \varepsilon\frac{dt}{d\tau}\frac{dx^\nu}{dt} = \varepsilon\frac{dx^0}{d\tau}\frac{dx^\nu}{dt}\,. \tag{4.116}$$

On general grounds of Lorentz covariance, it must therefore be that

$$\begin{aligned} T^{\mu\nu} &= \varepsilon\frac{dx^\mu}{d\tau}\frac{dx^\nu}{dt}\,, \\ &= \varepsilon\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{d\tau}{dt}\,. \end{aligned} \tag{4.117}$$

By writing it as we have done in the second line here, it becomes manifest that $T^{\mu\nu}$ for the particle is symmetric in μ and ν.

Consider now a system consisting of a particle with mass m and charge q, moving in an electromagnetic field. Clearly, since the particle interacts with the field, we should not expect either the energy-momentum tensor (4.98) for the electromagnetic field or the energy-momentum tensor (4.117) for the particle to be conserved separately. This is because energy, and momentum, is being exchanged between the particle and the field. We *can* expect, however, that the total energy-momentum tensor for the system, i.e. the sum of (4.98) and (4.117), to be conserved.

In order to distinguish clearly between the various energy-momentum tensors, let us define

$$T^{\mu\nu}_{\rm tot.} = T^{\mu\nu}_{\rm e.m.} + T^{\mu\nu}_{\rm part.}\,, \tag{4.118}$$

where $T^{\mu\nu}_{\rm e.m.}$ and $T^{\mu\nu}_{\rm part.}$ are the energy-momentum tensors for the electromagnetic field and the particle respectively:

$$
\begin{aligned}
T^{\mu\nu}_{\rm e.m.} &= \frac{1}{4\pi}\Big(F^\mu{}_\sigma\, F^{\nu\sigma} - \tfrac14 \eta^{\mu\nu}\, F_{\sigma\lambda} F^{\sigma\lambda}\Big)\,,\\
T^{\mu\nu}_{\rm part.} &= \varepsilon \frac{dx^\mu}{d\tau}\frac{dx^\nu}{dt}\,,
\end{aligned}
\tag{4.119}
$$

where $\varepsilon = m\delta^3(\vec r - \vec r_0(t))$.

Consider $T^{\mu\nu}_{\rm e.m.}$ first. Taking the divergence, we find

$$
\begin{aligned}
\partial_\nu T^{\mu\nu}_{\rm e.m.} &= \frac{1}{4\pi}\Big(\partial_\nu F^\mu{}_\sigma\, F^{\nu\sigma} + F^\mu{}_\sigma\, \partial_\nu F^{\nu\sigma} - \tfrac12 F^{\sigma\lambda}\partial^\mu F_{\sigma\lambda}\Big)\,,\\
&= \frac{1}{4\pi}\Big(\partial_\nu F^\mu{}_\sigma\, F^{\nu\sigma} + F^\mu{}_\sigma\, \partial_\nu F^{\nu\sigma} + \tfrac12 F^{\sigma\lambda}\partial_\sigma F_\lambda{}^\mu + \tfrac12 F^{\sigma\lambda}\partial_\lambda F^\mu{}_\sigma\Big)\,,\\
&= \frac{1}{4\pi}\Big(\partial_\nu F^\mu{}_\sigma\, F^{\nu\sigma} - \tfrac12 F^{\sigma\lambda}\partial_\sigma F^\mu{}_\lambda - \tfrac12 F^{\lambda\sigma}\partial_\lambda F^\mu{}_\sigma + F^\mu{}_\sigma\, \partial_\nu F^{\nu\sigma}\Big)\,,\\
&= \frac{1}{4\pi} F^\mu{}_\sigma\, \partial_\nu F^{\nu\sigma}\,,\\
&= -F^\mu{}_\nu\, J^\nu\,.
\end{aligned}
\tag{4.120}
$$

In getting to the second line we used the Bianchi identity on the last term in the top line. The third line is obtained by swapping indices on a field strength in the terms with the $\frac12$ factors, and this reveals that all except one term cancel, leading to the result. As expected, the energy-momentum tensor for the electromagnetic field by itself is not conserved when there are sources.

Now we want to show that this non-conservation is balanced by an equal and opposite non-conservation for the energy-momentum tensor of the particle, which is given in (4.119). We have

$$
\partial_\nu T^{\mu\nu}_{\rm part.} = \partial_\nu\Big(\varepsilon\frac{dx^\nu}{dt}\Big)\frac{dx^\mu}{d\tau} + \varepsilon\frac{dx^\nu}{dt}\partial_\nu\Big(\frac{dx^\mu}{d\tau}\Big)\,.
\tag{4.121}
$$

The first term is zero. This can be seen from the fact that the calculation is identical to the one which we used a while back in section 4.3 to show that the 4-current $J^\mu = \rho dx^\mu/dt$ for a charged particle is conserved. Thus we have

$$
\begin{aligned}
\partial_\nu T^{\mu\nu}_{\rm part.} &= \varepsilon\frac{dx^\nu}{dt}\partial_\nu\Big(\frac{dx^\mu}{d\tau}\Big) = \varepsilon\frac{dx^\nu}{dt}\partial_\nu U^\mu\,,\\
&= \varepsilon\frac{dU^\mu}{dt}\,.
\end{aligned}
\tag{4.122}
$$

By the Lorentz force equation $m dU^\mu/d\tau = qF^\mu{}_\nu U^\nu$, we have

$$
\varepsilon\frac{dU^\mu}{d\tau} = \rho F^\mu{}_\nu U^\nu = \rho F^\mu{}_\nu\frac{dx^\nu}{d\tau}\,,
\tag{4.123}
$$

and so

$$
\varepsilon\frac{dU^\mu}{dt} = \rho F^\mu{}_\nu\frac{dx^\nu}{dt} = F^\mu{}_\nu J^\nu\,,
\tag{4.124}
$$

since $J^\mu = \rho dx^\mu/dt$. Thus we conclude that

$$\partial_\nu T^{\mu\nu}_{\text{part.}} = F^\mu{}_\nu J^\nu\,, \tag{4.125}$$

and so, combining this with (4.120), we conclude that the total energy-momentum tensor for the particle plus electromagnetic field, defined in (4.118) is conserved,

$$\partial_\nu T^{\mu\nu}_{\text{tot.}} = 0\,. \tag{4.126}$$

5 Coulomb's Law

5.1 Potential of a point charges

Consider first a static point charge, for which the Maxwell equations therefore reduce to

$$\vec\nabla\times\vec E = 0\,,\qquad \vec\nabla\cdot\vec E = 4\pi\rho\,. \tag{5.1}$$

The first equation implies, of course, that we can write

$$\vec E = -\vec\nabla\phi\,, \tag{5.2}$$

and then the second equation implies that ϕ satisfies the Poisson equation

$$\nabla^2\phi = -4\pi\rho\,. \tag{5.3}$$

If the point charge is located at the origin, and the charge is e, then the charge density ρ is given by

$$\rho = e\,\delta^3(\vec r)\,. \tag{5.4}$$

Away from the origin, (5.3) implies that ϕ should satisfy the Laplace equation,

$$\nabla^2\phi = 0\,,\qquad |\vec r| > 0\,. \tag{5.5}$$

Since the charge density (5.4) is spherically symmetric, we can assume that ϕ will be spherically symmetric too, $\phi(\vec r) = \phi(r)$, where $r = |\vec r|$. From $r^2 = x^j x^j$ we deduce, by acting with ∂_i, that

$$\partial_i r = \frac{x^i}{r}\,. \tag{5.6}$$

From this it follows by the chain rule that

$$\partial_i\phi = \phi'\partial_i r = \phi'\frac{x^i}{r}\,, \tag{5.7}$$

where $\phi' \equiv d\phi/dr$, and hence

$$\begin{aligned}
\nabla^2\phi &= \partial_i\partial_i\phi = \partial_i\Big(\phi'\frac{x^i}{r}\Big) = \phi''\frac{x^i}{r}\frac{x^i}{r} + \phi'\frac{\partial_i x^i}{r} + \phi' x^i\partial_i\frac{1}{r}\,,\\
&= \phi'' + \frac{2}{r}\,\phi'\,. \qquad (5.8)
\end{aligned}$$

Thus the Laplace equation (5.5) can be written as

$$(r^2\phi')' = 0\,, \qquad r > 0\,, \tag{5.9}$$

which integrates to give

$$\phi = \frac{q}{r}\,, \tag{5.10}$$

where q is a constant, and we have dropped an additive constant of integration by using the gauge freedom to choose $\phi(\infty) = 0$.

To determine the constant q, we integrate the Poisson equation (5.3) over the interior V_R of a sphere of radius R centred on the origin, and use the divergence theorem:

$$\begin{aligned}
\int_{V_R}\nabla^2\phi d^3x &= -4\pi e\int_{V_R}\delta^3(\vec r)d^3x = -4\pi e\,,\\
&= \int_{S_R}\vec\nabla\phi\cdot d\vec S = \int_{S_R}\partial_i\Big(\frac{q}{r}\Big)dS_i\,,\\
&= -q\int_{S_R}\frac{x^i dS_i}{r^3} = -q\int_{S_R}\frac{n^i dS_i}{R^2}\,, \qquad (5.11)
\end{aligned}$$

where S_R is the surface of the sphere of radius R that bounds the volume V_R, and $n^i \equiv x^i/r$ is the outward-pointing unit vector. Clearly we have

$$n^i dS_i = R^2 d\Omega\,, \tag{5.12}$$

where $d\Omega$ is the area element on the unit-radius sphere, and so

$$-q\int_{S_R}\frac{n^i dS_i}{r^2} = -q\int d\Omega = -4\pi q\,, \tag{5.13}$$

and so we conclude that q is equal to e, the charge on the point charge at $r = 0$.

Note that if the point charge e were located at $\vec r\,'$, rather than at the origin, then by trivially translating the coordinate system we will have the potential

$$\phi(\vec r) = \frac{e}{|\vec r - \vec r\,'|}\,, \tag{5.14}$$

and this will satisfy

$$\nabla^2\phi = -4\pi e\delta^3(\vec r - \vec r\,')\,. \tag{5.15}$$

5.2 Electrostatic energy

In general, the energy density of an electromagnetic field is given by $W = 1/(8\pi)(\vec{E}^2 + \vec{B}^2)$. A purely electrostatic system therefore has a field energy U given by

$$
\begin{aligned}
U &= \frac{1}{8\pi}\int W d^3x = \frac{1}{8\pi}\int \vec{E}^2 d^3x\,,\\
&= -\frac{1}{8\pi}\int \vec{E}\cdot\vec{\nabla}\phi d^3x\,,\\
&= -\frac{1}{8\pi}\int \vec{\nabla}\cdot(\vec{E}\,\phi)d^3x + \frac{1}{8\pi}\int(\vec{\nabla}\cdot\vec{E})\phi d^3x\,,\\
&= -\frac{1}{8\pi}\int_S \vec{E}\,\phi\cdot d\vec{S} + \tfrac{1}{2}\int \rho\phi d^3x\,,\\
&= \tfrac{1}{2}\int \rho\phi d^3x\,. \qquad (5.16)
\end{aligned}
$$

Note that the surface integral over the sphere at infinity gives zero because the electric field is assumed to die away to zero there. Thus we conclude that the electrostatic field energy is given by

$$U = \tfrac{1}{2}\int \rho\phi d^3x\,. \qquad (5.17)$$

We can apply this formula to a system of N charges q_a, located at points $\vec{r}_a$, for which we shall have

$$\rho = \sum_{a=1}^{N} q_a \delta^3(\vec{r} - \vec{r}_a)\,. \qquad (5.18)$$

However, a naive application of (5.17) would give nonsense, since we find

$$U = \tfrac{1}{2}\sum_{a=1}^{N} q_a \int \delta^3(\vec{r} - \vec{r}_a)\,\phi(\vec{r})d^3x = \tfrac{1}{2}\sum_{a=1}^{N} q_a\phi(\vec{r}_a)\,, \qquad (5.19)$$

where $\phi(\vec{r})$ is given by (5.14),

$$\phi(\vec{r}) = \sum_{a=1}^{N} \frac{q_a}{|\vec{r} - \vec{r}_a|}\,, \qquad (5.20)$$

This means that (5.19) will give infinity since $\phi(\vec{r})$, not unreasonably, diverges at the location of each point charge.

This is the classic "self-energy" problem, which one encounters even for a single point charge. There is no totally satisfactory way around this in classical electromagnetism, and so one has to adopt a "fudge." The fudge consists of observing that the true self-energy of a charge, whatever that might mean, is a constant. Naively, it appears to be an infinite constant, but that is clearly the result of making the idealised assumption that the charge is literally located at a single point. In any case, one can argue that the constant self-energy will not be observable, as far as energy-conservation considerations are concerned, and so one might as well just drop it for now. Thus the way to make sense of the ostensibly divergent energy (5.19) for the system of point charges is to replace $\phi(\vec r_a)$, which means the potential at $\vec r = \vec r_a$ due to *all* the charges, by ϕ_a, which is defined to be the potential at $\vec r = \vec r_a$ due to all the charges *except the charge* q_a *that is itself located at* $\vec r = \vec r_a$. Thus we have

$$\phi_a \equiv \sum_{b\neq a} \frac{q_b}{|\vec r - \vec r_b|}\,, \tag{5.21}$$

and so (5.19) is now interpreted to mean that the total energy of the system of charges is

$$U = \tfrac12 \sum_a \sum_{b\neq a} \frac{q_a q_b}{|\vec r_a - \vec r_b|}\,. \tag{5.22}$$

5.3 Field of a uniformly moving charge

Suppose a charge e is moving with uniform velocity $\vec v$ in the Lorentz frame S. We may transform to a frame S', moving with velocity $\vec v$ relative to S, in which the charge is at rest. For convenience, we shall choose the origin of axes so that the charge is located at the origin of the frame S'.

It follows that in the frame S', the field due to the charge can be described purely by the electric scalar potential ϕ':

$$\text{In } S': \qquad \phi' = \frac{e}{r'}\,, \qquad \vec A' = 0\,. \tag{5.23}$$

(Note that the primes here all signify that the quantities are those of the primed frame S'.)

We know that $A^\mu = (\phi, \vec A)$ is a 4-vector, and so the components A^μ transform under Lorentz boosts in exactly the same way as the components of x^μ. Thus we shall have

$$\phi' = \gamma\,(\phi - \vec v\cdot\vec A)\,, \qquad \vec A' = \vec A + \frac{\gamma-1}{v^2}\,(\vec v\cdot\vec A)\vec v - \gamma\vec v\,\phi\,, \tag{5.24}$$

where $\gamma = (1-v^2)^{-1/2}$. Clearly the inverse Lorentz transformation is obtained by sending $\vec v \to -\vec v$, and so we shall have

$$\phi = \gamma\,(\phi' + \vec v\cdot\vec A')\,, \qquad \vec A = \vec A' + \frac{\gamma-1}{v^2}\,(\vec v\cdot\vec A')\vec v + \gamma\vec v\,\phi'\,. \tag{5.25}$$

From (5.23), we therefore find that the potentials in the frame S, in which the particle is moving with velocity $\vec v$, are given by

$$\phi = \gamma\phi' = \frac{e\gamma}{r'}\,, \qquad \vec A = \gamma\vec v\,\phi' = \frac{e\gamma\vec v}{r'}\,. \tag{5.26}$$

Note that we still have r' appearing in the denominator, which we would now like to express in terms of the unprimed coordinates.

Suppose, for example, that we orient the axes so that $\vec{v}$ lies along the x direction. Then we shall have

$$x' = \gamma(x - vt)\,, \qquad y' = y\,, \qquad z' = z\,, \tag{5.27}$$

and so

$$r'^2 = x'^2 + y'^2 + z'^2 = \gamma^2 (x - vt)^2 + y^2 + z^2\,. \tag{5.28}$$

It follows therefore from (5.26) that the scalar and 3-vector potentials in the frame S are given by

$$\phi = \frac{e}{R_*}\,, \qquad \vec{A} = \frac{e\vec{v}}{R_*}\,, \tag{5.29}$$

where we have defined

$$R_*^2 \equiv (x - vt)^2 + (1 - v^2)(y^2 + z^2)\,. \tag{5.30}$$

The electric and magnetic fields can now be calculated in the standard way from ϕ and $\vec{A}$, as in (2.8). Alternatively, and equivalently, we can first calculate $\vec{E}'$ and $\vec{B}'$ in the primed frame, and then Lorentz transform these back to the unprimed frame. In the frame S', we shall of course have

$$\vec{E}' = \frac{e\vec{r}\,'}{r'^3}\,, \qquad \vec{B}' = 0\,. \tag{5.31}$$

The transformation to the unprimed frame is then given by inverting the standard results (2.51) and (2.52) that express $\vec{E}'$ and $\vec{B}'$ in terms of $\vec{E}$ and $\vec{B}$. Again, this is simply achieved by interchanging the primed and unprimed fields, and sending $\vec{v}$ to $-\vec{v}$. This gives

$$\begin{aligned} \vec{E} &= \gamma(\vec{E}' - \vec{v}\times\vec{B}') - \frac{\gamma-1}{v^2}\,(\vec{v}\cdot\vec{E}')\,\vec{v}\,, \\ \vec{B} &= \gamma(\vec{B}' + \vec{v}\times\vec{E}') - \frac{\gamma-1}{v^2}\,(\vec{v}\cdot\vec{B}')\,\vec{v}\,, \end{aligned} \tag{5.32}$$

and so from (5.31), we find that $\vec{E}$ and $\vec{B}$ in the frame S are given by

$$\begin{aligned} \vec{E} &= \frac{e\gamma\vec{r}\,'}{r'^3} - \frac{\gamma-1}{v^2}\,\frac{e\vec{v}\cdot\vec{r}\,'}{r'^3}\,\vec{v}\,, \\ \vec{B} &= \gamma\vec{v}\times\vec{E}' = \frac{e\gamma\vec{v}\times\vec{r}\,'}{r'^3}\,. \end{aligned} \tag{5.33}$$

Let us again assume that we orient the axes so that $\vec{v}$ lies along the x direction. Then from the above we find that

$$E_x = \frac{ex'}{r'^3}\,, \qquad E_y = \frac{e\gamma y'}{r'^3}\,, \qquad E_z = \frac{e\gamma z'}{r'^3}\,, \tag{5.34}$$

and so

$$E_x = \frac{e\gamma(x - vt)}{r'^3}\,, \qquad E_y = \frac{e\gamma y}{r'^3}\,, \qquad E_z = \frac{e\gamma z}{r'^3}\,. \tag{5.35}$$

Since the charge is located at the point $(vt, 0, 0)$ in the frame S, it follows that the vector from the charge to the point $\vec{r} = (x, y, z)$ is

$$\vec{R} = (x - vt, y, z)\,. \tag{5.36}$$

From (5.35), we then find that the electric field is given by

$$\vec{E} = \frac{e\gamma\vec{R}}{r'^3} = \frac{e(1 - v^2)\vec{R}}{R_*^3}\,, \tag{5.37}$$

where R_* was defined in (5.30).

If we now define θ to be the angle between the vector $\vec{R}$ and the x axis, then the coordinates (x, y, z) of the observation point P will be such that

$$y^2 + z^2 = R^2 \sin^2\theta\,, \qquad \text{where} \qquad R^2 = |\vec{R}|^2 = (x - vt)^2 + y^2 + z^2\,. \tag{5.38}$$

This implies, from (5.30), that

$$R_*^2 = R^2 - v^2(y^2 + z^2) = R^2(1 - v^2 \sin^2\theta)\,, \tag{5.39}$$

and so the electric field due to the moving charge is

$$\vec{E} = \frac{e\vec{R}}{R^3}\,\frac{1 - v^2}{(1 - v^2 \sin^2\theta)^{3/2}}\,. \tag{5.40}$$

For an observation point P located on the x axis, the electric field will be $E_{||}$ (parallel to the x axis), and given by setting $\theta = 0$ in (5.40). On the other hand, we can define the electric field $E_\perp$ in the (y, z) plane (corresponding to $\theta = \pi/2$). From (5.40) we therefore have

$$E_{||} = \frac{e(1 - v^2)}{R^2}\,, \qquad E_\perp = \frac{e(1 - v^2)^{-1/2}}{R^2}\,. \tag{5.41}$$

Note that $E_{||}$ has the smallest magnitude, and $E_\perp$ has the largest magnitude, that $\vec{E}$ attains as a function of θ.

When the velocity is very small, the electric field is (as one would expect) more or less independent of θ. However, as v approaches 1 (the speed of light), we find that $E_{||}$ decreases to zero, while $E_\perp$ diverges. Thus for v near to the speed of light the electric field is very sharply peaked around $\theta = \pi/2$. If we set

$$\theta = \frac{\pi}{2} - \psi\,, \tag{5.42}$$

then

$$|\vec{E}| = \frac{e(1 - v^2)}{R^2(1 - v^2 \cos^2\psi)^{3/2}} \approx \frac{e(1 - v^2)}{(1 - v^2 + \frac{1}{2}\psi^2)^{3/2}} \tag{5.43}$$

if $v \approx 1$. Thus the angular width of the peak is of the order of

$$\psi \sim \sqrt{1-v^2}\,. \tag{5.44}$$

We saw previously that the magnetic field in the frame S is given by $\vec{B} = \gamma\vec{v} \times \vec{E}'$. From (5.33) we have $\vec{v} \times \vec{E} = \gamma\vec{v} \times \vec{E}'$, and so therefore

$$\vec{B} = \vec{v} \times \vec{E} = \frac{e(1-v^2)\vec{v} \times \vec{R}}{R_*^3}\,. \tag{5.45}$$

Note that if $|\vec{v}| << 1$ we get the usual non-relativistic expressions

$$\vec{E} \approx \frac{e\vec{R}}{R^3}\,, \qquad \vec{B} \approx \frac{e\vec{v} \times \vec{R}}{R^3}\,. \tag{5.46}$$

5.4 Motion of a charge in a Coulomb potential

We shall consider a particle of mass m and charge e moving in the field of a static charged Q. The classic "Newtonian" result is very familiar, with the orbit of the particle being a conic section; an ellipse, a parabola or a hyperbola, depending on the charges and the orbital parameters. In this section we shall consider the fully relativistic problem, when the velocity of the particle is not necessarily small compared with the speed of light.

The Lagrangian for the system is given by (2.79), with $\phi = Q/r$ and $\vec{A} = 0$:

$$L = -m(1-\dot{x}^i\dot{x}^i)^{1/2} - \frac{eQ}{r}\,, \tag{5.47}$$

where $\dot{x}^i = dx^i/dt$, and $r^2 = x^i x^i$. The charges occur in the combination eQ throughout the calculation, and so for convenience we shall define

$$q \equiv eQ\,. \tag{5.48}$$

It is convenient to introduce spherical polar coordinates in the standard way,

$$x = r\sin\theta\cos\varphi\,, \qquad y = r\sin\theta\sin\varphi\,, \qquad z = r\cos\theta\,, \tag{5.49}$$

and then the Lagrangian becomes

$$L = -m(1-\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2)^{1/2} - \frac{q}{r}\,. \tag{5.50}$$

The Lagrangian is of the form $L = L(q_i, \dot{q}_i)$ for coordinates q_i and velocities $\dot{q}_i$ (don't confuse the coordinates q_i with the product of charges $q = eQ$!). The Euler-Lagrange equations are

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) = 0\,. \tag{5.51}$$

Note that if L is independent of a particular coordinate, say q_j, there is an associated conserved quantity

$$\frac{\partial L}{\partial \dot{q}_j}\,. \tag{5.52}$$

The Euler-Lagrange equation for θ gives

$$r^2\sin\theta\cos\theta\dot\varphi^2(1-\dot r^2-r^2\dot\theta^2-r^2\sin^2\theta\dot\varphi^2)^{-1/2}-\frac{d}{dt}\Big(r^2\dot\theta(1-\dot r^2-r^2\dot\theta^2-r^2\sin^2\theta\dot\varphi^2)^{-1/2}\Big)=0\,. \tag{5.53}$$

It can be seen that a solution to this equation is to take $\theta=\pi/2$, and $\dot\theta=0$. In other words, if the particle starts out moving in the $\theta=\pi/2$ plane (i.e. the (x,y) plane at $z=0$), it will remain in this plane. This is just the familiar result that the motion of a particle moving under a central force lies in a plane. We may therefore assume now, without loss of generality, that $\theta=\pi/2$ for all time. We are left with just r and φ as polar coordinates in the (x,y) plane. The Lagrangian for the reduced system, where we consistently can set $\theta=\pi/2$, is then simply

$$L=-(1-\dot r^2-r^2\dot\varphi^2)^{1/2}-\frac{q}{r}\,. \tag{5.54}$$

We note that $\partial L/\partial\varphi=0$, and so there is a conserved quantity

$$\frac{\partial L}{\partial\dot\varphi}=mr^2\dot\varphi(1-\dot r^2-r^2\dot\varphi^2)^{-1/2}=\ell\,, \tag{5.55}$$

where ℓ is a constant. Since $(1-\dot r^2-r^2\dot\varphi^2)^{-1/2}=\gamma$, we simply have

$$m\gamma r^2\dot\varphi=\ell\,. \tag{5.56}$$

Note that we can also write this as

$$mr^2\frac{d\varphi}{d\tau}=\ell\,. \tag{5.57}$$

Since the Lagrangian does not depend explicitly on t, the total energy $\mathcal{E}$ is also conserved. Thus we have

$$\mathcal{E}=H=\sqrt{\vec p^{\,2}+m^2}+\frac{q}{r} \tag{5.58}$$

is a constant. Here,

$$\begin{aligned}\vec p^{\,2}&=m^2\gamma^2\vec v^{\,2}=m^2\gamma^2\dot r^2+m^2\gamma^2r^2\dot\varphi^2\,,\\&=m^2\Big(\frac{dr}{d\tau}\Big)^2+m^2r^2\Big(\frac{d\varphi}{d\tau}\Big)^2\,,\end{aligned} \tag{5.59}$$

since, as usual, coordinate time and proper time are related by $d\tau=dt/\gamma$.

We therefore have

$$\Big(\mathcal{E} - \frac{q}{r}\Big)^2 = \vec{p}^{\,2} + m^2 = m^2 \Big(\frac{dr}{d\tau}\Big)^2 + m^2 r^2 \Big(\frac{d\varphi}{d\tau}\Big)^2 + m^2\,. \tag{5.60}$$

We now perform the standard change of variables in orbit calculations, and let

$$r = \frac{1}{u}\,. \tag{5.61}$$

This implies

$$\frac{dr}{d\tau} = -\frac{1}{u^2}\frac{du}{d\tau} = -\frac{1}{u^2}\frac{du}{d\varphi}\frac{d\varphi}{d\tau} = -\frac{\ell}{m} u'\,, \tag{5.62}$$

where we have used (5.57) and also we have defined

$$u' \equiv \frac{du}{d\varphi}\,. \tag{5.63}$$

It now follows that (5.60) becomes

$$(\mathcal{E} - q u)^2 = \ell^2 u'^2 + \ell^2 u^2 + m^2\,. \tag{5.64}$$

This ordinary differential equation can be solved in order to find u as a function of φ, and hence r as a function of φ. The solution determines the shape of the orbit of the particle around the fixed charge Q.

Rewriting (5.64) as

$$\ell^2 u'^2 = \Big(u\sqrt{q^2-\ell^2} - \frac{q\mathcal{E}}{\sqrt{q^2-\ell^2}}\Big)^2 - m^2 - \frac{\mathcal{E}^2\ell^2}{q^2-\ell^2}\,, \tag{5.65}$$

we see that it is convenient to make a change of variable from u to w, defined by

$$u\sqrt{q^2-\ell^2} - \frac{q\mathcal{E}}{\sqrt{q^2-\ell^2}} = \pm\sqrt{m^2 + \frac{\mathcal{E}^2\ell^2}{q^2-\ell^2}}\,\cosh w\,, \tag{5.66}$$

where the $+$ sign is chosen if $q<0$, and the $-$ sign if $q>0$. We can then integrate (5.65), to obtain

$$\pm\frac{\ell}{\sqrt{q^2-\ell^2}} w = \varphi\,, \tag{5.67}$$

(making a convenient choice, without loss of generality, for the constant of integration), and hence we have

$$\sqrt{q^2-\ell^2}\,u = \pm\sqrt{m^2 + \frac{\mathcal{E}^2\ell^2}{q^2-\ell^2}}\,\cosh\Big[\Big(\frac{q^2}{\ell^2}-1\Big)^{1/2}\varphi\Big] + \frac{q\mathcal{E}}{\sqrt{q^2-\ell^2}}\,. \tag{5.68}$$

In other words, the orbit is given, in terms of $r = r(\varphi)$, by

$$\frac{q^2-\ell^2}{r} = \pm\sqrt{\mathcal{E}^2\ell^2 + m^2(q^2-\ell^2)}\,\cosh\Big[\Big(\frac{q^2}{\ell^2}-1\Big)^{1/2}\varphi\Big] + q\mathcal{E}\,. \tag{5.69}$$

The solution (5.69) is presented for the case where $|\ell| < |q|$. If instead $|\ell| > |q|$, it becomes

$$\frac{\ell^2 - q^2}{r} = \sqrt{\mathcal{E}^2\ell^2 - m^2(\ell^2 - q^2)}\,\cos\Big[\Big(1 - \frac{q^2}{\ell^2}\Big)^{1/2}\varphi\Big] - q\mathcal{E}\,. \tag{5.70}$$

Finally, if $|\ell| = |q|$, it is easier to go back to the equation (5.65) and re-solve it directly in this case, leading to

$$\frac{2q\mathcal{E}}{r} = \mathcal{E}^2 - m^2 - \mathcal{E}^2\,\varphi^2\,. \tag{5.71}$$

The situation described above for relativistic orbits should be contrasted with what happens in the non-relativistic case. In this limit, the Lagrangian (after restricting to motion in the (x, y) plane again) is simply given by

$$L = \tfrac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) - \frac{q}{r}\,. \tag{5.72}$$

Note that this can be obtained from the relativisitic Lagrangian (5.54) we studied above, by taking $\dot r$ and $r\dot\varphi$ to be small compared to 1 (the speed of light), and then expanding the square root to quadratic order in velocities. As discussed previously, one can ignore the leading-order term $-m$ in the expansion, since this is just a constant (the rest-mass energy of the particle) and so it does not enter in the Euler-Lagrange equations. The analysis of the Euler-Lagrange equations for the non-relativistic Lagrangian (5.72) is a standard one. There are conserved quantities

$$E = \tfrac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) + \frac{q}{r}\,, \qquad \ell = mr^2\dot\varphi\,. \tag{5.73}$$

Substituting the latter into the former give the standard radial equation, whose solution implies closed elliptical orbits given by

$$\frac{1}{r} = \frac{mq}{\ell^2}\Big(\sqrt{1 + \frac{2E\ell^2}{mq^2}}\,\cos\varphi - 1\Big)\,. \tag{5.74}$$

(This is for the case $E > -mq^2/(2\ell^2)$. If $E < -mq^2/(2\ell^2)$ the orbits are hyperbolae, while in the intermediate case $E = -mq^2/(2\ell^2)$ the orbits are parabolic.)

The key difference in the relativistic case is that the orbits are never closed, even when $|\ell| > |q|$, as in (5.70), for which the radius r is a trigonometric function of φ. The reason for this is that the argument of the trigonometric function is

$$\Big(1 - \frac{q^2}{\ell^2}\Big)^{1/2}\varphi\,, \tag{5.75}$$

and so φ has to increase through an angle $\Delta\varphi$ given by

$$\Delta\varphi = 2\pi\,\Big(1 - \frac{q^2}{\ell^2}\Big)^{-1/2} \tag{5.76}$$

before the cosine completes one cycle. If we assume that $|q/\ell|$ is small compared with 1, then the shape of the orbit is still approximately like an ellipse, except that the "perihelion" of the ellipse advances by an angle

$$\delta\varphi = 2\pi\,\Big[\Big(1-\frac{q^2}{\ell^2}\Big)^{-1/2} - 1\Big] \approx \frac{\pi q^2}{\ell^2} \tag{5.77}$$

per orbit.

If on the other hand $|\ell| \le |q|$, then if $q < 0$ (which means $eQ < 0$ and hence an attractive force between the charges), the particle spirals inwards and eventually reaches $r = 0$ within a finite time. This can never happen in the non-relativisitic case; the orbit of the particle can never reach the origin at $r = 0$, unless the angular momentum ℓ is exactly zero. The reason for this is that the centrifugal potential term ℓ^2/r^2 always throws the particle away from the origin if r tries to get too small. By contrast, in the relativisitic case the effect of the centrifugal term is reduced at large velocity, and it cannot prevent the collapse of the orbit to $r = 0$. This can be seen by looking at the conserved quantity $\mathcal{E}$ in the fully relativisitic analysis, which, from our discussion above, can be written as

$$\mathcal{E} = \Big(m^2 + m^2\Big(\frac{dr}{d\tau}\Big)^2 + \frac{\ell^2}{r^2}\Big)^{1/2} + \frac{q}{r}\,. \tag{5.78}$$

First, consider the non-relativistic limit, for which the rest-mass term dominates inside the square root:

$$\mathcal{E} \approx m + \tfrac{1}{2}m\Big(\frac{dr}{dt}\Big)^2 + \frac{\ell^2}{2mr^2} + \frac{q}{r}\,. \tag{5.79}$$

Here, we see that even if $q < 0$ (an attractive force), the repulsive centrifugal term always wins over the attractive charge term q/r at small enough r.

On the other hand, if we keep the full relativistic expression (5.78), then at small enough r the competition between the centrifugal term and the charge term becomes "evenly matched,"

$$\mathcal{E} \approx \frac{|\ell|}{r} + \frac{q}{r}\,, \tag{5.80}$$

and clearly if $q < -|\ell|$ the attraction between the charges wins the contest.

5.5 The multipole expansion

Consider the electrostatic potential of N point charges q_a, located at fixed positions $\vec r_a$. It is given by

$$\phi(\vec r) = \sum_{a=1}^{N} \frac{q_a}{|\vec r - \vec r_a|}\,. \tag{5.81}$$

In the continuum limit, the potential due to a charge distrubution characterised by the charge density $\rho(\vec r)$ is given by

$$\phi(\vec r) = \int \frac{\rho(\vec r\,')d^3\vec r\,'}{|\vec r - \vec r\,'|}\,. \tag{5.82}$$

Since we shall assume that the charges are confined to a finite region, it is useful to perform a *multipole expansion* of the potential far from the region where the charges are located. This amounts to an expansion in inverse powers of $r = |\vec r|$. This can be achieved by performing a Taylor expansion of $\phi(\vec r)$.

Recall that in one dimension, Taylor's theorem gives

$$f(x+a) = f(x) + a f'(x) + \frac{a^2}{2!} f''(x) + \frac{a^3}{3!} f'''(x) + \cdots\,. \tag{5.83}$$

In three dimensions, the analogous expansion is

$$f(\vec r + \vec a) = f(\vec r) + a_i\partial_i f(\vec r) + \frac{1}{2!} a_i a_j \partial_i\partial_j f(\vec r) + \frac{1}{3!} a_i a_j a_k \partial_i\partial_j\partial_k f(\vec r) + \cdots\,. \tag{5.84}$$

We now apply this 3-dimensional Taylor expansion to the function $f(\vec r) = 1/|\vec r| = 1/r$, taking $\vec a = -\vec r\,'$. This gives

$$\frac{1}{|\vec r - \vec r\,'|} = \frac{1}{r} - x_i'\partial_i\frac{1}{r} + \frac{1}{2!} x_i' x_j' \partial_i\partial_j \frac{1}{r} - \frac{1}{3!} x_i' x_j' x_k' \partial_i\partial_j\partial_k \frac{1}{r} + \cdots\,. \tag{5.85}$$

Now since $r^2 = x_j x_j$, it follows that $\partial_i r^2 = 2r\,\partial_i r = 2x_i$, and so

$$\partial_i r = \frac{x_i}{r}\,. \tag{5.86}$$

Note that we have (assuming $r \ne 0$) that

$$\partial_i\partial_i \frac{1}{r} = \partial_i\Big(-\frac{x_i}{r^3}\Big) = -\frac{3}{r^3} + \frac{3x_i}{r^4}\,\frac{x_i}{r} = 0\,, \tag{5.87}$$

or, in other words

$$\nabla^2\frac{1}{r} = 0\,. \tag{5.88}$$

A consequence of this is that the multiple derivatives

$$\partial_i\partial_j\frac{1}{r}\,,\qquad \partial_i\partial_j\partial_k\frac{1}{r}\,,\qquad \partial_i\partial_j\partial_k\partial_\ell\frac{1}{r}\,,\qquad \cdots \tag{5.89}$$

are all *traceless* on any pair of indices:

$$\delta_{ij}\partial_i\partial_j\frac{1}{r} = 0\,,\qquad \delta_{ij}\partial_i\partial_j\partial_k\frac{1}{r} = 0\,,\qquad \text{etc.} \tag{5.90}$$

We can use this property in order to replace the quantities

$$x_i' x_j'\,,\qquad x_i' x_j' x_k'\,,\qquad \cdots \tag{5.91}$$

that multiply the derivative terms in (5.85) by the totally tracefree quantities

$$(x'_i x'_j - \tfrac{1}{3}\delta_{ij} r'^2)\,, \qquad (x'_i x'_j x'_k - \tfrac{1}{5}[x'_i \delta_{jk} + x'_j \delta_{ik} + x'_k \delta_{ij}] r'^2)\,, \qquad \cdots \tag{5.92}$$

where $r'^2 = x'_i x'_i$. (We can do this because the trace terms that we are subtracting out here give zero when they are contracted onto the multiple derivatives of $1/r$ in (5.85).) It therefore follows from (5.82) and (5.85) that we have

$$\begin{aligned}\phi(\vec{r}) &= \frac{1}{r}\int \rho(\vec{r}\,')d^3\vec{r}\,' - \Big(\partial_i \frac{1}{r}\Big)\int x'_i \rho(\vec{r}\,')d^3\vec{r}\,' + \Big(\partial_i\partial_j \frac{1}{r}\Big)\int (x'_i x'_j - \tfrac{1}{3}\delta_{ij} r'^2)\rho(\vec{r}\,')d^3\vec{r}\,' \\ &\quad - \Big(\partial_i\partial_j\partial_k \frac{1}{r}\Big)\int (x'_i x'_j x'_k - \tfrac{1}{5}[x'_i \delta_{jk} + x'_j \delta_{ik} + x'_k \delta_{ij}] r'^2)\rho(\vec{r}\,')d^3\vec{r}\,' + \cdots\,. \end{aligned}\tag{5.93}$$

The expansion here can be written as

$$\phi(\vec{r}) = \frac{Q}{r} - p_i \partial_i \frac{1}{r} + \frac{1}{2!} Q_{ij}\partial_i\partial_j \frac{1}{r} - \frac{1}{3!} Q_{ijk}\partial_i\partial_j\partial_k \frac{1}{r} + \cdots \tag{5.94}$$

where

$$\begin{aligned} Q &= \int \rho(\vec{r}\,')d^3\vec{r}\,'\,, \\ p_i &= \int x'_i \rho(\vec{r}\,')d^3\vec{r}\,'\,, \\ Q_{ij} &= \int (x'_i x'_j - \tfrac{1}{3}\delta_{ij} r'^2)\rho(\vec{r}\,')d^3\vec{r}\,'\,, \\ Q_{ijk} &= \int (x'_i x'_j x'_k - \tfrac{1}{5}[x'_i \delta_{jk} + x'_j \delta_{ik} + x'_k \delta_{ij}] r'^2)\rho(\vec{r}\,')d^3\vec{r}\,'\,, \end{aligned}\tag{5.95}$$

and so on. The quantity Q is the total charge of the system, p_i is the dipole moment, Q_{ij} is the quadrupole moment, and Q_{ijk}, $Q_{ijk\ell}$, etc., are the higher multipole moments. Note that by construction, all the multipole moments with two or more indices are symmetric and traceless on all indices.

Note that the terms in the multipole expansion (5.94) do indeed fall off with increasing inverse powers of r. For example, the dipole term is given by

$$\phi_{\rm Dipole} = -p_i \partial_i \frac{1}{r} = \frac{p_i x_i}{r^3} = \frac{p_i n_i}{r^2}\,, \tag{5.96}$$

which falls off like $1/r^2$, since $n_i \equiv x_i/r$ is a unit-length vector. The quadrupole term is given by

$$\phi_{\rm Quadrupole} = \tfrac{1}{2} Q_{ij}\partial_i\partial_j \frac{1}{r} = \tfrac{1}{2} Q_{ij}\frac{(3x_i x_j - r^2\delta_{ij})}{r^5} = \tfrac{3}{2} Q_{ij}\frac{x_i x_j}{r^5} = \tfrac{3}{2} Q_{ij}\frac{n_i n_j}{r^3}\,, \tag{5.97}$$

which falls off like $1/r^3$. (The penultimate equality above follows because Q_{ij} is traceless.)

The total charge Q (the electric monopole moment) is of course a single quantity. The dipole moment p_i is a 3-vector, so it has three independent components in general. The quadrupole moment Q_{ij} is a symmetric 2-index tensor in three dimensions, which would mean $3 \times 4/2 = 6$ independent components. But it is also traceless, $Q_{ii} = 0$, which is one condition. Thus there are $6 - 1 = 5$ independent components.

The octopole moment Q_{ijk} is a 3-index symmetric tensor, which would mean $3 \times 4 \times 5/3! = 10$ independent components. But it is also traceless, $Q_{iij} = 0$, which is 3 conditions. Thus the octopole has in general $10 - 3 = 7$ independent components. It is straightforward to see in the same way that the 2^ℓ-pole moment

$$Q_{i_1 i_2 \cdots i_\ell} = \int (x'_{i_1} x'_{i_2} \cdots x'_{i_\ell} - \text{traces}) \rho(\vec{r}\,') d^3\vec{r}\,' \tag{5.98}$$

has $(2\ell + 1)$ independent components.

In fact, the multipole expansion (5.94) is equivalent to an expansion in spherical polar coordinates, using the spherical harmonics $Y_{\ell m}(\theta, \phi)$:

$$\phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m}\, Y_{\ell m}(\theta, \phi)\, \frac{1}{r^{\ell+1}}\,. \tag{5.99}$$

At a given value of ℓ the terms fall off like $r^{-\ell-1}$, and there are $(2\ell + 1)$ of them, with coefficients $C_{\ell m}$, since m ranges over the integers $-\ell \le m \le \ell$. For each value of ℓ, there is a linear relationship between the $(2\ell + 1)$ components of $C_{\ell m}$ and the $(2\ell + 1)$ components of the multipole moments Q, p_i. Q_{ij}, Q_{ijk}, etc. Likewise, for each ℓ there is a linear relationship between $r^{-\ell-1}\, Y_{\ell m}(\theta, \varphi)$ and the set of functions $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell} r^{-1}$.

Consider, for example, $\ell = 1$. The three functions $Z_i \equiv \partial_i r^{-1} = -x_i/r^3$ are given by

$$Z_1 = -\frac{\sin\theta \cos\varphi}{r^2}\,, \qquad Z_2 = -\frac{\sin\theta \sin\varphi}{r^2}\,, \qquad Z_3 = -\frac{\cos\theta}{r^2}\,, \tag{5.100}$$

when expressed in terms of spherical polar coordinates (see (5.49)). On the other hand, the $\ell = 1$ spherical harmonics are given by

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{\mathrm{i}\varphi}\,, \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\, \cos\theta\,, \qquad Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-\mathrm{i}\varphi}\,. \tag{5.101}$$

Thus we see that

$$Z_1 = \sqrt{\frac{8\pi}{3}}\, \frac{(Y_{11} - Y_{1,-1})}{2r^2}\,, \qquad Z_1 = \sqrt{\frac{8\pi}{3}}\, \frac{(Y_{11} + Y_{1,-1})}{2\mathrm{i}\, r^2}\,, \qquad Z_3 = -\sqrt{\frac{4\pi}{3}}\, \frac{Y_{10}}{r^2}\,. \tag{5.102}$$

Analogous relations can be seen for all higher values of ℓ.

6 Electromagnetic Waves

6.1 Wave equation

As discussed at the beginning of the course (see section 1.1), Maxwell's equations admit wave-like solutions. These solutions can esist in free space, in a region where there are no source currents, for which the equations take the form

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 0\,, \qquad \vec{\nabla}\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=0\,,\\ \vec{\nabla}\cdot\vec{B} &= 0\,, \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0\,.\end{aligned} \tag{6.1}$$

As discussed in section 1.1, taking the curl of the $\vec{\nabla}\times\vec{E}$ equation, and using the $\vec{\nabla}\times\vec{B}$ equation, one finds

$$\nabla^2\vec{E}-\frac{\partial^2\vec{E}}{\partial t^2}=0\,, \tag{6.2}$$

and similarly,

$$\nabla^2\vec{B}-\frac{\partial^2\vec{B}}{\partial t^2}=0\,. \tag{6.3}$$

Thus each component of $\vec{E}$ and each component of $\vec{B}$ satisfies d'Alembert's equation

$$\nabla^2 f-\frac{\partial^2 f}{\partial t^2}=0\,. \tag{6.4}$$

This can, of course, be written as

$$\Box f\equiv\partial^\mu\partial_\mu f=0\,, \tag{6.5}$$

which shows that d'Alembert's operator is Lorentz invariant.

The wave equation (6.4) admits plane-wave solutions, where f depends on t and on a single linear combination of the x, y and z coordinates. By choosing the orientation of the axes appropriately, we can make this linear combination become simply x. Thus we may seek solutions of (6.4) of the form $f=f(t,x)$. The function f will then satisfy

$$\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial t^2}=0\,, \tag{6.6}$$

which can be written in the factorised form

$$\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\Big)\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\Big)f(t,x)=0\,. \tag{6.7}$$

Now introduced "light-cone coordinates"

$$u=x-t\,,\qquad v=x+t\,. \tag{6.8}$$

We see that

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\,,\qquad \frac{\partial}{\partial t}=-\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\,, \tag{6.9}$$

and so (6.7) becomes

$$\frac{\partial^2 f}{\partial u\partial v}=0\,. \tag{6.10}$$

The general solution to this is

$$f = f_+(u) + f_-(v) = f_+(x - t) + f_-(x + t)\,, \tag{6.11}$$

where f_+ and f_- are arbitrary functions.

The functions $f_\pm$ determine the *profile* of a wave-like disturbance that propagates at the speed of light (i.e. at speed 1). In the case of a wave described by $f_+(x-t)$, the disturbance propagtes at the speed of light in the *positive* x direction. This can be seen from the fact that if we sit at a given point on the profile (i.e. at a fixed value of the argument of the function f_+), then as t increases the x value must increase too. This means that the disturbance moves, with speed 1, along the positive x direction. Likewise, a wave described by $f_-(x+t)$ moves in the negative x direction as time increases.

More generally, we can consider a plane-wave disturbance moving along the direction of a unit 3-vector $\vec{n}$:

$$f(t, \vec{r}) = f_+(\vec{n}\cdot\vec{r} - t) + f_-(\vec{n}\cdot\vec{r} + t)\,. \tag{6.12}$$

The f_+ wave moves in the direction of $\vec{n}$ as t increases, while the f_- wave moves in the direction of $-\vec{n}$. The previous case of propagation along the x axis, corresponds to taking $\vec{n} = (1, 0, 0)$.

Let us now return to the discussion of electromagnetic waves. Following the discussion above, there will exist plane-wave solutions of (6.2), propagating along the $\vec{n}$ direction, of the form

$$\vec{E} = \vec{E}(\vec{n}\cdot\vec{r} - t)\,. \tag{6.13}$$

From the Maxwell equation $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}$, we shall therefore have

$$\begin{aligned} \frac{\partial B_i}{\partial t} &= -\epsilon_{ijk}\partial_j\, E_k(n_\ell x_\ell - t)\,, \\ &= -\epsilon_{ijk} n_j\, E'(n_\ell x_\ell - t)\,, \end{aligned} \tag{6.14}$$

where E'_k denotes the derivative of E_k with respect to its argument. We also have that $\partial E_k(n_\ell x_\ell - t)/\partial t = -E'_k(n_\ell x_\ell - t)$, and so

$$\frac{\partial B_i}{\partial t} = \epsilon_{ijk}\, n_j \frac{\partial}{\partial t} E_k(n_\ell x_\ell - t)\,. \tag{6.15}$$

We can integrate this, and drop the constant of integration since an additional static $\vec{B}$ field term is of no interest to us when discussing electromagnetic waves. Thus we have

$$B_i = \epsilon_{ijk} n_j E_k\,, \qquad \text{i.e.} \qquad \vec{B} = \vec{n}\times\vec{E}\,. \tag{6.16}$$

The source-free Maxwell equation $\vec{\nabla}\cdot\vec{E}=0$ implies

$$\partial_i E_i(n_j x_j - t) = n_i E_i'(n_j x_j - t) = -\frac{\partial}{\partial t}\vec{n}\cdot\vec{E} = 0\,. \tag{6.17}$$

Again, we can drop the constant of integration, and conclude that for the plane wave

$$\vec{n}\cdot\vec{E} = 0\,. \tag{6.18}$$

Since $\vec{B}=\vec{n}\times\vec{E}$, it immediately follows that $\vec{n}\cdot\vec{B}=0$ and $\vec{E}\cdot\vec{B}=0$ also. Thus we see that for a plane electromagnetic wave propagating along the $\vec{n}$ direction, the $\vec{E}$ and $\vec{B}$ vectors are orthogonal to $\vec{n}$ and also orthogonal to each other:

$$\vec{n}\cdot\vec{E}=0\,,\qquad \vec{n}\cdot\vec{B}=0\,,\qquad \vec{E}\cdot\vec{B}=0\,. \tag{6.19}$$

It also follows from $\vec{B}=\vec{n}\times\vec{E}$ that

$$|\vec{E}| = |\vec{B}|\,,\qquad \text{i.e.}\qquad E=B\,. \tag{6.20}$$

Thus we find that the energy density W is given by

$$W = \frac{1}{8\pi}(E^2+B^2) = \frac{1}{4\pi}E^2\,. \tag{6.21}$$

The Poynting flux $\vec{S}=(\vec{E}\times\vec{B})/(4\pi)$ is given by

$$\begin{aligned} S_i &= \frac{1}{4\pi}\epsilon_{ijk}E_j\epsilon_{k\ell m}n_\ell E_m = \frac{1}{4\pi}n_i E_j E_j - \frac{1}{4\pi}E_i n_j E_j\,,\\ &= \frac{1}{4\pi}n_i E_j E_j\,, \end{aligned} \tag{6.22}$$

and so we have

$$W = \frac{1}{4\pi}E^2\,,\qquad \vec{S} = \frac{1}{4\pi}\vec{n}E^2 = \vec{n}W\,. \tag{6.23}$$

Note that the argument $\vec{n}\cdot\vec{r}-t$ can be written as

$$\vec{n}\cdot\vec{r}-t = n_\mu\, x^\mu\,, \tag{6.24}$$

where $n_\mu=(-1,\vec{n})$ and hence

$$n^\mu = (1,\vec{n})\,. \tag{6.25}$$

Since $\vec{n}$ is a unit vector, $\vec{n}\cdot\vec{n}=1$, we have

$$n^\mu n_\mu = \eta_{\mu\nu}\, n^\mu n^\nu = 0\,. \tag{6.26}$$

n^μ is called a *Null Vector.* This is a non-vanishing vector whose norm $n^\mu n_\mu$ vanishes. Such vectors can arise because of the minus sign in the η_{00} component of the 4-metric.

By contrast, in a metric of positive-definite signature, such as the 3-dimensional Euclidean metric δ_{ij}, a vector whose norm vanishes is itself necessarily zero.

We can now evaluate the various components of the energy-momentum tensor, which are given by (4.104) and the equations that follow it. Thus we have

$$
\begin{aligned}
T^{00} &= W = \frac{1}{4\pi} E^2 = \frac{1}{4\pi} B^2 \,, \\
T^{0i} &= T^{i0} = S_i = n_i W \,, \\
T^{ij} &= \frac{1}{4\pi}(-E_i E_j - B_i B_j + \tfrac{1}{2}(E^2 + B^2)\delta_{ij}) \,, \\
&= \frac{1}{4\pi}(-E_i E_j - \epsilon_{ik\ell}\epsilon_{jmn} n_k n_m E_\ell E_n + E^2 \delta_{ij}) \,, \\
&= \frac{1}{4\pi}(-E_i E_j - \delta_{ij} E^2 - n_i n_k E_k E_j - n_j n_\ell E_\ell E_i + \delta_{ij} n_k n_\ell E_k E_\ell \\
&\qquad + n_i n_j E_\ell E_\ell + n_k n_k E_i E_j + E^2 \delta_{ij}) \,, \\
&= \frac{1}{4\pi} n_i n_j E^2 = n_i n_j W \,. \qquad (6.27)
\end{aligned}
$$

Note that in deriving this last result, we have used the identity

$$\epsilon_{ik\ell}\epsilon_{jmn} = \delta_{ij}\delta_{km}\delta_{\ell n} + \delta_{im}\delta_{kn}\delta_{\ell j} + \delta_{in}\delta_{kj}\delta_{\ell m} - \delta_{im}\delta_{kj}\delta_{\ell n} - \delta_{ij}\delta_{kn}\delta_{\ell m} - \delta_{in}\delta_{km}\delta_{\ell j} \,. \qquad (6.28)$$

The expressions for T^{00}, T^{0i} and T^{ij} can be combined into the single Lorentz-covariant expression

$$T^{\mu\nu} = n^\mu \, n^\nu \, W \,. \qquad (6.29)$$

From this, we can compute the conserved 4-momentum

$$
\begin{aligned}
P^\mu &= \int_{t=\text{const.}} T^{\mu\nu} d\Sigma_\nu = \int T^{\mu 0} d^3x \,, \\
&= \int n^\mu W d^3x = n^\mu \int W d^3x \,, \qquad (6.30)
\end{aligned}
$$

and hence we have

$$P^\mu = n^\mu \, \mathcal{E} \,, \qquad (6.31)$$

where

$$\mathcal{E} = \int W d^3x \,, \qquad (6.32)$$

the total energy of the electromagnetic field. Note that P^μ is also a null vector,

$$P^\mu P_\mu = \mathcal{E}^2 \, n^\mu n_\mu = 0 \,. \qquad (6.33)$$

6.2 Monochromatic plane waves

In the discussion above, we considered plane electromagnetic waves with an arbitrary profile. A special case is to consider the situation when the plane wave has a definite frequency ω, so that its time dependence is of the form $\cos \omega t$. Thus we can write

$$\vec{E} = \vec{E}_0\, e^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\,, \qquad \vec{B} = \vec{B}_0\, e^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{6.34}$$

where $\vec{E}_0$ and $\vec{B}_0$ are (possibly complex) constants. The physical $\vec{E}$ and $\vec{B}$ fields are obtained by taking the real parts of $\vec{E}$ and $\vec{B}$. (Since the Maxwell equations are linear, we can always choose to work in such a complex notation, with the understanding that we take the real parts to get the physical quantities.)

As we shall discuss in some detail later, the more general plane-wave solutions discussed previously, with an arbitrary profile for the wave, can be built up as linear combinations of the monochromatic plane-wave solutions.

Of course, for the fields in (6.34) to solve the Maxwell equations, there must be relations among the constants $\vec{k}$, ω, $\vec{E}_0$ and $\vec{B}_0$. Specifically, since $\vec{E}$ and $\vec{B}$ must satisfy the wave equations (6.2) and (6.3), we must have

$$\vec{k}^2 = \omega^2\,, \tag{6.35}$$

and since $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$, we must have

$$\vec{k}\cdot\vec{E}_0 = 0\,, \qquad \vec{k}\cdot\vec{B}_0 = 0\,. \tag{6.36}$$

Finally, following the discussion in the more general case above, it follows from $\vec{\nabla}\times\vec{E} + \partial\vec{B}/\partial t = 0$ and $\vec{\nabla}\times\vec{B} - \partial\vec{E}/\partial t = 0$ that

$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}\,. \tag{6.37}$$

It is natural, therefore, to introduce the 4-vector

$$k^\mu = (\omega, \vec{k}) = \omega\, n^\mu\,, \tag{6.38}$$

where $n^\mu = (1, \vec{n})$ and $\vec{n} = \vec{k}/|\vec{k}| = \vec{k}/\omega$. Equation (6.35) then becomes simply the statement that k^μ is a null vector,

$$k^\mu k_\mu = 0\,. \tag{6.39}$$

Note that the argument of the exponentials in (6.34) can now be written as

$$\vec{k}\cdot\vec{r} - \omega t = k_\mu x^\mu\,, \tag{6.40}$$

which we shall commonly write as $k \cdot x$. Thus we may rewrite (6.34) more briefly as

$$\vec{E} = \vec{E}_0 \, e^{\mathrm{i}\, k\cdot x}\,, \qquad \vec{B} = \vec{B}_0 \, e^{\mathrm{i}\, k\cdot x}\,. \tag{6.41}$$

As usual, we have a plane transverse wave, propagating in the direction of the unit 3-vector $\vec{n} = \vec{k}/\omega$. The term "transverse" here signifies that $\vec{E}$ and $\vec{B}$ are perpendicular to the direction in which the wave is propagating. In fact, we have

$$\vec{n} \cdot \vec{E} = \vec{n} \cdot \vec{B} = 0\,, \qquad \vec{B} = \vec{n} \times \vec{E}\,, \tag{6.42}$$

and so we have also that $\vec{E}$ and $\vec{B}$ are perpendicular to each other, and that $|\vec{E}| = \vec{B}|$.

Consider the case where $\vec{E}_0$ is taken to be real, which means that $\vec{B}_0$ is real too. Then the physical fields (obtained by taking the real parts of the fields given in (6.34)), are given by

$$\vec{E} = \vec{E}_0 \, \cos(\vec{k} \cdot \vec{r} - \omega t)\,, \qquad \vec{B} = \vec{B}_0 \, \cos(\vec{k} \cdot \vec{r} - \omega t)\,. \tag{6.43}$$

The energy density is then given by

$$W = \frac{1}{8\pi}(E^2 + B^2) = \frac{1}{4\pi} E_0^2 \, \cos^2(\vec{k} \cdot \vec{r} - \omega t)\,. \tag{6.44}$$

If we define the time average of W by

$$\langle W \rangle \equiv \frac{1}{T} \int_0^T W dt\,, \tag{6.45}$$

where $T = 2\pi/\omega$ is the period of the oscillation, then we shall have

$$\langle W \rangle = \frac{1}{8\pi} E_0^2 = \frac{1}{8\pi} B_0^2\,. \tag{6.46}$$

Note that in terms of the *complex* expressions (6.34), we can write this as

$$\langle W \rangle = \frac{1}{8\pi} \vec{E} \cdot \vec{E}^* = \frac{1}{8\pi} \vec{B} \cdot \vec{B}^*\,, \tag{6.47}$$

where the $*$ denotes complex conjugation, since the time and position dependence of $\vec{E}$ or $\vec{B}$ is cancelled when multiplied by the complex conjugate field.[12]

In general, when $\vec{E}_0$ and $\vec{B}_0$ are not real, we shall also have the same expressions (6.47) for the time-averaged energy density.

In a similar manner, we can evaluate the time average of the Poynting flux vector $\vec{S} = (\vec{E} \times \vec{B})/(4\pi)$. If we first consider the case where $\vec{E}_0$ is real, we shall have

$$\vec{S} = \frac{1}{4\pi} \vec{E} \times \vec{B} = \frac{1}{4\pi} \vec{E}_0 \times \vec{B}_0 \cos^2(\vec{n} \cdot \vec{r} - \omega t) = \frac{1}{4\pi} \vec{n}\, E_0^2 \cos^2(\vec{n} \cdot \vec{r} - \omega t)\,, \tag{6.48}$$

[12] This "trick," of expressing the time-averaged energy density in terms of the dot product of the complex field with its complex conjugate, is rather specific to this specific situation, where the quantity being time-averaged is quadratic in the electric and magnetic fields.

and so

$$\langle \vec{S} \rangle = \frac{1}{8\pi}\,\vec{E}_0 \times \vec{B}_0 = \frac{1}{8\pi}\vec{n}\,E_0^2\,. \tag{6.49}$$

In general, even if $\vec{E}_0$ and $\vec{B}_0$ are not real, we can write $\langle \vec{S} \rangle$ in terms of the complex $\vec{E}$ and $\vec{B}$ fields as

$$\langle \vec{S} \rangle = \frac{1}{8\pi}\,\vec{E} \times \vec{B}^* = \frac{1}{8\pi}\,\vec{n}\,\vec{E}\cdot\vec{E}^*\,, \tag{6.50}$$

and so we have

$$\langle \vec{S} \rangle = \vec{n}\,\langle W \rangle\,. \tag{6.51}$$

6.3 Motion of a point charge in a linearly-polarised E.M. wave

Consider a plane wave propagating in the z direction, with

$$\vec{E} = (E_0\,\cos\omega(z-t), 0, 0)\,, \qquad \vec{B} = (0, E_0\,\cos\omega(z-t), 0)\,. \tag{6.52}$$

Suppose now that there is a particle of mass m and charge e in this field. By the Lorentz force equation we shall have

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v}\times\vec{B}\,. \tag{6.53}$$

For simplicity, we shall make the assumption that the motion of the particle can be treated non-relativistically, and so

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}\,. \tag{6.54}$$

Let us suppose that the particle is initially located at the point $z=0$, and that it moves only by a small amount in comparison to the wavelength $2\pi/\omega$ of the electromagnetic wave. Therefore, to a good approximation, we can assume that the particle is sitting in the uniform, although time-dependent, electromagnetic field obtained by setting $z=0$ in (6.52). Thus

$$\vec{E} = (E_0\,\cos\omega t, 0, 0)\,, \qquad \vec{B} = (0, E_0\,\cos\omega t, 0)\,, \tag{6.55}$$

and so the Lorentz force equation gives

$$\begin{aligned}
m\ddot{x} &= eE_0\cos\omega t - e\dot{z}E_0\cos\omega t \approx eE_0\cos\omega t\,,\\
m\ddot{y} &= 0\,,\\
m\ddot{z} &= e\dot{x}\,E_0\cos\omega t\,.
\end{aligned} \tag{6.56}$$

Note that the approximation in the first line follows from our assumption that the motion of the particle is non-relativistic, so $|\dot{z}| << 1$.

With convenient and inessential choices for the constants of integration, first obtain

$$\dot{x} = \frac{eE_0}{m\omega} \sin \omega t\,, \qquad x = -\frac{eE_0}{m\omega^2} \cos \omega t\,, \tag{6.57}$$

Substituting into the z equation then gives

$$\ddot{z} = \frac{e^2 E_0^2}{m^2 \omega} \sin \omega t\, \cos \omega t = \frac{e^2 E_0^2}{2m^2 \omega} \sin 2\omega t\,, \tag{6.58}$$

which integrates to give (dropping inessential constants of integration)

$$z = -\frac{e^2 E_0^2}{8m^2 \omega^3} \sin 2\omega t\,. \tag{6.59}$$

The motion in the y direction is purely linear, and since we are not interested in the case where the particle drifts uniformly through space, we can just focus on the solution where y is constant, say $y = 0$.

Thus the interesting motion of the particle in the electromagnetic field is of the form

$$x = \alpha \cos \omega t\,, \qquad z = \beta \sin 2\omega t = 2\beta \sin \omega t\, \cos \omega t\,, \tag{6.60}$$

which means

$$z = \frac{2\beta}{\alpha}\, x \sqrt{1 - \frac{x^2}{\alpha^2}}\,. \tag{6.61}$$

This describes a "figure of eight" lying on its side in the (x, z) plane. The assumptions we made in deriving this, namely non-relativistic motion and a small z displacement relative to the wavelength of the electromagnetic wave, can be seen to be satisfied provided the amplitude E_0 of the wave is sufficiently small.

The response of the charge particle to electromagnetic wave provides a model for how the electrons in a receiving antenna behave in the presence of an electromagnetic wave. This shows how the wave is converted into oscilliatory currents in the antenna, which are then amplified and processed into the final output signal in a radio receiver.

6.4 Circular and elliptical polarisation

The electromagnetic wave described in section 6.2 is *linearly polarised.* For example, we could consider the solution with

$$\vec{E}_0 = (0, E_0, 0)\,, \qquad \vec{B}_0 = (0, 0, B_0)\,, \qquad \vec{n} = (1, 0, 0)\,. \tag{6.62}$$

This corresponds to a linearly polarised electromagnetic wave propagating along the x direction.

By taking a linear superposition of waves propagating along a given direction $\vec{n}$, we can obtain circularly polarised, or more generally, elliptically polarised, waves. Let $\vec{e}$ and $\vec{f}$ be two orthogonal unit vectors, that are also both orthogonal to $\vec{n}$:

$$\begin{aligned} \vec{e}\cdot\vec{e} &= 1\,, \qquad \vec{f}\cdot\vec{f}=1\,, \qquad \vec{n}\cdot\vec{n}=1\,, \\ \vec{e}\cdot\vec{f} &= 0\,, \qquad \vec{n}\cdot\vec{e}=0\,, \qquad \vec{n}\cdot\vec{f}=0\,. \end{aligned} \tag{6.63}$$

Suppose now we consider a plane wave given by

$$\vec{E} = (E_0\,\vec{e} + \widetilde{E}_0\,\vec{f})\,e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,, \qquad \vec{B} = \vec{n}\times\vec{E}\,, \tag{6.64}$$

where E_0 and $\widetilde{E}_0$ are complex constants. If E_0 and $\widetilde{E}_0$ both have the same phase (i.e. $\widetilde{E}_0/E_0$ is real), then we again have a linearly-polarised electromagnetic wave. If instead the phases of E_0 and $\widetilde{E}_0$ are different, then the wave is in general elliptically polarised.

Consider as an example the case where

$$\widetilde{E}_0 = \pm\mathrm{i}\,E_0\,, \tag{6.65}$$

(with E_0 taken to be real, without loss of generality), for which the electric field will be given by

$$\vec{E} = E_0(\vec{e}\pm\mathrm{i}\,\vec{f})\,e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,. \tag{6.66}$$

Taking the real part, to get the physical electric field, we obtain

$$\vec{E} = E_0\vec{e}\,\cos(\vec{k}\cdot\vec{r}-\omega t) \mp E_0\vec{f}\,\sin(\vec{k}\cdot\vec{r}-\omega t)\,. \tag{6.67}$$

For example, if we choose

$$\vec{n} = (0,0,1)\,, \qquad \vec{e} = (1,0,0)\,, \qquad \vec{f} = (0,1,0)\,, \tag{6.68}$$

then the electric field is given by

$$E_x = E_0\cos\omega(z-t)\,, \qquad E_y = \mp E_0\sin\omega(z-t)\,. \tag{6.69}$$

It is clear from this that the magnitude of the electric field is constant,

$$|\vec{E}| = E_0\,. \tag{6.70}$$

If we fix a value of z, then the $\vec{E}$ vector can be seen to be rotating around the z axis (the direction of motion of the wave). This rotation is *anticlockwise* in the (x,y) plane if we choose the plus sign in (6.65), and *clockwise* if we choose the minus sign instead. These two choices correspond to having a circularly polarised wave of positive or negative helicity respectively. (Positive helicity means the rotation is parallel to the direction of propagation, while negative helicity means the rotation is anti-parallel to the direction of propagation.)

In more general cases, where the magnitudes of E_0 and $\widetilde{E}_0$ are unequal, or where the phase angle between them is not equal to 0 (linear polarisation) or 90 degrees, the electromagnetic wave will be elliptically polarised. Consider, for example, the case where the electric field is given by

$$\vec{E} = (a_1 e^{\mathrm{i}\,\delta_1}, a_2 e^{\mathrm{i}\,\delta_2}, 0)\, e^{\mathrm{i}\,\omega(z-t)}\,, \tag{6.71}$$

with the propagtion direction being $\vec{n} = (0,0,1)$. Then we shall have

$$\vec{B} = \vec{n} \times \vec{E} = (-a_2 e^{\mathrm{i}\,\delta_2}, a_1 e^{\mathrm{i}\,\delta_1}, 0)\, e^{\mathrm{i}\,\omega(z-t)}\,. \tag{6.72}$$

The real constants a_1, a_2, δ_1 and δ_2 determine the nature of this plane wave propagating along the z direction. Of course the overall phase is unimportant, so really it is only the *difference* $\delta_2 - \delta_1$ between the phase angles that is important.

The magnitude and phase information is sometimes expressed in terms of the *Stokes Parameters* (s_0, s_1, s_2, s_3), which are defined by

$$\begin{aligned} s_0 &= E_x E_x^* + E_y E_y^* = a_1^2 + a_2^2\,, \qquad s_1 = E_x E_x^* - E_y E_y^* = a_1^2 - a_2^2\,, \\ s_2 &= 2\Re(E_x^* E_y) = 2a_1 a_2 \cos(\delta_2 - \delta_1)\,, \qquad s_3 = 2\Im(E_x^* E_y) = 2a_1 a_2 \sin(\delta_2 - \delta_1)\,. \end{aligned} \tag{6.73}$$

(The last two involve the real and imaginary parts of $(E_x^* E_y)$ respectively.) The four Stokes parameters are not independent:

$$s_0^2 = s_1^2 + s_2^2 + s_3^2\,. \tag{6.74}$$

The parameter s_0 characterises the intensity of the electromagnetic wave, while s_1 characterises the amount of x polarisation versus y polarisation, with

$$-s_0 \le s_1 \le s_0\,. \tag{6.75}$$

The third independent parameter, which could be taken to be s_2, characterises the phase difference between the x and the y polarised waves. Circular polaristion with $\pm$ helicity corresponds to

$$s_1 = 0\,, \qquad s_2 = 0\,, \qquad s_3 = \pm s_0\,. \tag{6.76}$$

6.5 General superposition of plane waves

So far in the discussion of electromagnetic waves, we have considered the case where there is a single direction of propagation (i.e. a plane wave), and a single frequency (monochromatic). The most general wave-like solutions of the Maxwell equations can be expressed as linear cobinations of these basic monochromatic plane-wave solutions.

In order to discuss the general wave solutions, it is helpful to work with the gauge potential $A^\mu = (\phi, \vec{A})$. Recall that we have the freedom to make gauge transformations $A_\mu \to A_\mu + \partial_\mu \lambda$, where λ is an arbitrary function. For the present purposes, of describing wave solutions, a convenient choice of gauge is to set $\phi = 0$. Such a gauge choice would not be convenient when discussing solutions in electrostatics, but in the present case, where we know that the wave solutions are necessarily time-dependent, it is quite helpful.

Thus, we shall first write a single monochromatic plane wave in terms of the 3-vector potential, as

$$\vec{A} = a\vec{e}\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{6.77}$$

where $\vec{e}$ is a unit *polarisation vector*, and a is a constant. As usual, we must have $|\vec{k}|^2 = \omega^2$. The electric and magnetic fields will be given by

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \mathrm{i}\, a\omega\, \vec{e}\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,, \\ \vec{B} &= \vec{\nabla}\times\vec{A} = \mathrm{i}\, a\vec{k}\times\vec{e}\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)} = \frac{\vec{k}\times\vec{E}}{\omega}\,. \end{aligned} \tag{6.78}$$

We can immediately see that $\vec{E}$ and $\vec{B}$ satisfy the wave equation, and that we must impose $\vec{e}\cdot\vec{k} = 0$ in order to satisfy $\vec{\nabla}\cdot\vec{E} = 0$.

We have established, therefore, that (6.77) describes a monochromatic plane wave propagating along the $\vec{k}$ direction, with electric field along $\vec{e}$, provided that $\vec{e}\cdot\vec{k} = 0$ and $|\vec{k}| = \omega$. More precisely, the gauge potential that gives the physical (i.e. real) electric and magnetic fields is given by taking the real part of $\vec{A}$ in (6.77). Thus, when we want to describe the actual physical quantities, we shall write

$$\vec{A} = a\vec{e}\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)} + a^*\vec{e}\, e^{-\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,. \tag{6.79}$$

(We have absorbed a factor of $\frac{1}{2}$ here into a rescaling of a, in order to avoid carrying $\frac{1}{2}$ factors around in all the subsequent equations.) For brevity, we shall usually write the "physical" $\vec{A}$ as

$$\vec{A} = a\vec{e}\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)} + \text{c.c.}\,, \tag{6.80}$$

where c.c stands for "complex conjugate."

Now consider a general linear superposition of monochromatic plane waves, with different wave-vectors $\vec{k}$, different polarisation vectors $\vec{e}$, and different amplitudes a. We shall therefore label the polarisation vectors and amplitudes as follows:

$$\vec{e} \longrightarrow \vec{e}_\lambda(\vec{k})\,, \qquad a \longrightarrow a_\lambda(\vec{k})\,. \tag{6.81}$$

Here λ is an index which ranges over the values 1 and 2, which labels 2 real orthonormal vectors $\vec{e}_1(\vec{k})$ and $\vec{e}_2(\vec{k})$ that span the 2-plane perpendicular to $\vec{k}$. The general wave solution can then be written as the sum over all such monochromatic plane waves of the form (6.80). Since a continuous range of wave-vectors is allowed, the summation over these will be a 3-dimensional integral. Thus we can write

$$\vec{A} = \sum_{\lambda=1}^{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \Big[\vec{e}_\lambda(\vec{k})\, a_\lambda(\vec{k})\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)} + \text{c.c.}\Big]\,, \tag{6.82}$$

where $\omega = |\vec{k}|$, and

$$\vec{k}\cdot\vec{e}_\lambda(\vec{k}) = 0\,. \tag{6.83}$$

For many purposes, it will be convenient to expand $\vec{A}$ in a basis of circularly-polarised monochromatic plane waves, rather than linearly-polarised waves. In this case, we should choose the 2-dimensional basis of polarisation vectors $\vec{\epsilon}_\pm$, related to the previous basis by

$$\vec{\epsilon}_\pm = \frac{1}{\sqrt{2}}\,(\vec{e}_1 \pm \mathrm{i}\,\vec{e}_2)\,. \tag{6.84}$$

Since we have $\vec{e}_i\cdot\vec{e}_j = \delta_{ij}$, it follows that

$$\vec{\epsilon}_+\cdot\vec{\epsilon}_+ = 0\,, \qquad \vec{\epsilon}_-\cdot\vec{\epsilon}_- = 0\,, \qquad \vec{\epsilon}_+\cdot\vec{\epsilon}_- = 1\,. \tag{6.85}$$

Note that $\vec{\epsilon}_\pm{}^* = \vec{\epsilon}_\mp$. We can label the $\vec{\epsilon}_\pm$ basis vectors by $\vec{\epsilon}_\lambda$, where λ is now understood to take the two "values" $+$ and $-$. We then write the general wave solution as

$$\vec{A} = \sum_{\lambda=\pm} \int \frac{d^3\vec{k}}{(2\pi)^3} \Big[\vec{\epsilon}_\lambda(\vec{k})\, a_\lambda(\vec{k})\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)} + \text{c.c.}\Big]\,, \tag{6.86}$$

Of course, we also have $\vec{k}\cdot\vec{\epsilon}_\lambda = 0$, and $\omega = |\vec{k}|$.

6.5.1 Helicity and energy of circularly-polarised waves

The angular-momentum tensor $M^{\mu\nu}$ for the electromagnetic field is defined by

$$M^{\mu\nu} = \int_{t=\text{const}} (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) d\Sigma_\rho\,, \tag{6.87}$$

and so the three-dimensional components M^{ij} are

$$
\begin{aligned}
M^{ij} &= \int_{t=\text{const}} (x^i T^{j\rho} - x^j T^{i\rho}) d\Sigma_\rho = \int (x^i T^{j0} - x^j T^{i0}) d^3x\,, \\
&= \int (x^i S^j - x^j S^i) d^3x\,. \qquad (6.88)
\end{aligned}
$$

Thus, since $\vec S = (\vec E \times \vec B)/(4\pi)$, the three-dimensional angular momentum $L_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$ is given by

$$
L_i = \int \epsilon_{ijk} x^j S^k\, d^3x\,, \qquad (6.89)
$$

i.e.

$$
\vec L = \frac{1}{4\pi} \int \vec r \times (\vec E \times \vec B)\, d^3x\,. \qquad (6.90)
$$

Now, since $\vec B = \vec\nabla \times \vec A$, we have

$$
\begin{aligned}
[\vec r \times (\vec E \times \vec B)]_i &= \epsilon_{ijk}\epsilon_{k\ell m}\, x_j E_\ell B_m\,, \\
&= \epsilon_{ijk}\epsilon_{k\ell m}\epsilon_{mpq}\, x_j E_\ell \partial_p A_q\,, \\
&= \epsilon_{ijk}(\delta_{kp}\delta_{\ell q} - \delta_{kq}\delta_{\ell p})\, x_j E_\ell \partial_p A_q\,, \\
&= \epsilon_{ijk}\, x_j E_\ell \partial_k A_\ell - \epsilon_{ijk}\, x_j E_\ell \partial_\ell A_k\,, \qquad (6.91)
\end{aligned}
$$

and so

$$
\begin{aligned}
L_i &= \frac{1}{4\pi}\int (\epsilon_{ijk}\, x_j E_\ell \partial_k A_\ell - \epsilon_{ijk}\, x_j E_\ell \partial_\ell A_k) d^3x\,, \\
&= \frac{1}{4\pi}\int \Big(-\epsilon_{ijk}\, \partial_k (x_j E_\ell)\, A_\ell + \partial_\ell (x_j E_\ell)\, A_k \Big) d^3x\,, \\
&= \frac{1}{4\pi}\int \Big(-\epsilon_{ijk}\, x_j (\partial_k E_\ell)\, A_\ell + \epsilon_{ijk}\, E_j A_k \Big) d^3x\,. \qquad (6.92)
\end{aligned}
$$

Note that in performing the integrations by parts here, we have, as usual, assumed that the fields fall off fast enough at infinity that the surface term can be dropped. We have also used the source-free Maxwell equation $\partial_\ell E_\ell = 0$ in getting to the final line. Thus, we conclude that the angular momentum 3-vector can be expressed as

$$
\vec L = \frac{1}{4\pi}\int (\vec E \times \vec A - A_i\, (\vec r \times \vec\nabla) E_i) d^3x\,. \qquad (6.93)
$$

The two terms in (6.93) can be interpreted as follows. The second term can be viewed as an "orbital angular momentum," since it clearly depends on the choice of origin. It is rather analogous to an $\vec r \times \vec p$ contribution to the angular momentum of a system of particles. On the other hand, the first term in (6.93) can be viewed as an "intrinsic spin" term, since it is constructed purely from the electromagnetic fields themselves, and is independent of the choice of origin. We shall calculate this spin contribution,

$$
\vec L_{\text{spin}} = \frac{1}{4\pi}\int \vec E \times \vec A\, d^3x \qquad (6.94)
$$

to the angular momentum in the case of the sum over circularly-polarised waves that we introduced in the previous section. Recall that for this sum, the 3-vector potential is given by

$$\vec{A} = \sum_{\lambda'=\pm} \int \frac{d^3\vec{k}\,'}{(2\pi)^3} \Big[\vec{\epsilon}_{\lambda'}(\vec{k}\,')\, a_{\lambda'}(\vec{k}\,')\, e^{\mathrm{i}\,(\vec{k}\,'\cdot\vec{r}-\omega' t)} + \text{c.c.}\Big]\,, \tag{6.95}$$

The electric field is then given by

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \sum_{\lambda=\pm} \int \frac{d^3\vec{k}}{(2\pi)^3} \Big[\mathrm{i}\,\omega\, \vec{\epsilon}_{\lambda}(\vec{k})\, a_{\lambda}(\vec{k})\, e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)} + \text{c.c.}\Big]\,, \tag{6.96}$$

Note that we have put primes on the summation and integration variables λ and $\vec{k}$ in the expression for $\vec{A}$. This is so that we can take the product $\vec{E}\times\vec{A}$ and not have a clash of "dummy" summation variables, in what will follow below. We have also written the frequency as $\omega' \equiv |\vec{k}\,'|$ in the expression for $\vec{A}$.

Our interest will be to calculate the time average

$$\langle\vec{L}_{\text{spin}}\rangle \equiv \frac{1}{T}\int_0^T \vec{L}_{\text{spin}} dt\,. \tag{6.97}$$

Since we are considering a wave solution with an entire "chorus" of frequencies now, we define the time average by taking T to infinity. (It is easily seen that this coincides with the previous definition of the time average for a monomchromatic wave of frequency ω, where T was taken to be $2\pi/\omega$.) Note that the time average will be zero for any quantity whose time dependence is of the oscilliatory form $e^{\mathrm{i}\,\nu t}$, because we would have

$$\frac{1}{T}\int_0^T e^{\mathrm{i}\,\nu t} dt = \frac{1}{\mathrm{i}\,\nu T}(e^{\mathrm{i}\,\nu T}-1)\,, \tag{6.98}$$

which clearly goes to zero as T goes to infinity. Since the time dependence of all the quantities we shall consider is precisely of the form $e^{\mathrm{i}\nu t}$, it follows that in order to survive the time averaging, it must be that $\nu = 0$.

We are interested in calculating the time average of $\vec{E}\times\vec{A}$, where $\vec{A}$ and $\vec{E}$ are given by (6.95) and (6.96). The quantities ω appearing there are, by definition, positive, since we have defined $\omega \equiv |\vec{k}|$. The only way that we shall get terms in $\vec{E}\times\vec{A}$ that have zero frequency (i.e. $\nu = 0$) is from the product of one of the terms that is explicitly written times one of the "c.c." terms, since these, of course, have the opposite sign for their frequency dependence.

The upshot of this discussion is that when we evaluate the time average of $\vec{E}\times\vec{A}$, with $\vec{A}$ and $\vec{E}$ given by (6.95) and (6.96), the only terms that survive will be coming from the product of the explicitly-written terms for $\vec{E}$ times the "c.c." term for $\vec{A}$, plus the "c.c."

term for $\vec{E}$ times the explicitly-written term for $\vec{A}$. Furthermore, in order for the products to have zero frequency, and therefore survive the time averaging, it must be that $\omega' = \omega$. We therefore find

$$\begin{aligned}\langle \vec{E}\times\vec{A}\rangle &= \sum_{\lambda\,\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{d^3\vec{k}\,'}{(2\pi)^3}\mathrm{i}\,\omega\,\Big[\vec{\epsilon}_\lambda(\vec{k})\times\vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}\,')a_\lambda(\vec{k})a^*_{\lambda'}(\vec{k}\,')\,e^{\mathrm{i}(\vec{k}-\vec{k}\,')\cdot\vec{r}} \\ &\qquad -\vec{\epsilon}^{\,*}_\lambda(\vec{k})\times\vec{\epsilon}_{\lambda'}(\vec{k}\,')a^*_\lambda(\vec{k})a_{\lambda'}(\vec{k}\,')\,e^{-\mathrm{i}(\vec{k}-\vec{k}\,')\cdot\vec{r}}\Big]\,. \end{aligned} \tag{6.99}$$

We now need to integrate $\langle \vec{E}\times\vec{A}\rangle$ over all 3-space, which we shall write as

$$\int \langle \vec{E}\times\vec{A}\rangle\, d^3\vec{r}\,. \tag{6.100}$$

We now make use of the result from the theory of delta functions that

$$\int e^{\mathrm{i}(\vec{k}-\vec{k}\,')\cdot\vec{r}}\, d^3\vec{r} = (2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')\,. \tag{6.101}$$

Therefore, from (6.99) we find

$$\begin{aligned}\int \langle \vec{E}\times\vec{A}\rangle\, d^3\vec{r} &= \sum_{\lambda\,\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\mathrm{i}\,\omega\,\Big[\vec{\epsilon}_\lambda(\vec{k})\times\vec{\epsilon}^{\,*}_{\lambda'}(\vec{k})a_\lambda(\vec{k})a^*_{\lambda'}(\vec{k}) \\ &\qquad -\vec{\epsilon}^{\,*}_\lambda(\vec{k})\times\vec{\epsilon}_{\lambda'}(\vec{k})a^*_\lambda(\vec{k})a_{\lambda'}(\vec{k})\Big]\,. \end{aligned} \tag{6.102}$$

Finally, we recall that the polarization vectors $\vec{\epsilon}_\pm(\vec{k})$ span the 2-dimensional space orthogonal to the wave-vector $\vec{k}$. In terms of the original real basis unit vectors $\vec{e}_1(\vec{k})$ and $\vec{e}_2(\vec{k})$ we have

$$\vec{e}_1(\vec{k})\times\vec{e}_2(\vec{k}) = \frac{\vec{k}}{\omega}\,, \tag{6.103}$$

and so it follows from (6.84) that

$$\vec{\epsilon}_+(\vec{k})\times\vec{\epsilon}^{\,*}_+(\vec{k}) = -\frac{\mathrm{i}\,\vec{k}}{\omega}\,, \qquad \vec{\epsilon}_-(\vec{k})\times\vec{\epsilon}^{\,*}_-(\vec{k}) = \frac{\mathrm{i}\,\vec{k}}{\omega}\,. \tag{6.104}$$

From this, it follows that (6.102) becomes

$$\int \langle \vec{E}\times\vec{A}\rangle\, d^3\vec{r} = 2\int \frac{d^3\vec{k}}{(2\pi)^3}\,\vec{k}\,[a_+(\vec{k})a^*_+(\vec{k}) - a_-(\vec{k})a^*_-(\vec{k})]\,, \tag{6.105}$$

and so we have

$$\langle \vec{L}_{\rm spin}\rangle = \frac{1}{2\pi}\int \frac{d^3\vec{k}}{(2\pi)^3}\,\vec{k}\,\Big(|a_+(\vec{k})|^2 - |a_-(\vec{k})|^2\Big)\,. \tag{6.106}$$

It can be seen from this result that the modes associated with the coefficients $a_+(\vec{k})$ correspond to circularly-polarised waves of positive helicity; i.e. their spin is *parallel* to the wave-vector $\vec{k}$. Conversely, the modes with coefficients $a_-(\vec{k})$ correspond to circularly-polarised waves of negative helicity; i.e. with spin that is *anti-parallel* to the wave-vector $\vec{k}$.

In a similar fashion, we may evaluate the energy of the general wave solution as a sum over the individual modes. The total energy $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{1}{8\pi}\int (E^2 + B^2) d^3x = \frac{1}{4\pi}\int E^2 d^3x\,. \tag{6.107}$$

(Recall that $E^2 = B^2$ here.) Since $\vec{E} = -\partial\vec{A}/\partial t$ here, we have

$$\begin{aligned}\langle E^2\rangle &= \sum_{\lambda,\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{d^3\vec{k}\,'}{(2\pi)^3}\,\omega^2\Big[\vec{\epsilon}_\lambda(\vec{k})\cdot\vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}\,')\,a_\lambda(\vec{k})a^*_{\lambda'}(\vec{k}\,')\,e^{\mathrm{i}\,(\vec{k}-\vec{k}\,')\cdot\vec{r}} \\ &\qquad +\vec{\epsilon}^{\,*}_\lambda(\vec{k})\cdot\vec{\epsilon}_{\lambda'}(\vec{k}\,')\,a^*_\lambda(\vec{k})a_{\lambda'}(\vec{k}\,')\,e^{-\mathrm{i}\,(\vec{k}-\vec{k}\,')\cdot\vec{r}}\Big]\,, \end{aligned}\tag{6.108}$$

where again, the time-averaging has picked out only the terms whose total frequency adds to zero. The integration over all space then again gives a three-dimensional delta function $\delta^3(\vec{k}-\vec{k}\,')$, and so we find

$$\begin{aligned}\int\langle E^2\rangle d^3\vec{r} &= \sum_{\lambda,\lambda'}\int\frac{d^3\vec{k}}{(2\pi)^3}\omega^2\Big[\vec{\epsilon}_\lambda(\vec{k})\cdot\vec{\epsilon}^{\,*}_{\lambda'}(\vec{k})\,a_\lambda(\vec{k})a^*_{\lambda'}(\vec{k}) \\ &\qquad +\vec{\epsilon}^{\,*}_\lambda(\vec{k})\cdot\vec{\epsilon}_{\lambda'}(\vec{k})\,a^*_\lambda(\vec{k})a_{\lambda'}(\vec{k})\Big]\,, \end{aligned}\tag{6.109}$$

Finally, using the orthogonality relations (6.85), and the conjugation identity $\vec{\epsilon}_\pm = \vec{\epsilon}^{\,*}_\mp$, we obtain

$$\langle\mathcal{E}\rangle = \frac{1}{2\pi}\int\frac{d^3\vec{k}}{(2\pi)^3}\,\omega^2\Big(|a_+(\vec{k})|^2 + |a_-(\vec{k})|^2\Big)\,. \tag{6.110}$$

From the two results (6.106) and (6.110), we see that for a given mode characterised by helicity λ and wave-vector $\vec{k}$, we have

$$\begin{aligned}\langle\vec{L}_{\rm spin}\rangle_{\vec{k},\lambda} &= \frac{1}{2\pi}\,\vec{k}\,|a_\lambda(\vec{k})|^2\,(\mathrm{sign}\,\lambda)\,,\\ \langle\mathcal{E}\rangle_{\vec{k},\lambda} &= \frac{1}{2\pi}\,\omega^2\,|a_\lambda(\vec{k})|^2\,, \end{aligned}\tag{6.111}$$

where $(\mathrm{sign}\,\lambda)$ is $+1$ for $\lambda = +$ and -1 for $\lambda = -$. The helicity σ, which is the component of spin along the direction of the wave-vector $\vec{k}$, is therefore given by

$$\begin{aligned}\sigma &= \frac{1}{2\pi}\,|\vec{k}|\,|a_\lambda(\vec{k})|^2\,(\mathrm{sign}\,\lambda)\,,\\ &= \frac{1}{2\pi}\,\omega\,|a_\lambda(\vec{k})|^2\,(\mathrm{sign}\,\lambda)\,,\\ &= \frac{1}{\omega}\,\langle\mathcal{E}\rangle_{\vec{k},\lambda}\,(\mathrm{sign}\,\lambda)\,. \end{aligned}\tag{6.112}$$

In other words, we have that

$$\text{energy} = \pm(\text{helicity})\,\omega\,, \tag{6.113}$$

and so we can write

$$\mathcal{E} = |\sigma|\,\omega\,. \tag{6.114}$$

This can be compared with the result in quantum mechanics, that

$$E = \hbar\,\omega\,. \tag{6.115}$$

Planck's constant $\hbar$ has the units of angular momentum, and in fact the basic "unit" of angular momentum for the photon is one unit of $\hbar$. In the transition from classical to quantum physics, the helicity of the electromagnetic field becomes the spin of the photon.

6.6 Gauge invariance and electromagnetic fields

In the previous discussion, we described electromagnetic waves in terms of the gauge potential $A_\mu = (-\phi, \vec{A})$, working in the gauge where $\phi = 0$, i.e. $A_0 = 0$. Since the gauge symmetry of Maxwell's equations is

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda\,, \tag{6.116}$$

one might think that all the gauge freedom had been used up when we imposed the condition $\phi = 0$, on the grounds that one arbitrary function (the gauge parameter λ) has been used in order to set one function (the scalar potential ϕ) to zero. This is, in fact, not the case. To see this, recall that for the electromagnetic wave we wrote $\vec{A}$ as a superposition of terms of the form

$$\vec{A} = \vec{c}\,e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{6.117}$$

which implied that

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \mathrm{i}\,\omega\,\vec{c}\,e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,. \tag{6.118}$$

From this we have

$$\vec{\nabla}\cdot\vec{E} = -\omega\vec{k}\cdot\vec{c}\,e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{6.119}$$

and so the Maxwell equation $\vec{\nabla}\cdot\vec{E} = 0$ implies that $\vec{k}\cdot\vec{c} = 0$, and hence

$$\vec{k}\cdot\vec{A} = 0\,. \tag{6.120}$$

This means that as well as having $A_0 = -\phi = 0$, we also have a component of $\vec{A}$ vanishing, namely the projection along $\vec{k}$.

To see how this can happen, it is helpful to go back to a Lorentz-covariant gauge choice instead. First, consider the Maxwell field equation, in the absence of source currents:

$$\partial^\mu F_{\mu\nu} = 0\,. \tag{6.121}$$

Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, this implies

$$\partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = 0\,. \tag{6.122}$$

We now choose the Lorenz gauge condition,

$$\partial^\mu A_\mu = 0\,. \tag{6.123}$$

The field equation (6.122) then reduces to

$$\partial^\mu \partial_\mu A_\nu = 0\,, \qquad \text{i.e.} \qquad \Box A_\mu = 0\,. \tag{6.124}$$

One might again think that all the gauge symmetry had been "used up" in imposing the Lorenz gauge condition (6.123), on the grounds that the arbitrary function λ in the gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda \tag{6.125}$$

that allowed one to impose (6.123) would no longer allow any freedom to impose further conditions on A_μ. This is not quite true, however.

To see this, let us suppose we are already in Lorenz gauge, and then try performing a further gauge transformation, as in (6.125), insisting that we must remain in the Lorenz gauge. This means that λ should satisfy

$$\partial^\mu \partial_\mu \lambda = 0\,, \qquad \text{i.e.} \qquad \Box \lambda = 0\,. \tag{6.126}$$

Non-trivial such functions λ can of course exist; any solution of the wave equation will work.

To see what this implies, let us begin with a general solution of the wave equation (6.124), working in the Lorenz gauge (6.123). We can decompose this solution as a sum over plane waves, where a typical mode in the sum is

$$A_\mu = a_\mu\, e^{\mathrm{i}\,(\vec k\cdot\vec r - \omega t)} = a_\mu\, e^{\mathrm{i}\,k_\nu x^\nu} = a_\mu\, e^{\mathrm{i}\,k\cdot x}\,, \tag{6.127}$$

where a_μ and k_ν are constant. Substituting into the wave equation (6.124) we find

$$0 = \Box A_\mu = \partial^\sigma \partial_\sigma (a_\mu\, e^{\mathrm{i}\,k_\nu x^\nu}) = -k^\sigma k_\sigma\, a_\mu\, e^{\mathrm{i}\,k_\nu x^\nu}\,, \tag{6.128}$$

whilst the Lorenz gauge condition (6.123) implies

$$0 = \partial^\mu A_\mu = \partial^\mu (a_\mu \, e^{\mathrm{i}\, k_\nu x^\nu}) = \mathrm{i}\, k^\mu a_\mu \, e^{\mathrm{i}\, k_\nu x^\nu} \,. \tag{6.129}$$

In other words, k_μ and a_μ must satisfy

$$k^\mu k_\mu = 0\,, \qquad k^\mu a_\mu = 0\,. \tag{6.130}$$

The first of these equations implies that k^μ is a null vecor, as we had seen earlier. The second equation implies that 1 of the 4 independent components that a 4-vector a_μ generically has is restricted in this case, so that a_μ has only 3 independent components.

Now we perform the further gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$, where, as discussed above, $\Box\lambda = 0$ so that we keep the gauge-transformed A_μ in Lorenz gauge. Specifically, we shall choose

$$\lambda = \mathrm{i}\, h\, e^{\mathrm{i}\, k_\nu x^\nu}\,, \tag{6.131}$$

where h is a constant. Thus we shall have

$$A_\mu \longrightarrow A_\mu - h\, k_\mu \, e^{\mathrm{i}\, k_\nu x^\nu}\,. \tag{6.132}$$

With A_μ given by (6.127) this means we shall have

$$a_\mu \, e^{\mathrm{i}\, k_\nu x^\nu} \longrightarrow a_\mu \, e^{\mathrm{i}\, k_\nu x^\nu} - h\, k_\mu \, e^{\mathrm{i}\, k_\nu x^\nu}\,, \tag{6.133}$$

which implies

$$a_\mu \longrightarrow a_\mu - h\, k_\mu\,. \tag{6.134}$$

As a check, we can see that the redefined a_μ indeed still satisfies $k^\mu a_\mu = 0$, as it should, since k^μ is a null vector.

The upshot of this discussion is that the freedom to take the constant h to be anything we like allows us to place a *second* restriction on the components of a_μ. Thus not merely are its ostensible 4 components reduced to 3 by virtue of $k^\mu a_\mu = 0$, but a further component can be eliminated by means of the residual gauge freedom, leaving just 2 independent components in the polarisation vector a_μ. Since the physical degrees of freedom are, by definition, the independent quantities that cannot be changed by making gauge transformations, we see that there are 2 degrees of freedom in the electromagnetic wave, and not 3 as one might naively have supposed.

These 2 physical degrees of freedom can be organised as the $+$ and $-$ helicity states, just as we did in our earlier discussion. These are the circularly-polarised waves rotating anti-clockwise and clockwise, respectively. In other words, these are the states whose spin is eiether parallel, or anti-parallel, to the direction of propagation. One way of understanding why we have only 2, and not 3, allowed states is that the wave is travelling at the speed of light, and so it is not possible for it to have a helicity that projects other than fully parallel or anti-parallel to its direction of propagation.

We can make contact with the $\phi = 0$ gauge choice that we made in our previous discussion of electromagnetic waves. Starting in Lorenz gauge, we make use of the residual gauge transformation (6.134) by choosing h so that

$$a_0 - h\, k_0 = 0\,, \qquad \text{i.e.} \qquad h = -\frac{a_0}{\omega}\,. \tag{6.135}$$

this means that after performing the residual gauge transformation we shall have

$$a_0 = 0\,, \tag{6.136}$$

and so, from (6.127), we shall have

$$A_0 = 0\,, \qquad \text{i.e.} \qquad \phi = 0\,. \tag{6.137}$$

The original Lorenz gauge condition (6.123) then reduces to

$$\partial_i A_i = 0\,, \qquad \text{i.e.} \qquad \vec\nabla\cdot\vec A = 0\,. \tag{6.138}$$

This implies $\vec k\cdot\vec A = 0$, and so we have reproduced precisely the $\phi = 0$, $\vec k\cdot\vec A = 0$ gauge conditions that we used previously in our analysis of the general electromagnetic wave solutions. The choice $\phi = 0$ amd $\vec\nabla\cdot\vec A = 0$ is known as *Radiation Gauge.*

In D spacetime dimensions, the analogous result can easily be seen to be that the electromagnetic wave has $(D-2)$ degrees of freedom.

6.7 Fourier decomposition of electrostatic fields

We saw earlier in 6.5 that an electromagnetic wave, expressed in the radiation gauge in terms of the 3-vector potential $\vec A$, could be decomposed into Fourier modes as in (6.86). For each mode $\vec A_{(\vec k,\lambda)}$ in the sum, we have $\vec\epsilon_\lambda(\vec k)\cdot\vec k = 0$, and so each mode of the electric field $\vec E_{(\vec k,\lambda)} = -\partial\vec A_{(\vec k,\lambda)}/\partial t$ satisfies the transversality condition

$$\vec k\cdot\vec E_{(\vec k,\lambda)} = 0\,. \tag{6.139}$$

By constrast, an *electrostatic* field $\vec E$ is longitudinal. Consider, for example, a point charge at the origin, whose potential therefore satisfies

$$\nabla^2\phi = -4\pi e\,\delta^3(\vec r)\,. \tag{6.140}$$

We can express $\phi(\vec{r})$ in terms of its Fourier transform $\Phi(\vec{k})$ as

$$\phi(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \Phi(\vec{k})\, e^{\mathrm{i}\,\vec{k}\cdot\vec{r}}\,. \tag{6.141}$$

This is clearly a sum over zero-frequency waves, as one would expect since the fields are statsic.

It follows from (6.141) that

$$\nabla^2\phi(\vec{r}) = -\int \frac{d^3\vec{k}}{(2\pi)^3}\, \vec{k}^{\,2}\, \Phi(\vec{k})\, e^{\mathrm{i}\,\vec{k}\cdot\vec{r}}\,. \tag{6.142}$$

We also note that the delta-function in (6.140) can be written as

$$\delta^3(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{\mathrm{i}\,\vec{k}\cdot\vec{r}}\,. \tag{6.143}$$

It follows that if we substitute (6.141) into (6.140) we shall obtain $-\vec{k}^{\,2}\Phi(\vec{k}) = -4\pi e$, and hence

$$\Phi(\vec{k}) = \frac{4\pi e}{k^2}\,. \tag{6.144}$$

The electric field is given by $\vec{E} = -\vec{\nabla}\phi$, and so

$$\vec{E} = -\mathrm{i}\int \frac{d^3\vec{k}}{(2\pi)^3}\, \vec{k}\, \Phi(\vec{k})\, e^{\mathrm{i}\,\vec{k}\cdot\vec{r}}\,. \tag{6.145}$$

If we define $\vec{G}(\vec{k})$ to be the Fourier transform of $\vec{E}$, so that

$$\vec{E}(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \vec{G}(\vec{k})\, e^{\mathrm{i}\,\vec{k}\cdot\vec{r}}\,, \tag{6.146}$$

then we see that

$$\vec{G}(\vec{k}) = -\mathrm{i}\,\vec{k}\,\Phi(\vec{k}) = -\frac{4\pi\,\mathrm{i}\,e}{k^2}\,\vec{k}\,. \tag{6.147}$$

Thus we see that $\vec{G}(\vec{k})$ is parallel to $\vec{k}$, which proves that the electrostatic field is *Longitudinal.*

6.8 Waveguides

For our purposes, we shall define a waveguide to be a hollow, perfectly conducting, cylinder, essentially of infinite length. For convenience we shall take the axis of th cylinder to lie along the z direction. The cross-section of the cylinder, in the (x, y) plane, can for now be arbitrary, but it is the same for all values of z. Thus, the cross-section through the cylinder is a closed curve.

We shall consider an electromagnetic wave propagting down the cylinder, with angular frequency ω. It will therefore have z and t dependence of the form

$$e^{\mathrm{i}\,(kz-\omega t)}\,. \tag{6.148}$$

Note that k and ω will not in general be equal; i.e. , the wave will not propagate at the speed of light. Note that with the z and t dependence of the form (6.148), we shall have the replacements

$$\frac{\partial}{\partial T} \longrightarrow -\mathrm{i}\,\omega\,, \qquad \frac{\partial}{\partial z} \longrightarrow \mathrm{i}\,k\,. \tag{6.149}$$

The source-free Maxwell equations (which hold inside the waveguide), therefore imply

$$\begin{aligned} \vec\nabla\cdot\vec E &= 0\,, \qquad \vec\nabla\times\vec E = \mathrm{i}\,\omega\,\vec B\,,\\ \vec\nabla\cdot\vec B &= 0\,, \qquad \vec\nabla\times\vec B = -\mathrm{i}\,\omega\,\vec E\,. \end{aligned} \tag{6.150}$$

Because of the assumed form of the z dependence in (6.148), we may write

$$\vec E(x,y,z,t) = \vec E(x,y)\, e^{\mathrm{i}\,(kz-\omega t)}\,, \qquad \vec B(x,y,z,t) = \vec B(x,y)\, e^{\mathrm{i}\,(kz-\omega t)}\,. \tag{6.151}$$

It is convenient also to define certain transverse quantities, as follows:

$$\begin{aligned} \vec\nabla_\perp &\equiv \Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},0\Big)\,,\\ \vec E &\equiv (\vec E_\perp, E_z)\,, \qquad \vec B \equiv (\vec B_\perp, B_z)\,. \end{aligned} \tag{6.152}$$

From (6.150), the Maxwell equations become

$$\begin{aligned} \vec\nabla_\perp\cdot\vec E_\perp &= -\mathrm{i}\,k\,E_z\,,\\ \vec\nabla_\perp\cdot\vec B_\perp &= -\mathrm{i}\,k\,B_z\,,\\ \mathrm{i}\,k\vec E_\perp + \mathrm{i}\,\omega\,\vec m\times\vec B_\perp &= \vec\nabla_\perp E_z\,,\\ \vec m\cdot(\vec\nabla_\perp\times\vec E_\perp) &= \mathrm{i}\,\omega\,B_z\,,\\ \mathrm{i}\,k\vec B_\perp - \mathrm{i}\,\omega\,\vec m\times\vec E_\perp &= \vec\nabla_\perp B_z\,,\\ \vec m\cdot(\vec\nabla_\perp\times\vec B_\perp) &= -\mathrm{i}\,\omega\,E_z\,, \end{aligned} \tag{6.153}$$

where we have defined the unit vector $\vec m$ along the z direction (the axis of the waveguide):

$$\vec m = (0,0,1)\,. \tag{6.154}$$

Note that the cross product of any pair of transverse vectors, $\vec U_\perp\times\vec V_\perp$, lies purely in the z direction, i.e. parallel to $\vec m$.

6.8.1 TEM modes

There are various types of modes that can be considered. First, we may dispose of an "uninteresting" possibility, called *TEM modes.* The acronym stands for "transverse electric and magnetic," meaning that

$$E_z = 0\,, \qquad B_z = 0\,. \tag{6.155}$$

From the equations in (6.153) for $\vec{E}_\perp$, we see that

$$\vec{\nabla}_\perp \cdot \vec{E}_\perp = 0\,, \qquad \vec{\nabla}_\perp \times \vec{E}_\perp = 0\,. \tag{6.156}$$

These are the equations for electrostatics in the 2-dimensional (x, y) plane. The second equation implies we can write $\vec{E}_\perp = -\vec{\nabla}_\perp \phi$, and then the first equation implies that the electrostatic potential ϕ satisfies the 2-dimensional Laplace equation

$$\nabla^2_\perp \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,. \tag{6.157}$$

Since the cross-section of the waveguide in the (x, y) plane is a closed curve, at a fixed potential (since it is a conductor), we can deduce that ϕ is constant everywhere inside the conductor:

$$0 = \int dxdy\, \phi \nabla^2_\perp \phi = -\int dxdy\, |\vec{\nabla}_\perp \phi|^2\,, \tag{6.158}$$

which implies $\vec{\nabla}_\perp \phi = 0$ inside the wavguide, and hence $\phi =$ constant and so $\vec{E} = 0$. Similar considerations imply $\vec{B} = 0$ for the TEM mode also.[13]

6.8.2 TE and TM modes

In order to have non-trivial modes propagating in the waveguide, we must relax the TEM assumption. There are two basic types of non-trivial modes we may consider, where either $\vec{E}$ or $\vec{B}$ (but not both) are taken to be transverse. These are called TE modes and TM modes respectively.

To analyse these modes, we first need to consider the boundary conditions at the conducting surface of the cylinder. The component of $\vec{E}$ parallel to the surface must vanish (seen by integrating $\vec{E}$ around a loop comprising a line segment just inside the wavguide , and closed by a line segment just inside the conductor, where $\vec{E} = 0$ by definition). Then, if we define $\vec{n}$ to be the unit normal vector at the surface, we may say that $\vec{n} \times \vec{E} = 0$. Next, taking the scalar product of $\vec{n}$ with the $\vec{\nabla} \times \vec{E} = \mathrm{i}\,\omega\, \vec{B}$ Maxwell equation, we get

$$\mathrm{i}\,\omega\, \vec{n} \cdot \vec{B} = \vec{n} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \cdot (\vec{n} \times \vec{E}) = 0\,. \tag{6.159}$$

[13]If the waveguide were replaced by coaxial conducting cylinders then TEM modes *could* exist in the gap between the innner and outer cylinder, since the potentials on the two cylinder need not be equal.

Thus, we have

$$\vec{n} \times \vec{E} = 0\,, \qquad \vec{n} \cdot \vec{B} = 0 \tag{6.160}$$

on the surface of the waveguide. We may restate these boundary conditions as

$$E_z\Big|_S = 0\,, \qquad \vec{n} \cdot \vec{B}_\perp\Big|_S = 0\,, \tag{6.161}$$

where S denotes the surface of the cylindrical waveguide.

The two boundary conditions above imply also that

$$\vec{n} \cdot \vec{\nabla}_\perp B_z\Big|_S = 0\,. \tag{6.162}$$

This follows by taking the scalar product of $\vec{n}$ with the penultimate equation in (6.153):

$$\begin{aligned} \vec{n} \cdot \vec{\nabla}_\perp B_z &= \mathrm{i}\,k\,\vec{n} \cdot \vec{B}_\perp - \mathrm{i}\,\omega \vec{n} \cdot (\vec{m} \times \vec{E}_\perp)\,, \\ &= \mathrm{i}\,k\,\vec{n} \cdot \vec{B}_\perp + \mathrm{i}\,\omega \vec{m} \cdot (\vec{n} \times \vec{E}_\perp)\,, \end{aligned} \tag{6.163}$$

and then restricting to the surface S of the cylinder. The condition (6.162) may be rewritten as

$$\frac{\partial B_z}{\partial n}\Big|_S = 0\,, \tag{6.164}$$

where $\partial/\partial n \equiv \vec{n} \cdot \vec{\nabla}$ is the normal derivative.

With the assumption (6.148), the wave equations for $\vec{E}$ and $\vec{B}$ become

$$\nabla_\perp^2 \vec{E} + (\omega^2 - k^2)\,\vec{E} = 0\,, \qquad \nabla_\perp^2 \vec{B} + (\omega^2 - k^2)\,\vec{B} = 0\,, \tag{6.165}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2-dimensional Laplacian. The third and fifth equations in (6.153) become, in terms of components,

$$\begin{aligned} &\mathrm{i}\,k\,E_x - \mathrm{i}\,\omega\,B_y = \partial_x E_z\,, \qquad \mathrm{i}\,k\,E_y + \mathrm{i}\,\omega\,B_x = \partial_y E_z\,, \\ &\mathrm{i}\,k\,B_x + \mathrm{i}\,\omega\,E_y = \partial_x B_z\,, \qquad \mathrm{i}\,k\,B_y - \mathrm{i}\,\omega\,E_x = \partial_y B_z\,. \end{aligned} \tag{6.166}$$

These can be solved for E_x, E_y, B_x and B_y in terms of E_z and B_z, giving

$$\begin{aligned} E_x &= \frac{\mathrm{i}}{\omega^2 - k^2}\,(\omega\,\partial_y B_z + k\,\partial_x E_z)\,, \\ E_y &= \frac{\mathrm{i}}{\omega^2 - k^2}\,(-\omega\,\partial_x B_z + k\,\partial_y E_z)\,, \\ B_x &= \frac{\mathrm{i}}{\omega^2 - k^2}\,(-\omega\,\partial_y E_z + k\,\partial_x B_z)\,, \\ B_y &= \frac{\mathrm{i}}{\omega^2 - k^2}\,(\omega\,\partial_x E_z + k\,\partial_y B_z)\,. \end{aligned} \tag{6.167}$$

This means that we can concentrate on solving for E_z and B_z; after having done so, substitution into (6.167) gives the expressions for E_x, E_y, B_x and B_y.

As mentioned earlier, we can now distinguish two different categories of wave solution in the waveguide. These are

$$\begin{aligned}
\textbf{TE waves}: \qquad & E_z = 0\,, \qquad \text{and} \qquad \frac{\partial B_z}{\partial n}\Big|_S = 0\,,\\
& \vec{B}_\perp = \frac{\mathrm{i}\,k}{\omega^2 - k^2}\,\vec{\nabla} B_z\,, \qquad \vec{E} = -\frac{\omega}{k}\,\vec{m}\times\vec{B}_\perp\,,
\end{aligned} \tag{6.168}$$

$$\begin{aligned}
\textbf{TM waves}: \qquad & B_z = 0\,, \qquad \text{and} \qquad E_z\Big|_S = 0\,,\\
& \vec{E}_\perp = \frac{\mathrm{i}\,k}{\omega^2 - k^2}\,\vec{\nabla} E_z\,, \qquad \vec{B} = \frac{\omega}{k}\,\vec{m}\times\vec{E}_\perp\,.
\end{aligned} \tag{6.169}$$

Note that the vanishing of E_z or B_z in the two cases means *by definition* that this field component vanishes *everywhere* inside the waveguide, and not just on the cylindrical conductor. Note also that the second condition in each case is just the residual content of the boundary conditions in (6.161) and (6.162), after having imposed the transversality condition $E_z = 0$ or $B_z = 0$ respectively. The second line in each of the TE and TM cases gives the results from (6.167), written now in a slightly more compact way. In each case, the basic wave solution is given by solving the 2-dimensional Helmholtz equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \Omega^2\psi = 0\,, \tag{6.170}$$

where

$$\Omega^2 \equiv \omega^2 - k^2\,, \tag{6.171}$$

and ψ is equal to B_z or E_z in the case of TE or TM waves respectively. We also have the boundary conditions:

$$\textbf{TE waves}: \qquad \frac{\partial\psi}{\partial n}\Big|_S = 0\,, \tag{6.172}$$

$$\textbf{TM waves}: \qquad \psi\Big|_S = 0\,. \tag{6.173}$$

Equation (6.170), together with the boundary condition (6.172) or (6.173), defines an eigenfunction/eigenvalue problem. Since the the cross-section of the waveguide is a closed loop in the (x, y) plane, the equation (6.170) is to be solved in a compact closed region, and so the eigenvalue specture for Ω^2 will be discrete; there will be a semi-infinite number of eigenvalues, unbounded above, discretely separated from each other.

Consider, as an example, TM waves propagating down a waveguide with rectangular cross-section:

$$0 \le x \le a\,, \qquad 0 \le y \le b\,. \tag{6.174}$$

For TM waves, we must satisfy the boundary condition that ψ vanishes on the edges of the rectangle. It follows from an elementary calculation, in which one separates variables in (6.170) by writing $\psi(x,y) = X(x)Y(y)$, that the eigenfunctions and eigenvalues, labelled by integers (m,n), are given by[14]

$$\begin{aligned} \psi_{mn} &= e_{mn}\, \sin\Big(\frac{m\pi x}{a}\Big)\, \sin\Big(\frac{n\pi y}{b}\Big)\,, \\ \Omega_{mn}^2 &= \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\,. \end{aligned} \tag{6.175}$$

The wave-number k and the angular frequency ω for the (m,n) mode are then related by

$$k^2 = \omega^2 - \Omega_{mn}^2\,. \tag{6.176}$$

Notice that this means there is a *minimum frequency* $\omega_{\rm min} = \Omega_{mn}$ at which a wave can propagate down the waveguide in the (m,n) mode. If one tried to transmit a lower-frequency wave in this mode, it would have imaginary wave-number, and so from (6.151) it would die off exponentially with z. This is called an *evanescent wave.*

The absolute lowest bound on the angular frequency that can propagate down the waveguide is clearly given by $\Omega_{1,1}$. In other words, the lowest angular frequency of TM wave that can propagate down the rectangular waveguide is given by

$$\omega_{\rm min} = \pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}\,. \tag{6.177}$$

In view of the relation (6.171) between the angular frequency and the wave-number, we see that the *phase velocity* $v_{\rm ph}$ and the *group velocity* $v_{\rm gr}$ are given by

$$\begin{aligned} v_{\rm ph} &= \frac{\omega}{k} = \Big(1 - \frac{\Omega^2}{\omega^2}\Big)^{-1/2}\,, \\ v_{\rm gr} &= \frac{d\omega}{dk} = \Big(1 - \frac{\Omega^2}{\omega^2}\Big)^{1/2}\,. \end{aligned} \tag{6.178}$$

Note that because of the particular form of the *dispersion relation*, i.e. the equation (6.171) relating ω to k, it is the case here that

$$v_{\rm ph}\, v_{\rm gr} = 1\,. \tag{6.179}$$

[14]If we were instead solving for TE modes, we would have the boundary condition $\partial\psi/\partial n = 0$ on the edges of the rectangle, rather than $\psi = 0$ on the edges. This would give different eigenfunctions, involving cosines rather than sines.

We see that while the group velocity satisfies

$$v_{\rm gr} \le 1\,, \tag{6.180}$$

the phase velocity satisfies

$$v_{\rm ph} \ge 1\,. \tag{6.181}$$

There is nothing wrong with this, even though it means the phase velocity exceeds the speed of light, since nothing material, and no signal, is transferred faster than the speed of light. In fact, as we shall now verify, energy and information travel at the group velocity $v_{\rm gr}$, which is always less than or equal to the speed of light.

Note that the group velocity approaches the speed of light (from below) as ω goes to infinity. To be more precise, the group velocity approaches the speed of light as ω becomes large compared to the eigenvalue Ω associated with the mode of propagation under discussion. An example where this limit is (easily) approached is if you look through a length of metal drainpipe. Electromagnetic waves in the visible spectrum have a frequency vastly greater than the lowest TM or TE modes of the drainpipe, and they propagate through the pipe as if it wasn't there. The story would be different if one tried to channel waves from a microwave down the drainpipe.

Let us now investigate the flow of energy down the waveguide. This is obtained by working out the time average of the Poynting flux,

$$\langle \vec S\rangle = \Re\Big(\frac{1}{8\pi}\vec E\times\vec B^{\,*}\Big)\,. \tag{6.182}$$

Note that here the fields $\vec E$ and $\vec B$ are taken to be complex, and we are using the result discussed earlier about taking time averages of quadratic products of the physical $\vec E$ and $\vec B$ fields.

If we consider TM modes, then we shall have

$$\begin{aligned} \vec E_\perp &= \frac{{\rm i}\,k}{\Omega^2}\,\vec\nabla\psi\,, \qquad E_z = \psi\,,\\ \vec B_\perp &= \frac{\omega}{k}\,\vec m\times\vec E_\perp = \frac{{\rm i}\,\omega}{\Omega^2}\,\vec m\times\vec\nabla\psi\,, \qquad B_z = 0\,. \end{aligned} \tag{6.183}$$

(Recall that $\vec m=(0,0,1)$.) Note that the expressions for $\vec E$ and $\vec B$ can be condensed down to

$$\vec E = \frac{{\rm i}\,k}{\Omega^2}\,\vec\nabla\psi + \vec m\,\psi\,, \qquad \vec B = \frac{{\rm i}\,\omega}{\Omega^2}\,\vec m\times\vec\nabla\psi\,. \tag{6.184}$$

We therefore have

$$\vec E\times\vec B^{\,*} = \Big(\frac{{\rm i}\,k}{\Omega^2}\,\vec\nabla\psi + \vec m\,\psi\Big)\times\Big(-\frac{{\rm i}\,\omega}{\Omega^2}\,\vec m\times\vec\nabla\psi^*\Big)\,. \tag{6.185}$$

Using the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{A}\cdot\vec{B})\,\vec{C}$, we then find

$$\vec{E}\times\vec{B}^{\,*} = \frac{\omega k}{\Omega^4}\,(\vec{\nabla}\psi\cdot\vec{\nabla}\psi^*)\,\vec{m} + \frac{\mathrm{i}\,\omega}{\Omega^2}\,\psi\,\vec{\nabla}\psi^*\,, \tag{6.186}$$

since $\vec{m}\cdot\vec{\nabla}\psi = 0$. Along the z direction (i.e. along $\vec{m}$), we therefore have

$$\langle\vec{S}\rangle_z = \frac{\omega k}{8\pi\Omega^4}\,(\vec{\nabla}\psi\cdot\vec{\nabla}\psi^*) = \frac{\omega k}{8\pi\Omega^4}\,|\vec{\nabla}\psi|^2\,. \tag{6.187}$$

(The second term in (6.186) describes the circulation of energy within the cross-sectional plane of the waveguide.)

The total transmitted power P is obtained by integrating $\langle\vec{S}\rangle_z$ over the cross-sectional area Σ of the waveguide. This gives

$$\begin{aligned}
P &= \int_\Sigma dxdy\,\langle\vec{S}\rangle_z = \frac{\omega k}{8\pi\Omega^4}\int_\Sigma dxdy\,\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi\,,\\
&= \frac{\omega k}{8\pi\Omega^4}\int_\Sigma dxdy\,\Big(\vec{\nabla}\cdot(\psi^*\,\vec{\nabla}\psi) - \psi^*\,\nabla^2\psi\Big)\,,\\
&= \frac{\omega k}{8\pi\Omega^4}\oint_C \psi^*\frac{\partial\psi}{\partial n}\,d\ell - \frac{\omega k}{8\pi\Omega^4}\int_\Sigma dxdy\,\psi^*\,\nabla^2\psi\,,\\
&= -\frac{\omega k}{8\pi\Omega^4}\int_\Sigma dxdy\,\psi^*\,\nabla^2\psi = \frac{\omega k}{8\pi\Omega^2}\int_\Sigma dxdy\,\psi^*\,\psi\,,
\end{aligned} \tag{6.188}$$

and so we have

$$P = \frac{\omega k}{8\pi\Omega^2}\int_\Sigma dxdy\,|\psi|^2\,. \tag{6.189}$$

Not that in (6.188), the boundary term over the closed loop C that forms the boundary of the waveguide in the (x,y) plane gives zero because ψ vansihes everywhere on the cylinder. The remaining term was then simplified by using (6.170).

We may also work out the total energy per unit length of the waveguide. The total time-averaged energy density is given by

$$\begin{aligned}
\langle W\rangle &= \frac{1}{8\pi}\,\vec{E}\cdot\vec{E}^{\,*} = \frac{1}{8\pi}\Big(\frac{\mathrm{i}\,k}{\Omega^2}\,\vec{\nabla}\psi + \vec{m}\,\psi\Big)\cdot\Big(-\frac{\mathrm{i}\,k}{\Omega^2}\,\vec{\nabla}\psi^* + \vec{m}\,\psi^*\Big)\,,\\
&= \frac{k^2}{8\pi\Omega^4}\,\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi + \frac{1}{8\pi}\,\psi\psi^*\,.
\end{aligned} \tag{6.190}$$

The energy per unit length U is then obtained by integrating $\langle W\rangle$ over the cross-sectional area, which gives

$$\begin{aligned}
U &= \int_\Sigma dxdy\,\langle W\rangle = \frac{k^2}{8\pi\Omega^4}\int_\Sigma dxdy\,\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi + \frac{1}{8\pi}\int_\Sigma dxdy\,|\psi|^2\,,\\
&= \frac{k^2}{8\pi\Omega^2}\int_\Sigma dxdy\,|\psi|^2 + \frac{1}{8\pi}\int_\Sigma dxdy\,|\psi|^2\,,
\end{aligned} \tag{6.191}$$

where we have again integrated by parts in the first term, dropped the boundary term because ψ vanishes on the cylinder, and used (6.170) to simplify the result. Thus we find

$$U = \frac{\omega^2}{8\pi\Omega^2}\int_\Sigma dxdy\,|\psi|^2\,. \tag{6.192}$$

Having obtained the expression (6.189) for the power P passing through the waveguide, and the expression (6.192) for the energy per unit length in the waveguide, we may note that

$$P = \frac{k}{\omega}\,U = \frac{1}{v_{\rm ph}}\,U = v_{\rm gr}\,U\,. \tag{6.193}$$

This demonstrates that the energy flows down the waveguide at the group velocity $v_{\rm gr}$.

6.9 Resonant cavities

A resonant cavity is a hollow, closed conducting "container," inside which is an electromagnetic field. A simple example would be to take a length of waveguide of the sort we have considered in section 6.8, and turn it into a closed cavity by attaching conducting plates at each end of the cylinder. Let us suppose that the length of the cavity is d.

Consider, as an example, TM modes in the cavity. We solve the same 2-dimensional Helmholtz equation (6.170) as before,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \Omega^2\,\psi = 0\,, \tag{6.194}$$

subject again to the TM boundary condition that ψ must vanish on the surface of the cyliner. The $\vec E$ and $\vec B$ fields are given, as before, by

$$\begin{aligned}
\vec E_\perp &= \frac{{\rm i}\,k}{\Omega^2}\,e^{{\rm i}\,(\kappa z-\omega t)}\,\vec\nabla\psi\,, \qquad E_z = \psi\,e^{{\rm i}\,(\kappa z-\omega t)}\,,\\
\vec B_\perp &= \frac{\omega}{k}\,\vec m\times\vec E_\perp\,,
\end{aligned} \tag{6.195}$$

where $\vec m = (0,0,1)$. Now, however, we have the additional boundary conditions that $\vec E_\perp$ must vanish on the two conductiung plates, which we shall take to be at $z=0$ and $z=d$. This is because the component of $\vec E$ parallel to a conductor must vanish at the conducting surface.

In order to arrange that $\vec E_\perp$ vanish, for all t, at $z=0$ and $z=d$, it must be that there is a superposition of right-moving and left-moving waves. (These correspond to z and t dependences $e^{{\rm i}\,(\pm\kappa z-\omega t)}$ respectively.) Thus we need to take the combination that makes a standing wave,

$$\vec E_\perp = -\frac{k}{\Omega^2}\,\sin kz\,e^{-{\rm i}\,\omega t}\,\vec\nabla\psi\,, \tag{6.196}$$

in order to have $\vec{E}_\perp = 0$ at $z = 0$. Furthermore, in order to have also that $\vec{E}_\perp = 0$ at $z = d$, it must be that the wave-number k is now quantised, according to

$$k = \frac{p\pi}{d}\,, \tag{6.197}$$

where p is an integer. Note that we also have

$$E_z = \psi\, \cos kz\, e^{-\mathrm{i}\,\omega t}\,. \tag{6.198}$$

Recall that in the waveguide, we had already found that $\Omega^2 \equiv \omega^2 - k^2$ was quantised, being restricted to a semi-infinite discrete set of eigenvalues for the 2-dimensional Helmoholtz equation. In the waveguide, that still allowed k and ω to take continuous values, subject to the constraint (dispersion relation)

$$\omega^2 = \Omega^2 + k^2\,. \tag{6.199}$$

In the resonant cavity we now have the further restriction that k is quantised, according to (6.197). This means that the spectrum of allowed frequencies ω is now discrete, and given by

$$\omega^2 = \Omega^2 + \frac{p^2\pi^2}{d^2}\,. \tag{6.200}$$

If, for example, we consider the previous example of TM modes in a rectangular waveguide whose cross-section has sides of lengths a and b, but now with the added end-caps at $z = 0$ and $z = d$, then Ω^2 is given by (6.175), and so the resonant frequencies in the cavity are given by

$$\omega^2 = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}\Big)\,, \tag{6.201}$$

for positive integers (m, n, p).

7 Fields Due to Moving Charges

7.1 Retarded potentials

If we solve the Bianchi identity by writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the remaining Maxwell equation (i.e. the field equation)

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu \tag{7.1}$$

becomes

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = -4\pi J^\nu\,. \tag{7.2}$$

If we choose to work in the Lorenz gauge,

$$\partial_\mu A^\mu = 0\,, \tag{7.3}$$

then (7.2) becomes simply

$$\Box A^\mu = -4\pi J^\mu\,. \tag{7.4}$$

Since $A^\mu = (\phi, \vec{A})$ and $J^\mu = (\rho, \vec{J})$, this means we shall have

$$\Box\phi = -4\pi\,\rho\,, \qquad \Box\vec{A} = -4\pi\,\vec{J}\,, \tag{7.5}$$

or, in the three-dimensional language,

$$\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = -4\pi\,\rho\,, \qquad \nabla^2\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} = -4\pi\,\vec{J}\,. \tag{7.6}$$

In general, we can write the solutions to (7.6) as the sums of a *particular integral* of the inhomogeneous equation (i.e. the one with the source term on the right-hand side) plus the general solution of the homogeneous equation (the one with the right-hand side set to zero). Our interest now will be in finding the particular integral. Solving this problem in the case of static sources and fields will be very familiar from electrostatics and magnetostatics. Now, however, we wish to solve for the particular integral in the case where there is time dependence too. Consider the equation for ϕ first.

First consider the situation where there is just an infinitesimal amount of charge $de(t)$ in an infinitesimal volume. (We allow for it to be time dependent, in general.) Thus the charge density is

$$\rho = de(t)\,\delta^3(\vec{R})\,, \tag{7.7}$$

where $\vec{R}$ is the position vector from the origin to the location of the infinitesimal charge. We therefore wish to solve

$$\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = -4\pi\,de(t)\,\delta^3(\vec{R})\,. \tag{7.8}$$

When $\vec{R} \neq 0$. we have simply $\nabla^2\phi - \partial^2\phi/\partial t^2 = 0$.

Clearly, ϕ depends on $\vec{R}$ only through its magnitude $R \equiv |\vec{R}|$, and so $\phi = \phi(t, R)$. Now, with $\vec{R} = (x_1, x_2, x_3)$, we have $R^2 = x_i x_i$ and so $\partial_i R = x_i/R$. Consequently, we shall have

$$\partial_i\phi = \frac{x_i}{R}\,\phi'\,, \tag{7.9}$$

where $\phi' \equiv \partial\phi/\partial R$, and then

$$\nabla^2\phi = \partial_i\partial_i\phi = \phi'' + \frac{2}{R}\,\phi'\,. \tag{7.10}$$

Letting $\Phi = R\,\phi$, we have

$$\phi' = \frac{1}{R}\,\Phi' - \frac{1}{R^2}\,\Phi\,, \qquad \phi'' = \frac{1}{R}\,\Phi'' - \frac{2}{R^2}\,\Phi' + \frac{2}{R^3}\,\Phi\,. \tag{7.11}$$

This means that for $\vec R \ne 0$, we shall have

$$\frac{\partial^2\Phi}{\partial R^2} - \frac{\partial^2\Phi}{\partial t^2} = 0\,. \tag{7.12}$$

The general solution to this equation is

$$\Phi(t,R) = f_1(t-R) + f_2(t+R)\,, \tag{7.13}$$

where f_1 and f_2 are arbitrary functions.

The solution with f_1 is called the *retarded* solution, and the solution with f_2 is called the *advanced* solution. The reason for this terminology is that in the retarded solution, the "effect" occurs after the "cause," in the sense that the profile of the function f_1 propagates *outwards* from the origin where the charge $de(t)$ is located. By contrast, in the advanced solution the effect precedes the cause; the disturbance propagates inwards as time increases. The advanced solution is *acausal*, and therefore unphysical, and so we shall keep only the causal solution, i.e. the retarded solution. The upshot is that for $R \ne 0$, the solution is

$$\phi = \frac{1}{R}\,\Phi(t-R)\,. \tag{7.14}$$

We clearly expect that ϕ will go to infinity as R approaches zero, since the charge (albeit infinitesimal) is located there. Consequently, it will be the case that the derivatives $\partial/\partial R$ will dominate over the time derivatives $\partial/\partial t$ near to $R=0$, and so in that region we can write

$$\nabla^2\phi \approx -4\pi de(t)\,\delta^3(\vec R)\,. \tag{7.15}$$

This therefore has the usual solution that is familiar from electrostatics, namely

$$\phi \approx \frac{de(t)}{R}\,, \tag{7.16}$$

or, in other words,

$$\Phi \approx de(t) \tag{7.17}$$

near $R=0$. Since Φ is already eastablished to depend on t and R only through $\Phi = \Phi(t-R)$, we can therefore immediately write down the solution valid for all R, namely

$$\Phi(t-R) = de(t-R)\,. \tag{7.18}$$

From (7.14), we therefore have that

$$\phi(\vec{R},t) = \frac{de(t-R)}{R}\,. \tag{7.19}$$

This solution is valid for the particular case of an infinitesimal charge $de(t)$ located at $R=0$. For a general time-dependent charge distribution $\rho(\vec{r},t)$, we just exploit the linearity of the Maxwell equations and sum up the contributions from all the charges in the distribution. This therefore gives

$$\phi(\vec{r},t) = \int \frac{\rho(\vec{r}\,',t-R)}{R}\, d^3\vec{r}\,'\,, \tag{7.20}$$

where $\vec{R} \equiv \vec{r} - \vec{r}\,'$. This solution of the inhomogeneous equation is the one that is "forced" by the source term, in the sense that it vanishes if the source charge density ρ vanishes. The general solution is given by this particular integral plus an arbitrary solution of the homogeneous equation $\Box\phi = 0$. The solution (7.20) can be written as

$$\phi(\vec{r},t) = \int \frac{\rho(\vec{r}\,',t-|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'\,. \tag{7.21}$$

In an identical fashion, we can see that the solution for the 3-vector potential $\vec{A}$ in the presence of a 3-vector current source $\vec{J}(\vec{r},t)$ will be

$$\vec{A}(\vec{r},t) = \int \frac{\vec{J}(\vec{r}\,',t-|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'\,. \tag{7.22}$$

The solutions for $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ that we have obtained here are called the *Retarded Potentials.* The analogous "advanced potentials" would correspond to having $t+|\vec{r}-\vec{r}\,'|$ instead of $t-|\vec{r}-\vec{r}\,'|$ as the time argument of the charge and current densities inside the integrals. It is clear that the retarded potentials are the physically sensible ones, in that the potentials at the present time t depend upon the charge and current densities at times $\le t$. In the advanced potentials, by contrast, the potentials at the current time t would be influenced by what the charge and current densities *will be* in the future. This would be unphysical, since it would violate causality.

Since the procedure by which we arrived at the retarded potential solutions(7.21) and (7.22) may have seemed slightly "unrigorous," it is perhaps worthwhile to go back and check that they are indeed correct. This can be done straightforwardly, simply by substituting them into the original wave equations (7.6). One finds that they do indeed yield exact solutions of the equations. We leave this as an exercise for the reader.

7.2 Lienard-Wiechert potentials

We now turn to a discussion of the electromagnetic fields produced by a point charge e moving along an arbitrary path $\vec{r} = \vec{r}_0(t)$. We already considered a special case of this in section 5.3, where we worked out the fields produced by a charge in uniform motion (i.e. moving at constant velocity). In that case, we could work out the electromagnetic fields by using the trick of transforming to the Lorentz frame in which the particle was at rest, doing the very simple calculation of the fields in that frame, and then transforming back to the frame where the particle was in uniform motion.

Now, we are going to study the more general case where the particle can be accelerating; i.e. , where its velocity is not uniform. This means that there does not exist an inertial frame in which the particle is at rest for all time, and so we cannot use the previous trick.

It is worth emphasising that even though the particle is accelerating, this does not mean that we cannot solve the problem using special relativity. The point is that we shall only ever study the fields from the viewpoint of an observer who *is* in an inertial frame, and so for this observer, the laws of special relativity apply. Only if we wanted to study the problem from the viewpoint of an observer in an *accelerating* frame, such as the rest-frame of the particle, would we need to use the laws of general relativity.

Note that although we cannot use special relativity to study the problem in the *rest frame* of the accelerating particle, we can, and sometimes will, make use of an *instantaneous rest frame*. This is an inertial frame whose velocity just happens to match exactly the velocity of the particle at a particular instant of time. Since the particle is accelerating, then a moment later the particle will no longer be at rest in this frame. We could, if we wished, then choose an "updated" instantaneous rest frame, and use special relativity to study the problem (for an instant) in the new inertial frame. We shall find it expedient at times to make use of the concept of an instantaneous rest frame, in order to simply intermediate calculations. Ultimately, of course, we do not want to restrict ourselves to having to hop onto a new instantaneous rest frame every time we discuss the problem, and so the goal is to obtain results that are valid in *any* inertial frame.

Now, on with the problem. We might expect, on grounds of causality, that the electromagnetic fields at $(\vec{r}, t)$ will be determined by the position and state of motion of the particle at earlier times t', as measured in the chosen inertial frame, for which the time of propagation of information from $\vec{r}_0(t')$, where the particle was at time t', to $\vec{r}$ at the time t

is $t-t'$. (But see the comments after this derivation.) It is useful therefore to define

$$\vec{R}(t') \equiv \vec{r} - \vec{r}_0(t')\,. \tag{7.23}$$

This is the radius vector from the location $\vec{r}_0(t')$ of the charge at the time t' to the observation point r. The time t' is then determined by

$$t - t' = R(t')\,, \qquad \text{where} \qquad R(t') = |\vec{R}(t')|\,. \tag{7.24}$$

There is one solution for t', for each choice of t.

In the Lorentz frame where the particle is at rest at the particular instant t', the potential at time t will, according to the argument above, be given by

$$\phi = \frac{e}{R(t')}\,, \qquad \vec{A} = 0\,. \tag{7.25}$$

We can determine the 4-vector potential A^μ in an arbitrary Lorentz frame simply by inventing a 4-vector expression that reduces to (7.25) under the specialisation that the velocity $\vec{v} \equiv d\vec{r}\,'/dt'$ of the charge is zero at time t'.

Let the 4-velocity of the charge, in the observer's inertial frame, be U^μ. If the charge is at rest, its 4-velocity will be

$$U^\mu = (1, \vec{0})\,. \tag{7.26}$$

Thus to write a 4-vector expression for $A^\mu = (\phi, \vec{A})$ that reduces to (7.25) if U^μ is given by (7.26), we just have to find a scalar f such that

$$A^\mu = f\,U^\mu\,, \tag{7.27}$$

with f becoming $e/R(t')$ in the special case. Let us define the 4-vector

$$R^\mu = (t - t', \vec{r} - \vec{r}_0(t')) = (t - t', \vec{R}(t'))\,. \tag{7.28}$$

(This is clearly a 4-vector, because $(t, \vec{r})$ is a 4-vector, and $(t', \vec{r}_0(t')$, the spacetime coordinates of the particle, is a 4-vector.) Then, we can write f as the scalar

$$f = \frac{e}{(-U^\nu\,R_\nu)}\,, \qquad \text{and so} \qquad A^\mu = -\frac{eU^\mu}{(U^\nu R_\nu)}\,, \tag{7.29}$$

since clearly if U^μ is given by (7.26), we shall have $-U^\nu\,R_\nu = -R_0 = R^0 = t - t' = R(t')$.

Having written A^μ as a 4-vector expression that reduces to (7.25) under the specialisation (7.26), we know that it must be the correct expression in any Lorentz frame. Now, we have

$$U^\mu = (\gamma, \gamma\,\vec{v})\,, \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-v^2}}\,, \tag{7.30}$$

and so we see that

$$\begin{aligned}
\phi(\vec{r},t) &= A^0 = \frac{e\gamma}{(t-t')\gamma - \gamma\,\vec{v}\cdot\vec{R}} = \frac{e}{t-t'-\vec{v}\cdot\vec{R}} = \frac{e}{R-\vec{v}\cdot\vec{R}}\,,\\
\vec{A}(\vec{r},t) &= \frac{e\gamma\,\vec{v}}{(t-t')\gamma - \gamma\,\vec{v}\cdot\vec{R}} = \frac{e\,\vec{v}}{R-\vec{v}\cdot\vec{R}}\,.
\end{aligned} \tag{7.31}$$

To summarise, we have concluded that the gauge potentials for a charge e moving along the path $\vec{r}=\vec{r}_0(t')$, as seen from the point $\vec{r}$ at time t, are given by

$$\phi(\vec{r},t) = \frac{e}{R-\vec{v}\cdot\vec{R}}\,,\qquad \vec{A}(\vec{r},t) = \frac{e\,\vec{v}}{R-\vec{v}\cdot\vec{R}}\,, \tag{7.32}$$

where all quantities on the right-hand sides are evaluated at the time t', i.e. $\vec{R}$ means $\vec{R}(t')$ and $\vec{v}$ means $d\vec{r}_0(t')/dt'$, with

$$\vec{R}(t') = \vec{r} - \vec{r}_0(t')\,, \tag{7.33}$$

and t' is determined by solving the equation

$$R(t') = t - t'\,,\qquad \text{where}\qquad R(t') \equiv |\vec{R}(t')|\,. \tag{7.34}$$

These potentials are known as the Lienard-Wiechert potentials.

The next step will be to calculate the electric and magnetic fields from the Lienard-Wiechert potentials. However, before doing so, it is perhaps worthwhile to pause and give an alternative derivation of the result for the potentials. People's taste in what constitutes a satisfying proof of a result can differ, but I have to say that I personally find the derivation above rather unsatisfying. I would regard it as a bit of hand-waving argument, which one maybe would use after having first given a proper derivation, in order to try to give a physical picture of what is going on. The basic premise of the derivation above is that the potentials "here and now" will be given precisely by applying Coulomb's law to the position the particle was in "a light-travel time" ago. I find it far from obvious that this should give the right answer.[15] It is in fact very interesting that this does give the right answer, I would view this as a remarkable fact that emerges only after one has first given a proper derivation of the result, rather than as a solid derivation in its own right.

A "proper" derivation of the Lienard-Wiechert potentials can be given as follows. We take as the starting point the expressions (7.21) and (7.22) for the retarded potentials due to

[15] In particular, I think causality can really only be used to argue that the potentials at $(t,\vec{r})$ could, *a priori*, depend on the entire past history of the particle that is in causal contact with the spacetime point $(t,\vec{r})$. Thus, although one can say that any part of its history that lies "outside the light cone" cannot affect the potentials at $(t,\vec{r})$, I don't see that, based on causality, one can say that only the instant when the particle was a light-travel distance away could be relevant for determining the potentials at $(t,\vec{r})$.

a time-dependent charge and current source. These expressions can themselves be regarded as solid and rigorous, since one only has to verify by direct substitution into (7.6) that they are indeed correct. Consider first the retarded potential for ϕ, given in (7.21). We can rewrite this as a 4-dimensional integral by introducing a delta-function in the time variable, so that

$$\phi(\vec{r},t) = \int\int \frac{\rho(\vec{r}\,',t'')}{|\vec{r}-\vec{r}\,'|}\,\delta(t''-t+|\vec{r}-\vec{r}\,'|)\,dt''\,d^3\vec{r}\,'\,. \tag{7.35}$$

The charge density for a point charge e moving along the path $\vec{r}=\vec{r}_0(t)$ is given by

$$\rho(\vec{r},t) = e\,\delta^3(\vec{r}-\vec{r}_0(t))\,. \tag{7.36}$$

This means that we shall have

$$\phi(\vec{r},t) = \int\int \frac{e\,\delta^3(\vec{r}\,'-\vec{r}_0(t''))}{|\vec{r}-\vec{r}\,'|}\,\delta(t''-t+|\vec{r}-\vec{r}\,'|)\,dt''\,d^3\vec{r}\,'\,, \tag{7.37}$$

and so after performing the spatial integrations we obtain

$$\phi(\vec{r},t) = \int \frac{e}{|\vec{r}-\vec{r}_0(t'')|}\,\delta(t''-t+|\vec{r}-\vec{r}_0(t'')|)\,dt''\,. \tag{7.38}$$

To evaluate the time integral, we need to make use of a basic result about the Dirac delta-function, namely that if a function $f(x)$ has as zero at $x=x_0$, then[16]

$$\delta(f(x)) = \delta(x-x_0)\,\Big|\frac{df}{dx}\Big|^{-1}\,, \tag{7.39}$$

where df/dx is evaluated at $x=x_0$. (The result given here is valid if $f(x)$ vanishes only at the point $x=x_0$. If it vanishes at more than one point, then there will be a sum of terms of the type given in (7.39).)

To evaluate (7.38), we note that

$$\frac{\partial}{\partial t''}\Big(t''-t+|\vec{r}-\vec{r}_0(t'')|\Big) \;=\; 1+\frac{\partial}{\partial t''}\Big((\vec{r}-\vec{r}_0(t''))\cdot(\vec{r}-\vec{r}_0(t''))\Big)^{1/2}\,,$$

[16]To prove this, consider the integral $I=\int dx h(x)\delta(f(x))$ for an arbitrary function $h(x)$. Next, change variable to $z=f(x)$, so $dx=dz/(df/dx)$. Then we have

$$I=\int dz h(x)\frac{\delta(z)}{|df/dx|} = h(x_0)/|df/dx|\int \delta(z)dz = h(x_0)/|df/dx|\,,$$

where df/dx is evaluated at $x=x_0$. Thus we have

$$I = h(x_0)/|df/dx|_{x_0} = \int dx h(x)\,\frac{\delta(x-x_0)}{|df/dx|_{x_0}}\,,$$

which proves (7.39). (The reason for the absolute-value on $|df/dz|$ is that it is to be understood that the direction of the limits of the z integration should be the standard one (negative to positive). If the gradient of f is negative at $x=x_0$ then one has to insert a minus sign to achieve this. This is therefore handled by the absolute-value sign.)

$$\begin{aligned}
&= 1 + \Big((\vec r - \vec r_0(t'')) \cdot (\vec r - \vec r_0(t''))\Big)^{-1/2} (\vec r - \vec r_0(t'')) \cdot \frac{\partial(-\vec r_0(t''))}{\partial t''}\,, \\
&= 1 - \frac{\vec v \cdot (\vec r - \vec r_0(t''))}{|\vec r - \vec r_0(t'')|}\,, \\
&= 1 - \frac{\vec v \cdot \vec R(t'')}{R(t'')}\,,
\end{aligned} \tag{7.40}$$

where $\vec v = d\vec r_0(t'')/dt''$. Following the rule (7.39) for handling a "delta-function of a function," we therefore take the function in the integrand of (7.38) that multiplies the delta-function, evaluate it at the time t for which the argument of the delta-function vanishes, and divide by the absolute value of the derivative of the argument of the delta-function. This therefore gives

$$\phi(\vec r, t) = \frac{e}{R(t') - \vec v \cdot \vec R(t')}\,, \tag{7.41}$$

where t' is the solution of $t - t' = R(t')$, and so we have reproduced the previous expression for the Lienard-Wiechert potential for ϕ in (7.32). The derivation for $\vec A$ is very similar.

7.3 Electric and magnetic fields of a moving charge

Having obtained the Lienard-Wiechert potentials ϕ and $\vec A$ of a moving charge, the next step is to calculate the associated electric and magnetic fields,

$$\vec E = -\vec\nabla\phi - \frac{\partial \vec A}{\partial t}\,, \qquad \vec B = \vec\nabla \times \vec A\,. \tag{7.42}$$

To do this, we shall need the following results. First, we note that

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\,\frac{\partial t'}{\partial t}\,, \tag{7.43}$$

and so, since $R^2 = R_i R_i$ we have

$$\frac{\partial R}{\partial t'} = \frac{R_i}{R}\,\frac{\partial R_i}{\partial t'} = -\frac{v_i(t')\,R_i}{R} = -\frac{\vec v \cdot \vec R}{R}\,. \tag{7.44}$$

(Recall that $\vec R$ means $\vec R(t')$, and that it is given by (7.33).) Equation (7.43) therefore becomes

$$\frac{\partial R}{\partial t} = -\frac{\vec v \cdot \vec R}{R}\,\frac{\partial t'}{\partial t}\,, \tag{7.45}$$

and so, since we have from (7.34) that $R(t') = t - t'$, it follows that

$$1 - \frac{\partial t'}{\partial t} = -\frac{\vec v \cdot \vec R}{R}\,\frac{\partial t'}{\partial t}\,. \tag{7.46}$$

Solving for $\partial t'/\partial t$, we therefore have the results that

$$\frac{\partial t'}{\partial t} = \Big(1 - \frac{\vec v \cdot \vec R}{R}\Big)^{-1}\,, \tag{7.47}$$

$$\frac{\partial R}{\partial t} = -\frac{\vec v \cdot \vec R}{R - \vec v \cdot \vec R}\,. \tag{7.48}$$

Some other expressions we shall also need are as follows. First, from $t - t' = R(t')$ it follows that $\partial_i t' = -\partial_i R(t')$. Now $\vec{R}(t') = \vec{r} - \vec{r}_0(t')$, and so

$$R^2 = (x_j - x_j^0(t'))(x_j - x_j^0(t'))\,. \tag{7.49}$$

From this, by acting with ∂_i, we obtain

$$\begin{aligned} 2R\partial_i R &= 2(\delta_{ij} - \partial_i x_j^0(t'))(x_j - x_j^0(t'))\,, \\ &= 2R_i - 2\frac{\partial x_j^0(t')}{\partial t'}\,\frac{\partial t'}{\partial x_i}\,(x_j - x_j^0(t'))\,, \\ &= 2R_i - 2\vec{v}\cdot\vec{R}\,\partial_i t'\,. \end{aligned} \tag{7.50}$$

From this and $\partial_i t' = -\partial_i R(t')$ it follows that

$$\partial_i t' = -\frac{R_i}{R - \vec{v}\cdot\vec{R}}\,, \qquad \partial_i R = \frac{R_i}{R - \vec{v}\cdot\vec{R}}\,. \tag{7.51}$$

Further results that follow straightforwardly are

$$\begin{aligned} \partial_i R_j &= \partial_i(x_j - x_j^0(t')) = \delta_{ij} - \frac{\partial x_j^0(t')}{\partial t'}\,\partial_i t' = \delta_{ij} + \frac{v_j R_i}{R - \vec{v}\cdot\vec{R}}\,, \\ \partial_i v_j &= \frac{\partial v_j}{\partial t'}\,\partial_i t' = -\frac{\dot{v}_j\,R_i}{R - \vec{v}\cdot\vec{R}}\,, \\ \frac{\partial v_i}{\partial t} &= \frac{\partial v_i}{\partial t'}\,\frac{\partial t'}{\partial t} = \frac{\dot{v}_i R}{R - \vec{v}\cdot\vec{R}}\,, \\ \frac{\partial R}{\partial t} &= -\frac{\vec{v}\cdot\vec{R}}{R - \vec{v}\cdot\vec{R}}\,, \\ \frac{\partial \vec{R}}{\partial t} &= \frac{\partial \vec{R}}{\partial t'}\,\frac{\partial t'}{\partial t} = -\vec{v}\,\frac{\partial t'}{\partial t} = -\frac{\vec{v}\,R}{R - \vec{v}\cdot\vec{R}}\,. \end{aligned} \tag{7.52}$$

Note that $\dot{v}_i$ means $\partial v_i/\partial t'$; we shall define the acceleration $\vec{a}$ of the particle by

$$\vec{a} \equiv \frac{\partial \vec{v}}{\partial t'}\,. \tag{7.53}$$

We are now ready to evaluate the electric and magnetic fields. From (7.32) and the results above, we have

$$\begin{aligned} E_i &= -\partial_i \phi - \frac{\partial A_i}{\partial t}\,, \\ &= \frac{e}{(R - \vec{v}\cdot\vec{R})^2}\,(\partial_i R - \partial_i(v_j R_j)) - \frac{e}{R - \vec{v}\cdot\vec{R}}\,\frac{\partial v_i}{\partial t} + \frac{e v_i}{(R - \vec{v}\cdot\vec{R})^2}\Big(\frac{\partial R}{\partial t} - \frac{\partial(\vec{v}\cdot\vec{R})}{\partial t}\Big)\,, \\ &= \frac{e}{(R - \vec{v}\cdot\vec{R})^3}\Big\{R_i - v_i(R - \vec{v}\cdot\vec{R}) - v^2 R_i + \vec{a}\cdot\vec{R}\,R_i - a_i\,R\,(R - \vec{v}\cdot\vec{R}) \\ &\qquad\qquad -v_i\vec{v}\cdot\vec{R} - v_i\vec{a}\cdot\vec{R}\,R + v^2 v_i R\Big\}\,, \\ &= \frac{e(1 - v^2)(R_i - v_i\,R)}{(R - \vec{v}\cdot\vec{R})^3} + \frac{e[\vec{a}\cdot\vec{R}\,(R_i - v_i\,R) - a_i(R - \vec{v}\cdot\vec{R})\,R]}{(R - \vec{v}\cdot\vec{R})^3}\,. \end{aligned} \tag{7.54}$$

This can be rewritten as

$$\vec{E} = \frac{e(1-v^2)(\vec{R} - \vec{v}\, R)}{(R - \vec{v}\cdot\vec{R})^3} + \frac{e\vec{R}\times[(\vec{R} - \vec{v}\, R)\times\vec{a}]}{(R - \vec{v}\cdot\vec{R})^3}\,. \tag{7.55}$$

An analogous calculation of $\vec{B}$ shows that it can be written as

$$\vec{B} = \frac{\vec{R}\times\vec{E}}{R}\,. \tag{7.56}$$

Note that this means that $\vec{B}$ is perpendicular to $\vec{E}$.

The first term in (7.55) is independent of the acceleration $\vec{a}$, and so it represents a contribution that is present even if the charge is in uniform motion. It is easily seen that at large distance, where $R \to \infty$, it falls off like $1/R^2$. If the charge is moving with uniform velocity $\vec{v}$ then we shall have

$$\vec{r}_0(t) = \vec{r}_0(t') + \vec{v}\,(t - t')\,, \tag{7.57}$$

and so

$$\begin{aligned} \vec{R}(t') - \vec{v}\, R(t') &= \vec{r} - \vec{r}_0(t') - \vec{v}\,(t - t')\,, \\ &= \vec{r} - \vec{r}_0(t) + \vec{v}\,(t - t') - \vec{v}\,(t - t')\,, \\ &= \vec{R}(t)\,. \end{aligned} \tag{7.58}$$

In other words, in this case of uniform motion, $\vec{R}(t') - \vec{v}\, R(t')$ is equal to the vector $\vec{R}(t)$ that gives the line joining the charge to the point of observation *at the time the observation is made.* We shall also then have

$$\begin{aligned} R(t') - \vec{v}\cdot\vec{R}(t') &= R(t') - v^2 R(t') - \vec{v}\cdot\vec{R}(t)\,, \\ &= (1 - v^2) R(t') - \vec{v}\cdot\vec{R}(t)\,. \end{aligned} \tag{7.59}$$

If we now introduce the angle θ between $\vec{v}$ and $\vec{R}(t)$, we shall have $\vec{v}\cdot\vec{R}(t) = v\, R(t)\, \cos\theta$. Since, as we saw above, $\vec{R}(t') = \vec{v}\, R(t') + \vec{R}(t)$, we obtain, by squaring,

$$R^2(t') = v^2 R^2(t') + 2v R(t) R(t') \cos\theta + R^2(t)\,, \tag{7.60}$$

and this quadratic equation for $R(t')$ can be solved to give

$$R(t') = \frac{v R(t)\cos\theta + R(t)\sqrt{1 - v^2\sin^2\theta}}{1 - v^2}\,. \tag{7.61}$$

Equation (7.59) then gives

$$R(t') - \vec{v}\cdot\vec{R}(t') = v R(t)\cos\theta + R(t)\sqrt{1 - v^2\sin^2\theta} - v R(t)\cos\theta = R(t)\sqrt{1 - v^2\sin^2\theta}\,. \tag{7.62}$$

For a uniformly moving charge we therefore obtain the result

$$\vec{E} = \frac{e\vec{R}(t)}{R^3(t)} \, \frac{1-v^2}{(1-v^2\sin^2\theta)^{3/2}}\,, \tag{7.63}$$

which has reproduced the result (5.40) that we had previously obtained by boosting from the rest frame of the charged particle.

The second term in (7.55) is proportional to $\vec{a}$, and so it occurs only for an accelerating charge. At large distance, this term falls off like $1/R$, in other words, much less rapidly than the $1/R^2$ fall-off of the first term in (7.55). In fact the $1/R$ fall-off of the acceleration term is characteristic of an electromagnetic wave, as we shall now discuss.

7.4 Radiation by accelerated charges

A charge at rest generates a purely electric field, and if it is in uniform motion it generates both $\vec{E}$ and $\vec{B}$ fields. In neither case, of course, does it radiate any energy. However, if the charge is *accelerating*, then it actually emits electromagnetic radiation.

The easiest case to consider is when the velocity of the charge is small compared with the speed of light. In this case the acceleration term in (7.55) is approximated by

$$\vec{E} = \frac{e\vec{R}\times(\vec{R}\times a)}{R^3} = \frac{e\vec{n}\times(\vec{n}\times\vec{a})}{R}\,, \tag{7.64}$$

where

$$\vec{n} \equiv \frac{\vec{R}}{R}\,. \tag{7.65}$$

Note that $\vec{n}\cdot\vec{E} = 0$, and that $\vec{E}$ is also perpendicular to $\vec{n}\times\vec{a}$. This means that the polarisation of $\vec{E}$ lies in the plane containing $\vec{n}$ and $\vec{a}$, and is perpendicular to $\vec{n}$.

From (7.56) we shall also have

$$\vec{B} = \vec{n}\times\vec{E}\,. \tag{7.66}$$

As usual, all quantities here in the expressions for $\vec{E}$ and $\vec{B}$ are evaluated at the retarded time t'.

The energy flux, given by the Poynting vector, is given by

$$\vec{S} = \frac{1}{4\pi}\vec{E}\times\vec{B} = \frac{1}{4\pi}\,\vec{E}\times(\vec{n}\times\vec{E}) = \frac{1}{4\pi}E^2\,\vec{n} - \frac{1}{4\pi}(\vec{n}\cdot\vec{E})\,\vec{E}\,, \tag{7.67}$$

and so, since $\vec{n}\cdot\vec{E}=0$ we have

$$\vec{S} = \frac{1}{4\pi}\,E^2\,\vec{n}\,. \tag{7.68}$$

Let us define θ to be the angle between the unit vector $\vec{n}$ and the acceleration $\vec{a}$. Then we shall have

$$\vec{E} = \frac{e}{R}\,(\vec{n}\cdot\vec{a}\,\vec{n} - \vec{a}) = \frac{e}{R}\,(a\,\vec{n}\,\cos\theta - \vec{a})\,, \tag{7.69}$$

and so

$$E^2 = \frac{e^2}{R^2}\,(a^2\cos^2\theta - 2a^2\cos^2\theta + a^2) = \frac{e^2a^2\sin^2\theta}{R^2}\,, \tag{7.70}$$

implying that the energy flux is

$$\vec S = \frac{e^2a^2\sin^2\theta}{4\pi R^2}\,\vec n\,. \tag{7.71}$$

The area element $d\vec\Sigma$ can be written as

$$d\vec\Sigma = R^2\,\vec n\,d\Omega\,, \tag{7.72}$$

where $d\Omega = \sin\theta\,d\theta d\varphi$ is the area element on the unit-radius sphere (i.e. the solid angle element). The power radiated into the area element $d\vec\Sigma$ is $dP = \vec S\cdot d\vec\Sigma = R^2\vec n\cdot\vec S\,d\Omega$, and so we find that

$$\frac{dP}{d\Omega} = \frac{e^2a^2}{4\pi}\,\sin^2\theta \tag{7.73}$$

is the power radiated per unit solid angle.

The total power radiated in all directions is given by

$$\begin{aligned} P &= \int \frac{dP}{d\Omega}\,d\Omega = \frac{e^2a^2}{4\pi}\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi} d\varphi\,,\\ &= \tfrac12 e^2a^2\int_0^\pi \sin^3\theta\,d\theta = \tfrac12 e^2a^2\int_{-1}^1(1-c^2)dc = \tfrac23 e^2a^2\,, \end{aligned} \tag{7.74}$$

where, to evaluate the θ integral we change variable to $c = \cos\theta$. The expression

$$P = \tfrac23 e^2a^2 \tag{7.75}$$

is known as the *Larmor Formula* for a non-relativistic accelerating charge.

The Larmor formula can be generalised to the relativistic result fairly easily. In principle, we could simply repeat the argument given above, but without making the approximation that v is small compared to 1 (the speed of light). Note that in terms of the unit vector $\vec n = \vec R/R$, the expression (7.55) for the electric field becomes

$$\vec E = \frac{e(1-v^2)(\vec n - \vec v)}{R^2\,(1-\vec n\cdot\vec v)^3} + \frac{e\vec n\times[(\vec n-\vec v)\times\vec a]}{R\,(1-\vec n\cdot\vec v)^3}\,. \tag{7.76}$$

We can, in fact, obtain the relativisitic Larmor formula by a simple trick. First, we note from (7.76) that since $\vec S = (\vec E\times\vec B)/(4\pi)$ and $\vec B = \vec n\times\vec E$, the energy flux from the acceleration term must be quadratic in the acceleration $\vec a$. We can also note that the total radiated power P is a Lorentz scalar (since it is energy per unit time, and each of these quantities transforms as the 0 component of a 4-vector). Thus, the task is to find a Lorentz-invariant expression for P that reduces to the non-relativisitic Larmor result (7.75) in the limit when v goes to zero.

First, we note that the non-relativistic Larmor formula (7.75) can be written as

$$P = \tfrac{2}{3} e^2 a^2 = \frac{2e^2}{3m^2} \Big(\frac{d\vec p}{dt}\Big)^2 . \tag{7.77}$$

There is only one Lorentz-invariant quantity, quadratic in $\vec a$, that reduces to this expression in the limit that v goes to zero. It is given by

$$P = \frac{2e^2}{3m^2} \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau} , \tag{7.78}$$

where p^μ is the 4-momentum of the particle and τ is the proper time along its path. Noting that $p^\mu = m(\gamma, \gamma\vec v)$, we see that

$$\frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt} = m\gamma(\gamma^3 \vec v\cdot\vec a, \gamma^3 (\vec v\cdot\vec a)\, \vec v + \gamma\vec a) , \tag{7.79}$$

and so

$$\begin{aligned} \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau} &= m^2\gamma^2[-\gamma^6 (\vec v\cdot\vec a)^2 + \gamma^6 v^2 (\vec v\cdot\vec a)^2 + 2\gamma^4 (\vec v\cdot\vec a)^2 + \gamma^2 a^2] , \\ &= m^2\gamma^2[\gamma^4 (\vec v\cdot\vec a)^2 + \gamma^2 a^2) . \end{aligned} \tag{7.80}$$

Now consider the quantity

$$\begin{aligned} a^2 - (\vec v\times\vec a)^2 &= a^2 - \epsilon_{ijk}\epsilon_{i\ell m} v_j a_k v_\ell a_m , \\ &= a^2 - v^2 a^2 + (\vec v\cdot\vec a)^2 = \frac{a^2}{\gamma^2} + (\vec v\cdot\vec a)^2 , \end{aligned} \tag{7.81}$$

which shows that we can write

$$\frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau} = m^2\gamma^6 \Big(\frac{a^2}{\gamma^2} + (\vec v\cdot\vec a)^2\Big) = m^2\gamma^6\, [a^2 - (\vec v\times\vec a)^2] . \tag{7.82}$$

Thus we see that the scalar P given in (7.78) is given by

$$P = \tfrac{2}{3} e^2\gamma^6 [a^2 - (\vec v\times\vec a)^2] . \tag{7.83}$$

This indeed reduces to the non-relativistic Larmor formula (7.75) if the velocity $\vec v$ is sent to zero. For the reasons we described above, it must therefore be the correct fully-relativistic Larmor result for the total power radiated by an accelerating charge.

7.5 Applications of Larmor formula

7.5.1 Linear accelerator

In a linear accelerator, a charged massive particle is accelerated along a straight-line trajectory, and so its velocity $\vec v$ and acceleration $\vec a$ are parallel. Defining $p = |\vec p| = m\gamma|\vec v|$, we have

$$\frac{dp}{dt} = m\gamma \frac{dv}{dt} + mv\frac{d\gamma}{dt} , \tag{7.84}$$

where $v = |\vec{v}|$ and $\gamma = (1 - v^2)^{-1/2}$. Clearly we have

$$v \frac{dv}{dt} = \vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \vec{a} = va\,, \qquad \frac{d\gamma}{dt} = \gamma^3 \vec{v} \cdot \frac{d\vec{v}}{dt} = \gamma^3 va\,, \tag{7.85}$$

and so

$$\frac{dp}{dt} = m\gamma^3 a\,. \tag{7.86}$$

With $\vec{v}$ and $\vec{a}$ parallel, the relativisitic Larmor formula (7.83) gives $P = \frac{2}{3} e^2 \gamma^6 a^2$, and so we have

$$P = \frac{2e^2}{3m^2} \Big(\frac{dp}{dt}\Big)^2\,. \tag{7.87}$$

The expression (7.87) gives the power that is radiated by the charge as it is accelerated along a straight line trajectory. In a particle accelerator, the goal, obviously, is to accelerate the particles to as high a velocity as possible. Equation (7.87) describes the the power that is lost through radiation when the particle is being accelerated. The energy $\mathcal{E}$ of the particle is related to its rest mass m and 3-momentum $\vec{p}$ by the standard formula

$$\mathcal{E}^2 = p^2 + m^2\,. \tag{7.88}$$

The rate of change of energy with distance travelled, $d\mathcal{E}/dx$, is therefore given by

$$\mathcal{E} \frac{d\mathcal{E}}{dx} = p \frac{dp}{dx}\,, \tag{7.89}$$

and so we have

$$\frac{d\mathcal{E}}{dx} = \frac{p}{\mathcal{E}} \frac{dp}{dx} = \frac{m\gamma v}{m\gamma} \frac{dp}{dx} = v \frac{dp}{dx} = \frac{dx}{dt} \frac{dp}{dx} = \frac{dp}{dt}\,. \tag{7.90}$$

This means that (7.87) can be rewritten as

$$P = \frac{2e^2}{3m^2} \Big(\frac{d\mathcal{E}}{dx}\Big)^2\,. \tag{7.91}$$

The "energy-loss factor" of the accelerator can be judged by taking the ratio of the power radiated divided by the power supplied. By energy conservation, the power supplied is equal to the rate of change of energy of the particle, $d\mathcal{E}/dt$. Thus we have

$$\begin{aligned} \frac{\text{Power radiated}}{\text{Power supplied}} &= \frac{P}{(d\mathcal{E}/dt)} = \frac{P}{(d\mathcal{E}/dx)} \frac{dt}{dx} = \frac{P}{v\,(d\mathcal{E}/dx)}\,, \\ &= \frac{2e^2}{3m^2 v} \frac{d\mathcal{E}}{dx}\,. \end{aligned} \tag{7.92}$$

In the relativistic limit, where v is very close to the speed of light (as is typically achieved in a powerful linear accelerator), we therefore have

$$\frac{\text{Power radiated}}{\text{Power supplied}} \approx \frac{2e^2}{3m^2} \frac{d\mathcal{E}}{dx}\,. \tag{7.93}$$

A typical electron linear accelerator achieves an energy input of about 10 MeV per metre, and this translates into an energy-loss factor of about 10^{-13}. In other words, very little of the applied power being used to accelerate the electron is lost through Larmor radiation.

7.5.2 Circular accelerator

The situation is very different in the case of a circular accelerator, since the transverse acceleration necessary to keep the particle in a circular orbit is typically very much larger than the linear acceleration discussed above. In other words, the direction of the 3-momemtum $\vec{p}$ is changing rapidly, while, by contrast, the energy, and hence the magnitude of $\vec{p}$, is relatively slowly-changing. In fact the change in $|\vec{p}|$ per revolution is rather small, and we can study the power loss by assuming that the particle is in an orbit of fixed angular frequency ω. This means that we shall have

$$\left|\frac{d\vec{p}}{dt}\right| = \omega\,|\vec{p}|\,, \tag{7.94}$$

and so

$$\left|\frac{d\vec{p}}{d\tau}\right| = \gamma\omega\,|\vec{p}|\,, \tag{7.95}$$

where $d\tau = dt/\gamma$ is the proper-time interval. Since the energy is constant in this approximation, we therefore have

$$\frac{dp^0}{d\tau} = 0\,, \qquad \text{and so} \qquad \frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \Big(\frac{d\vec{p}}{d\tau}\Big)^2 = \gamma^2\omega^2 p^2\,. \tag{7.96}$$

Using equation (7.78) for the Larmor power radiation, we therefore have

$$P = \frac{2e^2}{3m^2}\gamma^2\omega^2 p^2 = \tfrac{2}{3}e^2\gamma^4\omega^2 v^2\,. \tag{7.97}$$

If the radius of the accelerator is R then the angular and linear velocities of the particle are related by $\omega = v/R$ and so the power loss is given by

$$P = \frac{2e^2\gamma^4 v^4}{3R^2}\,. \tag{7.98}$$

The radiative energy loss per revolution, $\Delta\mathcal{E}$, is given by the product of P with the period of the orbit, namely

$$\Delta\mathcal{E} = \frac{2\pi R P}{v} = \frac{4\pi e^2\gamma^4 v^3}{3R}\,. \tag{7.99}$$

A typical example would be a 10 GeV electron synchrotron, for which the radius R is about 100 metres. Plugging in the numbers, this implies an energy loss of about 10 MeV per revolution, or about 0.1% of the energy of the particle. Bearing in mind that the time taken to complete an orbit is very small (the electron is travelling at nearly the speed of light), it is necessary to supply energy at a very high rate in order to replenish the radiative loss. It also implies that there will be a considerable amount of radiation being emitted by the accelerator.

7.6 Angular distribution of the radiated power

We saw previously that for a non-relativistic charged particle whose acceleration $\vec{a}$ makes an angle θ with respect to the position vector $\vec{R}$, the angular distribution of the radiated power is given by (see (7.73))

$$\frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi}\,\sin^2\theta\,. \tag{7.100}$$

In the general (i.e. relativistic) case, where the velocity $\vec{v}$ is large, the we have, from (7.76), that at large R the electric and magnetic fields are dominated by the radiation-field term:

$$\vec{E} = \frac{e\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{R\,(1-\vec{n}\cdot\vec{v})^3}\,,\qquad \vec{B} = \vec{n}\times\vec{E}\,. \tag{7.101}$$

The radial component of the Poynting vector, $\vec{n}\cdot\vec{S}$, is therefore given by

$$\begin{aligned}\vec{n}\cdot\vec{S} &= \frac{1}{4\pi}\,\vec{n}\cdot(\vec{E}\times\vec{B}) = \frac{1}{4\pi}\vec{n}\cdot[\vec{E}\times(\vec{n}\times\vec{E})]\,,\\ &= \frac{1}{4\pi}\vec{n}\cdot[E^2\,\vec{n}-(\vec{n}\cdot\vec{E})\,\vec{E}] = \frac{1}{4\pi}E^2\,,\end{aligned} \tag{7.102}$$

since $\vec{n}\cdot\vec{E}=0$. Thus we have

$$\vec{n}\cdot\vec{S} = \frac{e^2}{4\pi R^2}\,\Big|\frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^3}\Big|^2\,, \tag{7.103}$$

where as usual all quantities on the right-hand side are evaluated at the retarded time t' calculated from the equation $t-t'=R(t')$, with $\vec{R}(t')=\vec{r}-\vec{r}_0(t')$. It is conventional to denote the quantity in (7.103) by $[\vec{n}\cdot\vec{S}]_{\rm ret.}$, to indicate that it is evaluated at the retarded time t'.

The associated energy radiated during the time interval from $t=T_1$ to $t=T_2$ is therefore given by

$$\mathcal{E} = \int_{T_1}^{T_2}[\vec{n}\cdot\vec{S}]_{\rm ret.}dt\,, \tag{7.104}$$

Defining the corresponding retarded times $t'=T_i'$, the integral can therefore be rewritten as[17]

$$\mathcal{E} = \int_{T_1'}^{T_2'}[\vec{n}\cdot\vec{S}]_{\rm ret.}\frac{dt}{dt'}\,dt'\,. \tag{7.105}$$

The quantity $[\vec{n}\cdot\vec{S}]_{\rm ret.}(dt/dt')$ is the power radiated per unit area as measured with respect to the charge's retarded time t', and so we have the result that

$$\frac{dP(t')}{d\Omega} = R^2[\vec{n}\cdot\vec{S}]_{\rm ret.}\frac{dt}{dt'} = R^2(1-\vec{n}\cdot\vec{v})[\vec{n}\cdot\vec{S}]_{\rm ret.}\,. \tag{7.106}$$

(Note that we used the result (7.47) here.)

[17] All that is really being said here is that we can relate the previously-defined quantity $dP/d\Omega$ (power per unit solid angle as measured by the observer at time t) to $dP(t')/d\Omega$ (power per unit solid angle as measured at the particle, at retarded time t') by $dP/d\Omega\,dt = dP/d\Omega\,(dt/dt')\,dt' \equiv dP(t')/d\Omega\,dt'$.

7.6.1 Angular power distribution for linear acceleration

As an example, consider the situation when the charge is accelerated uniformly for only a short time, so that $\vec{v}$ as well as $\vec{a}$ are approximately constant during the time interval of the acceleration. This means that $\vec{n}$ and R are approximately constant, and so from (7.103) and (7.106) we obtain the angular distribution

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi} \frac{|\vec{n} \times [(\vec{n} - \vec{v}) \times \vec{a}]|^2}{(1 - \vec{n} \cdot \vec{v})^5} \,. \tag{7.107}$$

If we now suppose that the acceleration is linear, i.e. that $\vec{v}$ and $\vec{a}$ are parallel, then we obtain

$$\frac{dP(t')}{d\Omega} = \frac{e^2 a^2}{4\pi} \frac{\sin^2\theta}{(1 - v\cos\theta)^5} \,, \tag{7.108}$$

where as before we define θ to be the angle between $\vec{a}$ and $\vec{n}$.

When $|v| << 1$, the expression (7.108) clearly reduces to the non-relativistic result given in (7.73). In this limit, the angular radiated power distribution is described by a figure-of-eight, oriented perpendicularly to the direction of the acceleration. As the velocity becomes larger, the two lobes of the figure-of-eight start to tilt forwards, along the direction of the acceleration. This is illustrated for the non-relativistic and relativisitic cases in Figures 1 and 2 below. In each case, the acceleration is to the right along the horizontal axis.

The angle at which the radiated power is largest is found by solving $d(dP/d\Omega)/d\theta = 0$. This gives

$$2(1 - v\cos\theta)\cos\theta - 5v\sin^2\theta = 0 \,, \tag{7.109}$$

and hence

$$\theta_{\text{max.}} = \arccos\Big(\frac{\sqrt{1 + 15v^2} - 1}{3v}\Big) \,. \tag{7.110}$$

In the case of a highly relativistic particle, for which v is very close to the speed of light, the velocity itself is not a very convenient parameter, and instead we can more usefully characterise it by $\gamma = (1 - v^2)^{-1/2}$, which becomes very large in the relativistic limit. Thus, substituting $v = \sqrt{1 - \gamma^{-2}}$ into (7.110), we obtain

$$\theta_{\text{max.}} = \arccos\Big(\frac{4\sqrt{1 - \frac{15}{16}\gamma^{-2}} - 1}{3\sqrt{1 - \gamma^{-2}}}\Big) \,. \tag{7.111}$$

At large γ we can expand the argument as a power series in γ^{-2}, finding that

$$\theta_{\text{max.}} \approx \arccos(1 - \tfrac{1}{8}\gamma^{-2}) \,. \tag{7.112}$$

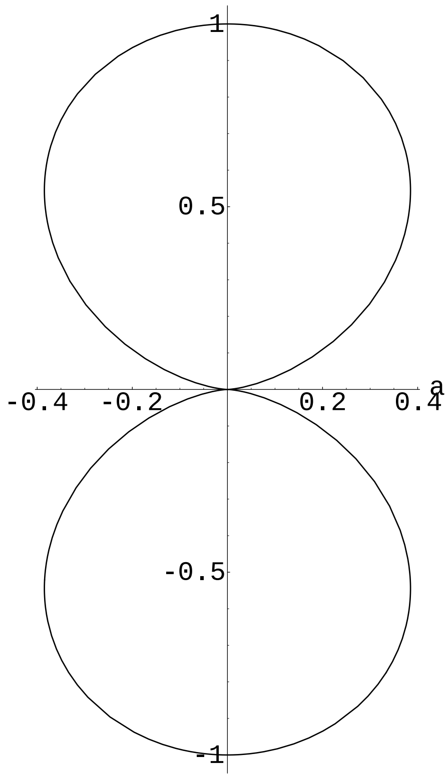

Figure 1: The angular power distribution in the non-relativistic case

This implies that $\theta_{\rm max.}$ is close to 0 when γ is very large. In this regime we have $\cos\theta_{\rm max.} \approx 1 - \frac{1}{2}\theta^2_{\rm max.}$, and so in the highly relativistic case we have

$$\theta_{\rm max.} \approx \frac{1}{2\gamma}\,. \tag{7.113}$$

We see that the lobes of the angular power distribution tilt forward sharply, so that they are directed nearly parallel to the direction of acceleration of the particle.

Continuing with the highly-relativistic limit, we may consider the profile of the angular power distribution for all small angles θ. Substituting

$$v = \sqrt{1-\gamma^{-2}}\,, \qquad \sin\theta \approx \theta\,, \qquad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2 \tag{7.114}$$

into (7.108), and expanding in inverse powers of γ, we find that

$$\frac{dP(t')}{d\Omega} \approx \frac{e^2 a^2 \theta^2}{4\pi\Big(1 - \sqrt{1-\gamma^{-2}}\sqrt{1-\frac{1}{2}\theta^2}\Big)^5} \approx \frac{8e^2 a^2 \theta^2}{\pi(\gamma^{-2}+\theta^2)^5}\,, \tag{7.115}$$

which can be written as

$$\frac{dP(t')}{d\Omega} \approx \frac{8e^2 a^2 \gamma^2}{\pi}\, \frac{(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5}\,. \tag{7.116}$$

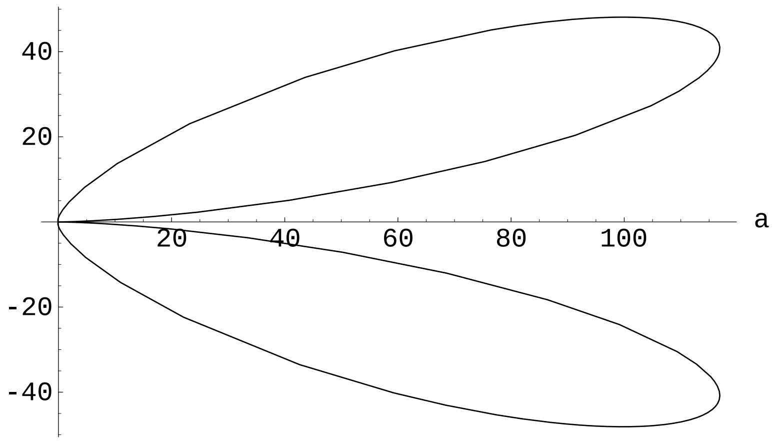

Figure 2: The angular power distribution in the relativistic case ($v = 4/5$)

This shows that indeed there are two lobes, of characteristic width $\Delta\theta \sim 1/\gamma$, on each side of $\theta = 0$. The radiated power is zero in the exactly forward direction $\theta = 0$.

We can straightforwardly integrate our result (7.108) for the angular power distribution for a linearly-accelerated particle, to find the total radiated power. We obtain

$$P = \int \frac{dP(t')}{d\Omega} d\Omega = \frac{e^2 a^2}{4\pi}\, 2\pi \int_0^\pi \frac{\sin^2\theta}{(1 - v\cos\theta)^5}\, \sin\theta\, d\theta = \tfrac{1}{2} e^2 a^2 \int_{-1}^1 \frac{(1-c^2)dc}{(1-vc)^5}\,, \qquad (7.117)$$

where $c = \cos\theta$. The integral is elementary, giving the result

$$P = \tfrac{2}{3} e^2 \gamma^6 a^2\,. \qquad (7.118)$$

This can be seen to be in agreement with our earlier result (7.83), under the specialisation that $\vec{a}$ and $\vec{v}$ are parallel.

7.6.2 Angular power distribution for circular motion

For a second example, consider the situation of a charge that is in uniform circular motion. For these purposes, we need only assume that it is instantaneously in such motion; the complete path of the particle could be something more complicated than a circle, but such that at some instant it can be described by a circular motion.

Circular motion implies that the velocity $\vec{v}$ and the acceleration $\vec{a}$ are perpendicular. At the instant under consideration, we may choose a system of Cartesian axes oriented so that the velocity $\vec{v}$ lies along the z direction, and the acceleration lies along the x direction. The unit vector $\vec{n} = \vec{R}/R$ can then be parameterised by spherical polar coordinates (θ, φ) defined in the usual way; i.e. θ measures the angle between $\vec{n}$ and the z axis, and φ is the azimuthal angle, measured from the x axis, of the projection of $\vec{n}$ onto the (x,y) plane. Thus we shall have

$$\vec{n} = (\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta)\,, \qquad \vec{v} = (0,0,v)\,, \qquad \vec{a} = (a,0,0)\,. \tag{7.119}$$

Of course, in particular, we have $\vec{n}\cdot\vec{v} = v\cos\theta$.

From (7.103) and (7.106), we have the general expression

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi}\,\frac{|\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]|^2}{(1-\vec{n}\cdot\vec{v})^5}\,, \tag{7.120}$$

for the angular distribution of the radiated power. Using the fact that $\vec{v}\cdot\vec{a}=0$ in the case of circular motion, we have

$$\begin{aligned}
|\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]|^2 &= |(\vec{n}\cdot\vec{a})(\vec{n}-\vec{v}) - (1-\vec{n}\cdot\vec{v})\vec{a}|^2\,,\\
&= (\vec{n}\cdot\vec{a})^2(1-2\vec{n}\cdot\vec{v}+v^2) + (1-\vec{n}\cdot\vec{v})^2 a^2 - 2(\vec{n}\cdot\vec{a})^2(1-\vec{n}\cdot\vec{v})\,,\\
&= -(\vec{n}\cdot\vec{a})^2(1-v^2) + (1-\vec{n}\cdot\vec{v})^2 a^2\,,\\
&= (1-v\cos\theta)^2 a^2 - \gamma^{-2}\,a^2\sin^2\theta\,\cos^2\varphi\,,
\end{aligned} \tag{7.121}$$

and so for instantaneous circular motion we have

$$\frac{dP(t')}{d\Omega} = \frac{e^2 a^2}{4\pi(1-v\cos\theta)^3}\Big[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2\,(1-v\cos\theta)^2}\Big]\,. \tag{7.122}$$

We see that as v tends to 1, the angular distribution is peaked in the forward direction i.e. in the direction of the velocity $\vec{v}$, meaning that θ is close to 0.

The total power is obtained by integrating $\frac{dP(t')}{d\Omega}$ over all solid angles:

$$\begin{aligned}
P(t') &= \int \frac{dP(t')}{d\Omega} d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \frac{dP(t')}{d\Omega}\,,\\
&= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \frac{e^2 a^2}{4\pi(1-v\cos\theta)^3}\Big[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2\,(1-v\cos\theta)^2}\Big]\,,\\
&= \int_0^{\pi} \sin\theta d\theta \frac{e^2 a^2}{2(1-v\cos\theta)^3}\Big[1 - \frac{\sin^2\theta}{2\gamma^2\,(1-v\cos\theta)^2}\Big]\,,\\
&= \int_{-1}^{1} \frac{e^2 a^2}{2(1-vc)^3}\Big[1 - \frac{1-c^2}{2\gamma^2(1-vc)^2}\Big]\,dc\,,
\end{aligned} \tag{7.123}$$

where $c=\cos\theta$. After performing the integration, we obtain

$$P(t') = \tfrac{2}{3} e^2\gamma^4 a^2\,. \tag{7.124}$$

This expression can be compared with the general result (7.83), specialised to the case where $\vec{v}$ and $\vec{a}$ are perpendicular. Noting that then

$$(\vec{v}\times\vec{a})^2 = \epsilon_{ijk}\epsilon_{i\ell m} v_j a_k v_\ell a_m = v_j v_j a_k a_k - v_j a_j v_k a_k = v_j v_j a_k a_k = v^2 a^2\,, \tag{7.125}$$

we see that (7.83) indeed agrees with (7.124) in this case.

The total power radiated in the case of linear acceleration, with its γ^6 factor as in (7.118), is larger by a factor of γ^2 than the total power radiated in the case of circular motion, *provided we take the acceleration a to be the same in the two cases.* However, this is not always the most relevant comparison to make. Another way to make the comparison is to take the magnitude of the applied force, $|d\vec{p}/dt|$, to be the same in the two cases. For circular motion we have that v is constant, and so

$$\frac{d\vec{p}}{dt} = m\gamma \frac{d\vec{v}}{dt} = m\gamma \vec{a}\,. \tag{7.126}$$

Thus for circular motion, we have from (7.124) that

$$P(t') = \frac{2e^2\gamma^2}{3m^2} \Big|\frac{d\vec{p}}{dt}\Big|^2 . \tag{7.127}$$

By contrast, for linear acceleration, where $\vec{v}$ is parallel to $\vec{a}$, we have

$$\frac{d\vec{p}}{dt} = m\gamma\vec{a} + m\gamma^3(\vec{v}\cdot\vec{a})\vec{v} = m\gamma^3\vec{a}\,, \tag{7.128}$$

and so this gives

$$P(t') = \frac{2e^2}{3m^2} \Big|\frac{d\vec{p}}{dt}\Big|^2 . \tag{7.129}$$

Thus if we hold $|d\vec{p}/dt|$ fixed when compariung the two, we see that it is the particle in circular motion whose radiated power is larger than that of the linearly-accelerated particle, by a factor of γ^2.

7.7 Frequency distribution of radiated energy

In this section, we shall discuss the spectrum of frequencies of the electromagnetic radiation emitted by an accelerating charge. The basic technique for doing this will be to perform a Fourier transform of the time dependence of the radiated power.

In general, we have

$$\frac{dP(t)}{d\Omega} = [R^2\vec{n}\cdot\vec{S}]_{\rm ret} = \frac{1}{4\pi}\,|[R\vec{E}]_{\rm ret}|^2 . \tag{7.130}$$

Let

$$\vec{G}(t) = \frac{1}{\sqrt{4\pi}}\,[R\vec{E}]_{\rm ret}\,, \tag{7.131}$$

so that we shall have

$$\frac{dP(t)}{d\Omega} = |\vec{G}(t)|^2 . \tag{7.132}$$

Note that here $dP(t)/d\Omega$ is expressed in the *observer's* time t, and not the retarded time t'. This is because our goal here will be to determine the frequency spectrum of the electromagnetic radiation as measured by the observer.

Suppose that the acceleration of the charge occurs only for a finite period of time, so that the total energy emitted is finite. We shall assume that the observation point is far enough away from the charge that the spatial region spanned by the charge while it is accelerating subtends only a small angle as seen by the observer.

The total energy radiated per unit solid angle is given by

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega}\, dt = \int_{-\infty}^{\infty} |\vec{G}(t)|^2 dt\,. \tag{7.133}$$

We now define the Fourier transform $\vec{g}(\omega)$ of $\vec{G}(t)$:

$$\vec{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{G}(t)\, e^{\mathrm{i}\,\omega t}\, dt\,. \tag{7.134}$$

In the usual way, the inverse transform is then

$$\vec{G}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{g}(\omega)\, e^{-\mathrm{i}\,\omega t}\, d\omega\,. \tag{7.135}$$

It follows that

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{G}(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \vec{g}^{\,*}(\omega') \cdot \vec{g}(\omega)\, e^{\mathrm{i}\,(\omega'-\omega)t}\,. \tag{7.136}$$

The t integration can be performed, using

$$\int_{-\infty}^{\infty} dt\, e^{\mathrm{i}\,(\omega'-\omega)t} = 2\pi\delta(\omega'-\omega)\,, \tag{7.137}$$

and so

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \vec{g}^{\,*}(\omega') \cdot \vec{g}(\omega)\, \delta(\omega'-\omega) = \int_{-\infty}^{\infty} d\omega \vec{g}^{\,*}(\omega) \cdot \vec{g}(\omega)\,, \tag{7.138}$$

i.e.

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega |\vec{g}(\omega)|^2\,. \tag{7.139}$$

(The result that (7.133) can be expressed as (7.139) is known as *Parseval's Theorem* in Fourier transform theory.)

We can re-express (7.139) as

$$\frac{dW}{d\Omega} = \int_0^{\infty} d\omega\, \frac{d^2 I(\omega, \vec{n})}{d\omega d\Omega}\,, \tag{7.140}$$

where

$$\frac{d^2 I(\omega, \vec{n})}{d\omega d\Omega} = |\vec{g}(\omega)|^2 + |\vec{g}(-\omega)|^2 \tag{7.141}$$

is the energy emitted per unit solid angle per unit frequency interval. If $\vec{G}(t) = [R\vec{E}]_{\rm ret}/\sqrt{4\pi}$ is real, then

$$\vec{g}(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \vec{G}(t)\, e^{-{\rm i}\,\omega t} = \vec{g}^{\,*}(\omega)\,, \tag{7.142}$$

and then

$$\frac{d^2 I(\omega,\vec{n})}{d\omega d\Omega} = 2|\vec{g}(\omega)|^2\,. \tag{7.143}$$

Using the expression for $\vec{E}$ in (7.101), the Fourier transform $\vec{g}(\omega)$, given by (7.134) with (7.131), is

$$\vec{g}(\omega) = \frac{e}{2\sqrt{2}\,\pi} \int_{-\infty}^{\infty} e^{{\rm i}\,\omega t} \Big[\frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^3}\Big]_{\rm ret}\, dt\,, \tag{7.144}$$

where as usual, the subscript "ret" is a reminder that the quantity is evaluated at the retarded time t'. Since

$$dt = \frac{dt}{dt'}\, dt' = (1-\vec{n}\cdot\vec{v})\, dt'\,, \tag{7.145}$$

we therefore have

$$\vec{g}(\omega) = \frac{e}{2\sqrt{2}\,\pi} \int_{-\infty}^{\infty} e^{{\rm i}\,\omega(t'+R(t'))}\, \frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^2}\, dt'\,. \tag{7.146}$$

(We have now dropped the "ret" reminder, since everything inside the integrand now depends on the retarded time t'.)

We are assuming that the observation point is far away from the accelerating charge, and that the period over which the acceleration occurs is short enough that the the vector $\vec{n} = \vec{R}(t')/R(t')$ is approximately constant during this time interval. It is convenient to choose the origin to be near to the particle during its period of acceleration. With the observer being far away, at position vector $\vec{r}$, it follows from $\vec{R}(t') = \vec{r} - \vec{r}_0(t')$ that to a good approximation we have

$$R^2(t') \approx r^2 - 2\vec{r}\cdot\vec{r}_0(t')\,, \tag{7.147}$$

and so

$$R(t') \approx r\Big(1 - \frac{2\vec{r}\cdot\vec{r}_0(t')}{r^2}\Big)^{1/2} \approx r - \frac{\vec{r}\cdot\vec{r}_0(t')}{r}\,. \tag{7.148}$$

Furthermore, we can also approximate $\vec{n} \equiv \vec{R}(t')/R(t')$ by $\vec{r}/r$, and so

$$R(t') \approx r - \vec{n}\cdot\vec{r}_0(t')\,. \tag{7.149}$$

Substituting this into (7.146), there will be a phase factor $e^{{\rm i}\,\omega r}$ that can be taken outside the integral, since it is independent of t'. This overall phase factor is unimportant (it will cancel out when we calculate $|\vec{g}(\omega)|^2$, and so we may drop it and write

$$\vec{g}(\omega) = \frac{e}{2\sqrt{2}\,\pi} \int_{-\infty}^{\infty} e^{{\rm i}\,\omega(t'-\vec{n}\cdot\vec{r}_0(t'))}\, \frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^2}\, dt'\,. \tag{7.150}$$

From (7.143) we therefore have

$$\frac{d^2 I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2}{4\pi^2}\Big|\int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega(t'-\vec{n}\cdot\vec{r}_0(t'))}\,\frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^2}\,dt'\Big|^2\,, \tag{7.151}$$

as the energy per unit solid angle per unit frequency interval.

The integral can be neatened up by observing that we can write

$$\frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^2} = \frac{d}{dt'}\Big[\frac{\vec{n}\times(\vec{n}\times\vec{v})}{1-\vec{n}\cdot\vec{v}}\Big]\,, \tag{7.152}$$

under the assumption that $\vec{n}$ is a constant. This can be seen be distributing the derivative, to obtain

$$\begin{aligned}\frac{d}{dt'}\Big[\frac{\vec{n}\times(\vec{n}\times\vec{v})}{1-\vec{n}\cdot\vec{v}}\Big] &= \frac{\vec{n}\times(\vec{n}\times\vec{a})}{1-\vec{n}\cdot\vec{v}} + \frac{\vec{n}\times(\vec{n}\times\vec{v})\,(\vec{n}\cdot\vec{a})}{(1-\vec{n}\cdot\vec{v})^2}\,,\\ &= \frac{(1-\vec{n}\cdot\vec{v})(\vec{n}\,(\vec{n}\cdot\vec{a})-\vec{a}) + (\vec{n}\,(\vec{n}\cdot\vec{v})-\vec{v})(\vec{n}\cdot\vec{a})}{(1-\vec{n}\cdot\vec{v})^2}\,,\\ &= \frac{(\vec{n}\cdot\vec{a})(\vec{n}-\vec{v})-(1-\vec{n}\cdot\vec{v})\vec{a}}{(1-\vec{n}\cdot\vec{v})^2}\,,\\ &= \frac{\vec{n}\times[(\vec{n}-\vec{v})\times\vec{a}]}{(1-\vec{n}\cdot\vec{v})^2}\,.\end{aligned} \tag{7.153}$$

This allows us to integrate (7.151) by parts, to give

$$\frac{d^2 I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2}{4\pi^2}\Big|-\int_{-\infty}^{\infty}\frac{\vec{n}\times(\vec{n}\times\vec{v})}{1-\vec{n}\cdot\vec{v}}\,\frac{d}{dt'}e^{\mathrm{i}\,\omega(t'-\vec{n}\cdot\vec{r}_0(t'))}\,dt'\Big|^2\,, \tag{7.154}$$

and hence

$$\frac{d^2 I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2}\Big|\int_{-\infty}^{\infty}\vec{n}\times(\vec{n}\times\vec{v})\,e^{\mathrm{i}\,\omega(t'-\vec{n}\cdot\vec{r}_0(t'))}\,dt'\Big|^2\,, \tag{7.155}$$

It should be remarked here that the effect of having integrated by parts is that the acceleration $\vec{a}$ no longer appears in the expression (7.155). Prior to the integration by parts, the fact that we were taking the acceleration to be non-zero for only a finite time interval ensured that the integration over all t' from $-\infty$ to ∞ would be cut down to an integration over only the finite time interval during which $\vec{a}$ was non-zero. After the integration by parts, the integrand in (7.155) no longer vanishes outside the time interval of the non-zero acceleration, and so one might worry about issues of convergence, and the validity of having dropped the boundary terms at $t'=\pm\infty$ coming from the integration by parts. In fact, it can be verified that all is well, and any problem with convergence can be handled by introducing a convergence factor $e^{-\epsilon|t'|}$, and then sending ϵ to zero.

We shall make use of the result (7.155) in two applications. In the first, we shall calculate the frequency spectrum for a relativistic particle in instantaneous circular motion.

7.8 Frequency spectrum for relativistic circular motion

Consider a particle which, at some instant, is following a circular arc of radius ρ. We shall choose axes so that the arc lies in the (x, y) plane, and choose the origin so that at $t = 0$ the particle is located at the origin, $x = y = 0$. Without loss of generality, we may choose the unit vector $\vec{n}$ (which points in the direction of the observation point) to lie in the (x, z) plane. We shall, for notational convenience, drop the prime from the time t', so from now on t will denote the retarded time.

The position vector of the particle at time t will be given by

$$\vec{r}_0 = \Big(\rho \sin\frac{vt}{\rho}, \rho\cos\frac{vt}{\rho} - \rho, 0\Big)\,, \tag{7.156}$$

where $v = |\vec{v}|$ is its speed. Since $\vec{v} = d\vec{r}_0(t)/dt$, we shall have

$$\vec{v} = \Big(v\cos\frac{vt}{\rho}, -v\sin\frac{vt}{\rho}, 0\Big)\,. \tag{7.157}$$

We may parameterise the unit vector $\vec{n}$, which we are taking to lie in the (x, z) plane, in terms of the angle θ between $\vec{n}$ and the x axis:

$$\vec{n} = (\cos\theta, 0, \sin\theta)\,. \tag{7.158}$$

We then have

$$\vec{n}\times(\vec{n}\times\vec{v}) = (\vec{n}\cdot\vec{v})\,\vec{n} - \vec{v} = \Big(-v\sin^2\theta\,\cos\frac{vt}{\rho}, -v\sin\frac{vt}{\rho}, v\sin\theta\,\cos\theta\,\cos\frac{vt}{\rho}\Big)\,. \tag{7.159}$$

We shall write this as

$$\vec{n}\times(\vec{n}\times\vec{v}) = -v\sin\frac{vt}{\rho}\,\vec{e}_{||} + v\sin\theta\,\cos\frac{vt}{\rho}\,\vec{e}_{\perp}\,, \tag{7.160}$$

where

$$\vec{e}_{||} = (0, 1, 0) \qquad \text{and} \qquad \vec{e}_{\perp} = \vec{n}\times\vec{e}_{||} = (-\sin\theta, 0, \cos\theta)\,. \tag{7.161}$$

We shall consider a particle whose velocity is highly-relativistic. It will be recalled from our earlier discussions that for such a particle, the electromagnetic radiation will be more or less completely concentrated in the range of angles θ very close to 0. Thus, to a good approximation we shall have $\vec{e}_{\perp} \approx (0, 0, 1)$, which is the unit normal to the plane of the circular motion. In what follows, we shall make approximations that are valid for small θ, and also for small t. We shall also assume that v is very close to 1 (the speed of light).

From (7.156) and (7.158), we find

$$\begin{aligned}
t - \vec{n}\cdot\vec{r}_0(t) &= t - \rho\,\cos\theta\,\sin\frac{vt}{\rho} \approx t - \rho(1-\tfrac{1}{2}\theta^2)\Big[\frac{vt}{\rho} - \tfrac{1}{6}\Big(\frac{vt}{\rho}\Big)^3\Big]\,,\\
&\approx (1-v)t + \tfrac{1}{2}\theta^2 vt + \frac{v^3 t^3}{6\rho^2}\,,\\
&\approx \tfrac{1}{2}(1+v)(1-v)t + \tfrac{1}{2}\theta^2\,t + \frac{t^3}{6\rho^2}\,,\\
&= \frac{t}{2\gamma^2} + \tfrac{1}{2}\theta^2\,t + \frac{t^3}{6\rho^2}\,.
\end{aligned} \tag{7.162}$$

From (7.160), we find

$$\vec{n}\times(\vec{n}\times\vec{v}) \approx -\frac{t}{\rho}\,\vec{e}_{||} + \theta\,\vec{e}_\perp\,. \tag{7.163}$$

We therefore find from (7.155) that

$$\begin{aligned}
\frac{d^2 I}{d\omega d\Omega} &\approx \frac{e^2\omega^2}{4\pi^2}\,\Big| - g_{||}(\omega)\,\vec{e}_{||} + g_\perp(\omega)\,\vec{e}_\perp\Big|^2\,,\\
&= \frac{e^2\omega^2}{4\pi^2}\,\Big(|g_{||}(\omega)|^2 + |g_\perp(\omega)|^2\Big)\,,
\end{aligned} \tag{7.164}$$

where

$$\begin{aligned}
g_{||}(\omega) &= \frac{1}{\rho}\int_{-\infty}^{\infty} t\,e^{\mathrm{i}\,\omega[(\gamma^{-2}+\theta^2)t + \frac{1}{3}t^3\rho^{-2}]/2}\,dt\,,\\
g_\perp(\omega) &= \theta\int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega[(\gamma^{-2}+\theta^2)t + \frac{1}{3}t^3\rho^{-2}]/2}\,dt\,.
\end{aligned} \tag{7.165}$$

Letting

$$u = \frac{t}{\rho}\,(\gamma^{-2}+\theta^2)^{-1/2}\,, \qquad \xi = \tfrac{1}{3}\omega\rho(\gamma^{-2}+\theta^2)^{3/2}\,, \tag{7.166}$$

leads to

$$\begin{aligned}
g_{||}(\omega) &= \rho(\gamma^{-2}+\theta^2)\int_{-\infty}^{\infty} u e^{3\mathrm{i}\,\xi(u+u^3/3)/3}\,du\,,\\
g_\perp(\omega) &= \rho\theta(\gamma^{-2}+\theta^2)^{1/2}\int_{-\infty}^{\infty} e^{3\mathrm{i}\,\xi(u+u^3/3)/3}\,du\,.
\end{aligned} \tag{7.167}$$

These integrals are related to Airy integrals, or modified Bessel functions:

$$\int_0^\infty u\sin[3\xi(u+u^3/3)/2]\,du = \frac{1}{\sqrt{3}}\,K_{2/3}(\xi)\,, \qquad \int_0^\infty \cos[3\xi(u+u^3/3)/2]\,du = \frac{1}{\sqrt{3}}\,K_{1/3}(\xi)\,, \tag{7.168}$$

and so we have

$$\frac{d^2 I}{d\omega d\Omega} \approx \frac{e^2\omega^2\rho^2}{3\pi^2}\,(\gamma^{-2}+\theta^2)^2\Big[(K_{2/3}(\xi))^2 + \frac{\theta^2}{\gamma^{-2}+\theta^2}\,(K_{1/3}(\xi))^2\Big]\,. \tag{7.169}$$

The asymptotic forms of the modified Bessel functions $K_\nu(x)$, for small x and large x, are

$$
\begin{aligned}
K_\nu(x) &\longrightarrow \tfrac{1}{2}\Gamma(\nu)\left(\frac{2}{x}\right)^\nu; \qquad x \longrightarrow 0\,,\\
K_\nu(x) &\longrightarrow \sqrt{\frac{\pi}{2x}}\,e^{-x}; \qquad x \longrightarrow \infty\,.
\end{aligned}
\tag{7.170}
$$

It therefore follows from (7.169) that $d^2I/(d\omega d\Omega)$ falls off rapidly when ξ becomes large. Bearing in mind that γ^{-2} is small (since the velocity of the particle is very near to the speed of light), and that θ has been assumed to be small, we see from (7.166) that there is a regime where ξ can be large, whilst still fulfilling our assumptions, if $\omega\rho$ is large enough. The value of ξ can then become very large if θ increases sufficiently (whilst still being small compared to 1), and so the radiation is indeed concentrated around very small angles θ.

If ω becomes sufficiently large that $\omega\rho\gamma^{-3}$ is much greater than 1 then ξ will be very large even if $\theta = 0$. Thus, there is an effective high-frequency cut-off for all angles. It is convenient to define a "cut-off" frequency ω_c for which $\xi = 1$ at $\theta = 0$:

$$
\omega_c = \frac{3\gamma^3}{\rho} = \frac{3}{\rho}\left(\frac{\mathcal{E}}{m}\right)^3. \tag{7.171}
$$

If the particle is following a uniform periodic circular orbit, with angular frequency $\omega_0 = vrho \approx 1/\rho$, then we shall have

$$
\omega_c = 3\omega_0\left(\frac{\mathcal{E}}{m}\right)^3. \tag{7.172}
$$

The radiation in this case of a charged particle in a highly relativistic circular orbit is known as "Synchrotron Radiation."

Consider the frequency spectrum of the radiation in the orbital plane, $\theta = 0$. In the two regimes $\omega << \omega_c$ and $\omega >> \omega_c$ we shall therefore have

$$
\begin{aligned}
\omega << \omega_c: &\qquad \frac{d^2I}{d\omega d\Omega}\Big|_{\theta=0} \approx e^2\left(\frac{\Gamma(2/3)}{\pi}\right)^2\left(\frac{3}{4}\right)^{1/3}(\omega\rho)^{2/3}\,,\\
\omega >> \omega_c: &\qquad \frac{d^2I}{d\omega d\Omega}\Big|_{\theta=0} \approx \frac{3e^2\gamma^2}{2\pi}\,\frac{\omega}{\omega_c}\,e^{-2\omega/\omega_c}\,.
\end{aligned}
\tag{7.173}
$$

This shows that the power per unit solid angle per unit frequency increases from 0 like $\omega^{2/3}$ for small ω, reaches a peak around $\omega = \omega_c$, and then falls off exponentially rapidly one ω is significantly greater than ω_c.

It is clear that one could continue with the investigation of the properties of the synchrotron radiation in considerably more depth. For example, would could consider the detailed angular distibution of the radiation as a function of θ, and one could consider the total power per unit frequency interval, obtained by integrating over all solid angles:

$$
\frac{dI}{d\omega} = \int \frac{d^2I}{d\omega d\Omega}\,d\Omega\,. \tag{7.174}
$$

A discussion of further details along these lines can be found in almost any of the advanced electrodynamics textbooks.

7.9 Frequency spectrum for periodic motion

Suppose that the motion of the charged particle is exactly periodic, with period $T = 2\pi/\omega_0$, where ω_0 is the angular frequency of the particle's motion. This means that $\vec{n}\cdot\vec{r}_0(t)$ will be periodic with period T, and so the factor $e^{-\mathrm{i}\,\omega\,\vec{n}\cdot\vec{r}_0(t)}$ in (7.155) will have time dependence of the general form

$$H(t) = \sum_{n\ge 1} b_n\, e^{-\mathrm{i}\,n\omega_0\,t}\,. \tag{7.175}$$

(We are again using t to denote the retarded time here, to avoid a profusion of primes.) The Fourier transform $h(\omega)$ of the function $H(t)$ is zero except when ω is an integer multiple of ω_0, and for these values it is proportional to a delta function:

$$\begin{aligned} h(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega t}\, H(t) dt = \sum_{n\ge 1} b_n \int_{-\infty}^{\infty} e^{\mathrm{i}\,(\omega-n\omega_0)t}\, dt\,,\\ &= 2\pi \sum_{n\ge 1} b_n\, \delta(\omega - n\omega_0)\,. \end{aligned} \tag{7.176}$$

In fact, it is more appropriate to work with Fourier series, rather than Fourier transforms, in this situation with a discrete frequency spectrum.

Going back to section 7.7, we therefore now expand $\vec{G}(t)$ in the Fourier series

$$\vec{G}(t) = \sum_{n\ge 1} \vec{a}_n\, e^{-\mathrm{i}\,n\omega_0 t}\,. \tag{7.177}$$

Multiplying by $e^{\mathrm{i}\,m\omega_0 t}$ and integrating over the period $T = 2\pi/\omega_0$ gives

$$\frac{1}{T}\int_0^T e^{\mathrm{i}\,m\omega_0 t}\vec{G}(t) dt = \frac{1}{T}\sum_{n\ge 1} \vec{a}_n \int_0^T e^{\mathrm{i}\,(m-n)\omega_0 t} dt = \vec{a}_m\,, \tag{7.178}$$

since the integral of $e^{\mathrm{i}\,(m-n)\omega_0 t}$ vanishes unless $n = m$:

$$\frac{1}{T}\int_0^T e^{\mathrm{i}\,(m-n)\omega_0 t} dt = \delta_{m,n}\,. \tag{7.179}$$

Thus the coefficients $\vec{a}_n$ in the Fourier series (7.177) are given by

$$\vec{a}_n = \frac{1}{T}\int_0^T e^{\mathrm{i}\,n\omega_0 t}\vec{G}(t) dt\,. \tag{7.180}$$

The analogue of Parseval's theorem for the case of the discrete Fourier series is now given by considering

$$\frac{1}{T}\int_0^T |\vec{G}(t)|^2 dt = \frac{1}{T}\int_0^T \sum_{m,n} \vec{a}_n\cdot\vec{a}^{\,*}_m\, e^{\mathrm{i}\,(m-n)\omega_0 t} dt = \sum_n |\vec{a}_n|^2\,. \tag{7.181}$$

The time average of the power per unit solid angle is therefore given by

$$\Big\langle \frac{dP}{d\Omega}\Big\rangle = \frac{1}{T}\int_0^T \frac{dP}{d\Omega}\, dt = \frac{1}{T}\int_0^T |\vec G(t)|^2 dt = \sum_n |\vec a_n|^2 \,. \tag{7.182}$$

The term $|\vec a_n|^2$ in this summation therefore has the interpretation of being the time-averaged power per unit solid angle in the n'th mode, which we shall denote by $dP_n/d\Omega$:

$$\frac{dP_n}{d\Omega} = |\vec a_n|^2 \,. \tag{7.183}$$

It is now a straightforward matter, using (7.180), to obtain an expression for $|\vec a_n|^2$ in terms of the integral of the retarded electric field. The steps follow exactly in parallel with those we described in section 7.7, except that the integral $\int_{-\infty}^\infty dt$ is now replaced by $T^{-1}\int_0^T dt$. The upshot is that the expression (7.155) for $d^2I/(d\omega d\Omega)$ is replaced by[18]

$$\frac{dP_n}{d\Omega} = \frac{e^2 n^2 \omega_0^4}{4\pi^2}\Big|\int_0^T \vec n\times(\vec n\times\vec v)\, e^{\mathrm{i}\, n\omega_0 (t-\vec n\cdot\vec r_0(t))}\, dt\Big|^2 \,, \tag{7.184}$$

where $T = 2\pi/\omega_0$. This gives the expression for the time-averaged power per unit solid angle in the n'th Fourier mode.

Since we are assuming the observer (at $\vec r$) is far away from the particle, and since the integral in (7.184) is taken over the finite time interval $T = 2\pi/\omega_0$, it follows that to a good approximation we can freely take the unit vector $\vec n$ outside the integral. Thus we may make the replacement

$$\int_0^T \vec n\times(\vec n\times\vec v)\, e^{\mathrm{i}\, n\omega_0 (t-\vec n\cdot\vec r_0(t))}\, dt \longrightarrow \vec n\times\Big(\vec n\times\int_0^T \vec v\, e^{\mathrm{i}\, n\omega_0 (t-\vec n\cdot\vec r_0(t))}\Big)\, dt \,. \tag{7.185}$$

Now, for any vector $\vec V$, we have that

$$|\vec n\times(\vec n\times\vec V)|^2 = |\vec n\cdot\vec V - \vec V|^2 = V^2 - (\vec n\cdot\vec V)^2 \,, \tag{7.186}$$

and on the other hand we also have

$$|\vec n\times\vec V|^2 = (\vec n\times\vec V)\cdot(\vec n\times\vec V) = \vec n\cdot[\vec V\times(\vec n\times\vec V)] = \vec n\cdot[V^2\vec n - (\vec n\cdot\vec V)\vec V] = V^2 - (\vec n\cdot\vec V)^2 \,. \tag{7.187}$$

Thus $|\vec n\times(\vec n\times\vec V)|^2 = |\vec n\times\vec V|^2$, and so we can re-express (7.184) as

$$\frac{dP_n}{d\Omega} = \frac{e^2 n^2 \omega_0^4}{4\pi^2}\Big|\int_0^T \vec v\times\vec n\, e^{\mathrm{i}\, n\omega_0 (t-\vec n\cdot\vec r_0(t))}\, dt\Big|^2 \,, \tag{7.188}$$

where $T = 2\pi/\omega_0$ and ω_0 is the angular frequency of the periodic motion. Recall that throughout this section, we are using t to denote the retarded time, in order to avoid writing the primes on t' in all the formulae.

[18]The integer n labelling the modes is not to be confused with the unit vector $\vec n$, of course!

7.10 Cerenkov radiation

So far, all the situations we have considered have involved electromagnetic fields in a vacuum, i.e. in the absence of any dielectric or magnetically permeable media. In this section, we shall take a brief foray into a situation where there is a dielectric medium.

It will be recalled that if a medium has permittivity ϵ and permeability μ, then electromagnetic waves in the medium will propagate with speed $\tilde{c} = 1/\sqrt{\epsilon\,\mu}$. This means in general that the "speed of light" in the medium will be less than the speed of light in vacuum. A consequence of this is that a charged particle, such as an electron, can travel faster than the local speed of light inside the medium. This leads to an interesting effect, known as *Cerenkov Radiation.* In practice, the types of media of interest are those that are optically transparent, such as glass or water, and these have magnetic permeability μ very nearly equal to 1, while the electric permittivity ϵ can be quite significantly greater than 1. Thus for the purposes of our discussion, we shall assume that $\mu = 1$ and that the local speed of light is reduced because ϵ is significantly greater than 1.

We shall make use of the result (7.155) for the radiated power spectrum, in order to study the Cerenkov radiation. First, we shall need to introduce the dielectric constant into the formula. This can be done by a simple scaling argument. We shall also, just for the purposes of this section, restore the explicit symbol c for the speed of light. This can be done by sending

$$t \longrightarrow ct\,, \qquad \omega \longrightarrow \frac{\omega}{c}\,. \tag{7.189}$$

(Of course any other quantity that involves time will also need to be rescaled appropriately too. This is just dimensional analysis.)

Referring back to the discussion in section 2.1, it can be seen that the dielectric constant ϵ can be introduced into the vacuum Maxwell equations by making the scalings

$$\rho \longrightarrow \frac{\rho}{\sqrt{\epsilon}}\,, \quad \vec{E} \longrightarrow \sqrt{\epsilon}\,\vec{E}\,, \quad \vec{B} \longrightarrow \vec{B}\,, \quad c \longrightarrow \frac{c}{\sqrt{\epsilon}}\,. \tag{7.190}$$

Of course the scaling of the charge density ρ implies that we must also rescale the charge e of the particle, according to

$$e \longrightarrow \frac{e}{\sqrt{\epsilon}}\,. \tag{7.191}$$

Note that c continues to mean the speed of light in vacuum. The speed of light inside the dielectric medium is given by

$$\tilde{c} = \frac{c}{\sqrt{\epsilon}}\,. \tag{7.192}$$

The expression (7.155) for the radiated power per unit solid angle per unit frequency interval now becomes

$$\frac{d^2I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\omega^2\sqrt{\epsilon}}{4\pi^2c^3}\Big|\int_{-\infty}^{\infty}\vec{n}\times(\vec{n}\times\vec{v})\,e^{{\rm i}\,\omega(t'-\sqrt{\epsilon}\,\vec{n}\cdot\vec{r}_0(t')/c)}\,dt'\Big|^2, \tag{7.193}$$

For a charge moving at constant velocity $\vec{v}$, we shall have

$$\vec{r}_0(t') = \vec{v}\,t', \tag{7.194}$$

and so (7.193) gives

$$\frac{d^2I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\omega^2\sqrt{\epsilon}}{4\pi^2c^3}\,|\vec{n}\times\vec{v}|^2\,\Big|\int_{-\infty}^{\infty}e^{{\rm i}\,\omega t'(1-\sqrt{\epsilon}\,\vec{n}\cdot\vec{v}/c)}\,dt'\Big|^2, \tag{7.195}$$

since $|\vec{n}\times(\vec{n}\times\vec{v})|^2 = |(\vec{n}\cdot\vec{v})\,\vec{n}-\vec{v}|^2 = v^2-(\vec{n}\cdot\vec{v})^2 = |\vec{n}\times\vec{v}|^2$.

The integration over t' produces a delta-function.[19] Defining θ to be the angle between $\vec{n}$ and $\vec{v}$, so that $\vec{n}\cdot\vec{v} = v\cos\theta$, we therefore have

$$\frac{d^2I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\omega^2\sqrt{\epsilon}}{c^3}\,v^2\,\sin^2\theta\,|\delta(\omega(1-\sqrt{\epsilon}\,(v/c)\cos\theta))|^2, \tag{7.196}$$

and so (using $\delta(ax) = \delta(x)/a)$)

$$\frac{d^2I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\sqrt{\epsilon}}{c^3}\,v^2\,\sin^2\theta\,|\delta(1-\sqrt{\epsilon}\,(v/c)\cos\theta)|^2. \tag{7.197}$$

This expression shows that all the radiation is emitted at a single angle θ_c, known as the *Cerenkov Angle*, given by

$$\cos\theta_c = \frac{c}{v\sqrt{\epsilon}}. \tag{7.198}$$

Note that in terms of $\tilde{c}$, the speed of light in the medium, as given in (7.192), we have

$$\cos\theta_c = \frac{\tilde{c}}{v}. \tag{7.199}$$

This makes clear that the phenomenon of Cerenkov radiation occurs only if $v > \tilde{c}$, i.e. if the charged particle is moving through the medium at a velocity that is greater than the local velocity of light in the medium. In fact one can understand the Cerenkov radiation as a kind of "shock wave," very like the acoustic shock wave that occurs when an aircraft is travelling faster than the speed of sound. The Cerenkov angle θ_c is given by a very simple geometric construction, shown in Figure 3 below. The circles show the light-fronts of light emmitted by the particle. Since the particle is travelling faster than the speed of light in the medium, it "outruns" the circles, leaving a trail of light-fronts tangent to the angled line in the figure. This is the light-front of the Cerenkov radiation.

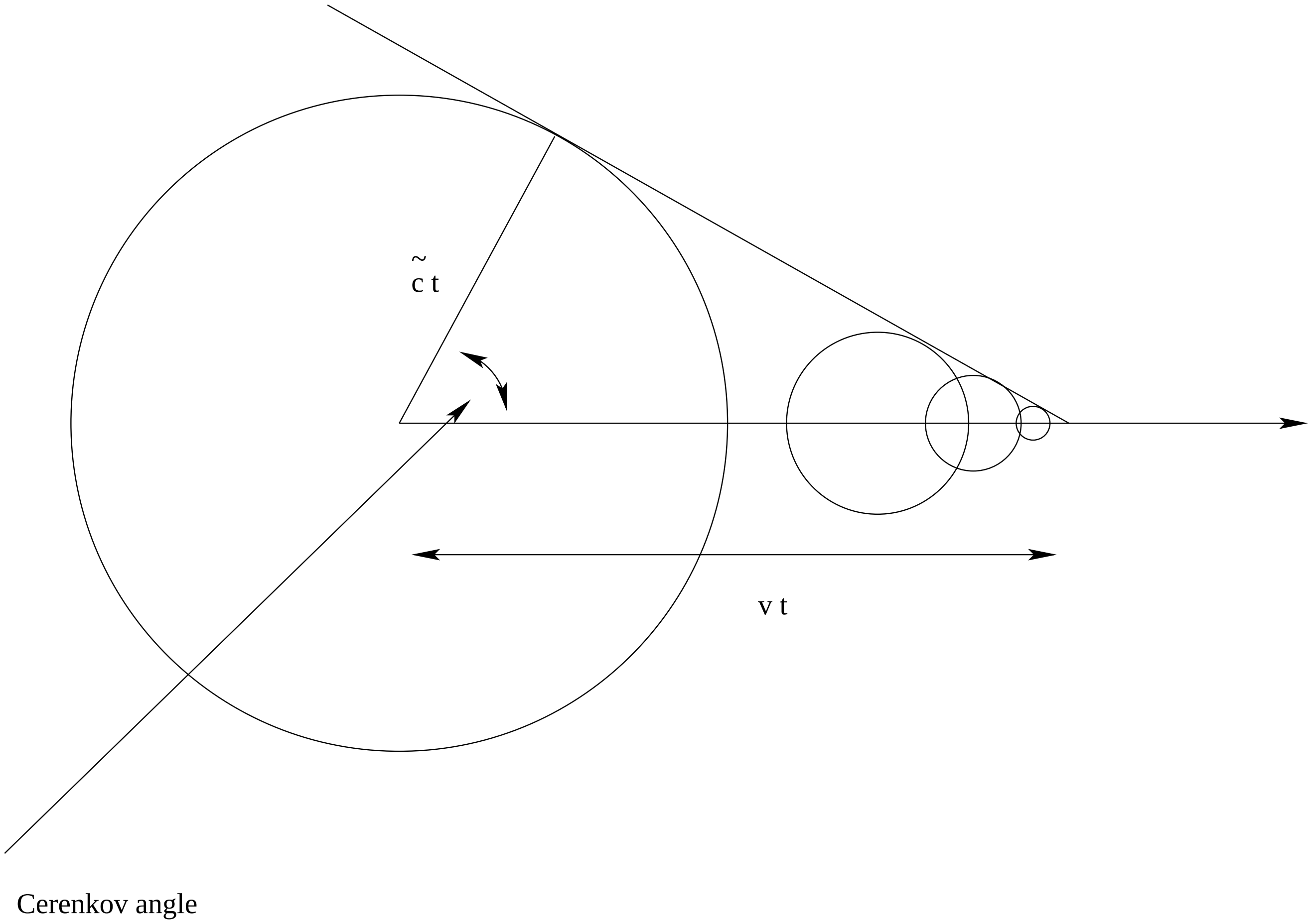

Figure 3: The Cerenkov angle θ_c is given by $\cos\theta_c = (\tilde{c}t)/(vt) = \tilde{c}/v$.

As mentioned above, the squared delta-function in (7.197) is the result of making the unrealistic assumption that the particle has been ploughing through the medium for ever, at a speed greater than the local speed of light. A more realistic situation would be to consider a charged particle entering a thin slab of dielectric medium, such that it enters at time $t' = -T$ and exits at $t' = +T$. The expression (7.195) is then replaced by

$$\frac{d^2 I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\omega^2\sqrt{\epsilon}}{4\pi^2 c^3}\,|\vec{n}\times\vec{v}|^2\,\Big|\int_{-T}^{T} e^{\mathrm{i}\,\omega t'(1-\sqrt{\epsilon}\,\vec{n}\cdot\vec{v}/c)}\,dt'\Big|^2\,, \tag{7.200}$$

which, using $\int_{-T}^{T} dt e^{\mathrm{i}\,bt} = 2b^{-1}\,\sin bT$, therefore implies that

$$\frac{d^2 I(\omega,\vec{n})}{d\omega d\Omega} = \frac{e^2\omega^2\sqrt{\epsilon}\,v^2 T^2\sin^2\theta}{\pi^2 c^3}\Big(\frac{\sin[\omega T(1-\sqrt{\epsilon}\,(v/c)\cos\theta)]}{\omega T(1-\sqrt{\epsilon}\,(v/c)\cos\theta}\Big)^2\,. \tag{7.201}$$

This is sharply peaked around the Cerenkov angle θ_c given by (7.198).

[19]The occurrence of the delta-function is because of the unphysical assumption that the particle has been moving in the medium forever. Below, we shall obtain a more realistic expression by supposing that the particle travels through a slab of mdeium of finite thickness.

Integrating over all angles we obtain the total energy per unit frequency interval

$$\frac{dI}{d\omega} = \int \frac{d^2 I}{d\omega d\Omega}\, d\Omega \approx \frac{2e^2\omega^2\sqrt{\epsilon}\, v^2 T^2\, \sin^2\theta_c}{\pi c^3} \int_0^\pi \Big(\frac{\sin[\omega T(1-\sqrt{\epsilon}\,(v/c)\cos\theta)]}{\omega T(1-\sqrt{\epsilon}\,(v/c)\cos\theta}\Big)^2 \sin\theta\, d\theta\,. \tag{7.202}$$

(The integrand is peaked sharply around $\theta=\theta_c$, so to a good approximation we can take the $\sin^2\theta$ factor outside the integral, calling it $\sin^2\theta_c$.) Letting $x=\cos\theta$, the remaining integral can be written as

$$\int_{-1}^{1} \Big(\frac{\sin[\omega T(1-\sqrt{\epsilon}\,(v/c)x)]}{\omega T(1-\sqrt{\epsilon}\,(v/c)x}\Big)^2 dx \approx \int_{-\infty}^{\infty} \Big(\frac{\sin[\omega T(1-\sqrt{\epsilon}\,(v/c)x)]}{\omega T(1-\sqrt{\epsilon}\,(v/c)x}\Big)^2 dx\,. \tag{7.203}$$

(The limits of integration can, to a good approximation, be extended to $\pm\infty$ because the integrand is peaked around $x=\cos\theta_c$.) Letting $\omega T - \omega T\sqrt{ep}\, x/c = -y$, the inetgral becomes

$$\frac{c}{\omega T\sqrt{\epsilon}\, v}\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2}\, dy = \frac{\pi c}{\omega T\sqrt{\epsilon}\, v}\,, \tag{7.204}$$

and so expression (7.202) for the total energy per unit frequency interval becomes

$$\frac{dI}{d\omega} \approx \frac{2e^2 v\omega T\sin^2\theta_c}{c^2}\,. \tag{7.205}$$

The distance through the slab is given by $2vT$, and so dividing by this, we obtain an expression for the total energy of Cerenkov radiation per unit frequency interval per unit path length:

$$\frac{d^2 I}{d\omega d\ell} = \frac{e^2\omega}{c^2}\sin^2\theta_c = \frac{e^2\omega}{c^2}\Big(1-\frac{c^2}{v^2\epsilon}\Big)\,. \tag{7.206}$$

This is known as the Frank-Tamm relation. Note that this expression grows linearly with ω, which means that the bulk of the energy is concentrated in the higher frequencies of electromagnetic radiation. Of course there must be some limit, which arises because the dielectric constant will fall off with increasing frequency, and so the Cerenkov effect will cease to operate at high enough frequencies.[20] In practice, the peak of the frequency spectrum for Cerenkov radiation is in the ultra-violet.

The bluish-green glow visible in pictures of nuclear fuel rods immersed in water is a familiar example of Cerenkov radiation. Apart from looking nice, the Cerenkov effect is also of practical use, for measuring the velocity of charged particles moving at relativistic speeds. One can determine the velocity by allowing the particles to move through a slab of suitably-chosen dielectric material, and measuring the Cerenkov angle.

[20] At sufficiently high frequencies, which implies very small wavelengths, the approximation in which the medium is viewed as a continuum with an effective dielectric constant breaks down, and it looks more and more like empty space with isolated charges present. At such length scales the electron is more or less propagating through a vacuum, and so there is no possibility of its exceeding the local speed of light. Thus the Cerenkov effect tails off at sufficiently high frequencies.

7.11 Thompson scattering

Another application of the Larmor formula is in the phenomenon known as *Thompson scattering*. Consider a plane electromagnetic wave incident on a particle of charge e and mass m. The charge will oscillate back and forth in the electric field of the wave, and so it will therefore emit electromagnetic radiation itself. The net effect is that the electron "scatters" some of the incoming wave.

In most circumstances, we can assume that the induced oscillatory motion of the electron will be non-relativistic. As we saw in (7.73), if Θ is the angle between the acceleration $\vec{a}$ and the unit vector $\vec{n}$ (which lies along the line from the electron to the observation point), then the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi} \sin^2\Theta\,. \tag{7.207}$$

Let us suppose that the plane electromagnetic wave has electric field given by (the real part of)

$$\vec{E} = E_0\,\vec{\epsilon}\,e^{\mathrm{i}\,(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{7.208}$$

that the wave-vector $\vec{k}$ lies along the z axis. The unit polarisation vector $\vec{\epsilon}$, which must therefore lie in the (x, y) plane, may be parameterised as

$$\vec{\epsilon} = (\cos\psi, \sin\psi, 0)\,. \tag{7.209}$$

Using standard spherical polar coordinates, the unit vector $\vec{n}$ will be given by

$$\vec{n} = (\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta)\,. \tag{7.210}$$

In particular, this means

$$\vec{n}\cdot\vec{\epsilon} = \sin\theta\,(\cos\varphi\cos\psi + \sin\varphi\sin\psi) = \sin\theta\,\cos(\varphi-\psi)\,. \tag{7.211}$$

The acceleration of the electron will be given by

$$m\vec{a} = e\vec{E}\,, \qquad \text{so} \qquad \vec{a} = \frac{e}{m}E_0\,\vec{\epsilon}\,e^{\mathrm{i}\,\omega(z-t)}\,. \tag{7.212}$$

Note that this means

$$\vec{n}\cdot\vec{a} = \frac{e}{m}E_0\,\vec{n}\cdot\vec{\epsilon}e^{\mathrm{i}\,\omega(z-t)} = \frac{e}{m}E_0\,e^{\mathrm{i}\,\omega(z-t)}\sin\theta\,\cos(\varphi-\psi)\,. \tag{7.213}$$

Since $\vec{n}\cdot\vec{a} = a\cos\Theta$, it follows that (7.207) becomes

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi}\Big(a^2 - (\vec{n}\cdot\vec{a})^2\Big)\,, \tag{7.214}$$

and so the time average will be given by

$$\Big\langle \frac{dP}{d\Omega} \Big\rangle = \frac{e^4}{8\pi m^2}\, |E_0|^2 [1 - (\vec{n} \cdot \vec{\epsilon})^2] \,. \tag{7.215}$$

Thus we find

$$\Big\langle \frac{dP}{d\Omega} \Big\rangle = \frac{e^4}{8\pi m^2}\, |E_0|^2\, [1 - \sin^2\theta\, \cos^2(\varphi - \psi)] \,. \tag{7.216}$$

The direction of the polarisation (in the (x, y) plane) of the incoming electromagnetic wave is parameterised by the angle ψ. For unpolarised incoming waves, we should average over all angles ψ. Thus we obtain

$$\begin{aligned} \Big\langle \Big\langle \frac{dP}{d\Omega} \Big\rangle \Big\rangle_\psi &\equiv \frac{1}{2\pi} \int_0^{2\pi} d\psi \Big\langle \frac{dP}{d\Omega} \Big\rangle = \frac{e^4}{8\pi m^2}\, |E_0|^2\, (1 - \tfrac{1}{2} \sin^2\theta) \,, \\ &= \frac{e^4}{16\pi m^2}\, |E_0|^2\, (1 + \cos^2\theta) \,. \end{aligned} \tag{7.217}$$

The scattering cross section $d\sigma/d\Omega$ is then defined by

$$\frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/unit time/unit solid angle}}{\text{Incident energy flux/unit area/unit time}} \,. \tag{7.218}$$

The denominator here will just be $|E_0|^2/(8\pi)$, which is the time average of the Poynting flux for the incoming wave. Thus we arrive at the *Thompson Formula* for the cross section:

$$\frac{d\sigma}{d\Omega} = \frac{e^4(1 + \cos^2\theta)}{2m^2} \,. \tag{7.219}$$

The total scattering cross section is obtained by integrating $d\sigma/d\Omega$ over all solid angles, which gives

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_0^\pi \frac{d\sigma}{d\Omega} \sin\theta d\theta \,, \\ &= \frac{\pi e^4}{m^2} \int_0^\pi \sin^3\theta d\theta = \frac{\pi e^4}{m^2} \int_{-1}^1 (1 + c^2) dc \,, \end{aligned} \tag{7.220}$$

and so we find

$$\sigma = \frac{8\pi e^4}{3m^2} \,. \tag{7.221}$$

8 Radiating Systems

8.1 Fields due to localised oscillating sources

Consider a localised system of oscillating charge, at a single frequency ω. The charge density and current density can therefore be written as

$$\rho(\vec{r}, t) = \rho(\vec{r})\, e^{-\mathrm{i}\,\omega t} \,, \qquad \vec{J}(\vec{r}, t) = \vec{J}(\vec{r})\, e^{-\mathrm{i}\,\omega t} \,. \tag{8.1}$$

From the expressions (7.21) and (7.22) for the retarded potentials, we shall have

$$\begin{aligned}\phi(\vec r,t) &= \int \frac{\rho(\vec r\,',t-|\vec r-\vec r\,'|)}{|\vec r-\vec r\,'|}\, d^3\vec r\,'\,,\\ &= e^{-\mathrm{i}\,\omega t}\int \frac{\rho(\vec r\,')}{|\vec r-\vec r\,'|}\, e^{\mathrm{i}\,k|\vec r-\vec r\,'|}\, d^3\vec r\,'\,. \end{aligned} \tag{8.2}$$

Note that here k is simply equal to ω, and we have switched to the symbol k in the exponential inside the integral because it looks more conventional. In a similar fashion, we shall have

$$\vec A(\vec r,t) = e^{-\mathrm{i}\,\omega t}\int \frac{\vec J(\vec r\,')}{|\vec r-\vec r\,'|}\, e^{\mathrm{i}\,k|\vec r-\vec r\,'|}\, d^3\vec r\,'\,. \tag{8.3}$$

From these expressions for ϕ and $\vec A$, we can calculate $\vec E = -\vec\nabla\phi - \partial\vec A/\partial t$ and $\vec B = \vec\nabla\times\vec A$. In fact, because of the simple monochromatic nature of the time dependence, we can calculate $\vec E$ easily, once we know $\vec B$, from the Maxwell equation

$$\vec\nabla\times\vec B - \frac{\partial\vec E}{\partial t} = 4\pi\vec J\,. \tag{8.4}$$

Away from the localised source region we have $\vec J=0$. From the time dependence we have $\partial\vec E/\partial t = -\mathrm{i}\,\omega\vec E = -\mathrm{i}\,k\vec E$, and so we shall have

$$\vec E = \frac{\mathrm{i}}{k}\,\vec\nabla\times\vec B\,. \tag{8.5}$$

Let us suppose that the region where the source charges and currents are non-zero is of scale size d. The wavelength of the monochromatic waves that they generate will be given by

$$\lambda = \frac{2\pi}{\omega} = \frac{2\pi}{k}\,. \tag{8.6}$$

We shall assume that $d << \lambda$, i.e. the scale size of the source region is very small compared with the wavelength of the electromagnetic waves that are produced. It will be convenient to choose the origin of the coordinate system to lie within the neighbourhood of the source region, so that we may therefore assume

$$|\vec r\,'| << \lambda \tag{8.7}$$

for all integration points $\vec r\,'$ in the expressions (8.2) and (8.3).

The discussion of the electromagnetic fields generated by these sources can then be divided, like all Gaul, into three parts:

$$\begin{aligned} \text{Near zone, or }\textbf{Static zone}: &\quad d << r << \lambda\,,\\ \text{Intermediate zone, or }\textbf{Induction zone}: &\quad d << r \sim \lambda\,,\\ \text{Far zone, or }\textbf{Radiation zone}: &\quad d << \lambda << r\,. \end{aligned} \tag{8.8}$$

8.1.1 The static zone

First, let us consider the near zone, where $r << \lambda$. Equivalently, we may say that

$$\text{Static zone :} \qquad kr << 1\,. \tag{8.9}$$

Since we are also assuming $d << \lambda$, and that the origin of the coordinate system is located in the neighbourhood of the source region, it follows that in the near zone, we can just approximate $e^{\mathrm{i}\,k|\vec{r}-\vec{r}\,'|}$ by 1. Thus we shall have

$$\vec{A}(\vec{r},t) \approx e^{-\mathrm{i}\,\omega t} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'\,, \tag{8.10}$$

in the near zone. Aside from the time-dependent factor $e^{-\mathrm{i}\,\omega t}$, this is just like the expression for the magnetostatic case. We can make a standard expansion, in terms of spherical harmonics:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, \frac{r_<^\ell}{r_>^{\ell+1}}\, Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)\,, \tag{8.11}$$

where $r_>$ is the larger of $r = |\vec{r}|$ and $r' = |\vec{r}\,'|$, and $r_<$ is the smaller of r and r', and (θ,φ) and (θ',φ') are the spherical polar angles for $\vec{r}$ and $\vec{r}\,'$ respectively. Thus we shall have, since in our case $r' << r$,

$$\vec{A}(\vec{r},t) = e^{-\mathrm{i}\,\omega t} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, \frac{1}{r^{\ell+1}}\, Y_{\ell m}(\theta,\varphi) \int \vec{J}(\vec{r}\,')\, r'^{\ell} Y_{\ell m}^*(\theta',\varphi')\, d^3\vec{r}\,'\,. \tag{8.12}$$

In the near zone, therefore, the electromagnetic fields will just be like static fields, except that they are oscillating in time.

8.1.2 The radiation zone

Next, we shall consider the far zone, or radiation zone. Here, we shall have $kr >> 1$, while at the same time $kr' << 1$. In other words, the source is (as always) small compared with the wavelength, but that the fields are being observed from a large distance which is much larger than the wavelength. This means that

$$|\vec{r}-\vec{r}\,'|^2 = r^2 - 2\vec{r}\cdot\vec{r} + r'^2 \approx r^2 - 2\vec{r}\cdot\vec{r}\,, \tag{8.13}$$

and so

$$|\vec{r}-\vec{r}\,'| \approx r\left(1 - \frac{2\vec{r}\cdot\vec{r}\,'}{r^2}\right)^{1/2} \approx r - \frac{\vec{r}\cdot\vec{r}\,'}{r}\,, \tag{8.14}$$

and hence

$$|\vec{r}-\vec{r}\,'| \approx r - \vec{n}\cdot\vec{r}\,'\,, \tag{8.15}$$

where $\vec{n}$ is the unit vector along $\vec{r}$:

$$\vec{n} = \frac{\vec{r}}{r} \,. \tag{8.16}$$

For a more complete and comprehensive treatment, see section 8.3

From (8.3) we shall therefore have

$$\vec{A}(\vec{r},t) \approx e^{\mathrm{i}\,(kr-\omega t)} \int \frac{\vec{J}(\vec{r}\,')}{r - \vec{n}\cdot\vec{r}\,'}\, e^{-\mathrm{i}\,k\vec{n}\cdot\vec{r}\,'}\, d^3\vec{r}\,' \,, \tag{8.17}$$

in the far zone, and hence, to a very good approximation,

$$\vec{A}(\vec{r},t) \approx \frac{1}{r}\, e^{\mathrm{i}\,(kr-\omega t)} \int \vec{J}(\vec{r}\,')\, e^{-\mathrm{i}\,k\vec{n}\cdot\vec{r}\,'}\, d^3\vec{r}\,' \,. \tag{8.18}$$

(Recall that k and ω are just two names for the same quantity here.)

The magnetic field is given by $\vec{B} = \vec{\nabla}\times\vec{A}$, or $B_i = \epsilon_{ijk}\partial_j A_k$. We are interested in the contribution that dominates at large distance, and this therefore comes from the term where the derivative lands on the $e^{\mathrm{i}\,kr}$ factor rather than the $1/r$ factor. Thus we shall have

$$\begin{aligned} B_i &\approx \epsilon_{ijk}\,\frac{1}{r}(\partial_j e^{\mathrm{i}\,kr}) e^{-\mathrm{i}\,\omega t} \int \vec{J}(\vec{r}\,')\, e^{-\mathrm{i}\,k\vec{n}\cdot\vec{r}\,'}\, d^3\vec{r}\,' \,,\\ &= \mathrm{i}\,k\,\frac{x_j}{r}\,\epsilon_{ijk}\,\frac{1}{r} e^{\mathrm{i}\,kr} e^{-\mathrm{i}\,\omega t} \int \vec{J}(\vec{r}\,')\, e^{-\mathrm{i}\,k\vec{n}\cdot\vec{r}\,'}\, d^3\vec{r}\,' \end{aligned} \tag{8.19}$$

and so

$$\vec{B} \approx \mathrm{i}\,k\,\vec{n}\times\vec{A} \,. \tag{8.20}$$

Note that the magnetic field in this leading-order approximation falls off like $1/r$. This is characteristic of electromagnetic radiation, as we have seen previously.

The electric field can be calculated from the magnetic field using (8.5), and again the leading-order behaviour comes from the term where the gradient operator lands on the $e^{\mathrm{i}\,kr}$ factor. The rule again is therefore that $\vec{\nabla} \to \mathrm{i}\,k\,\vec{n}$, and so we find

$$\vec{E} \approx -k\,\vec{n}\times(\vec{n}\times\vec{A}) \,. \tag{8.21}$$

Note that (8.20) and (8.21) imply

$$\vec{n}\cdot\vec{B} = 0 \,, \qquad \vec{n}\cdot\vec{E} = 0 \,, \qquad \vec{E}\cdot\vec{B} = 0 \,. \tag{8.22}$$

Thus $\vec{E}$ and $\vec{B}$ are transverse and orthogonal. This is characteristic of radiation fields.

Since we are assuming that the characteristic size d of the source is very small compared with the wavelength, $d << \lambda = 2\pi/k$, it follows that $kd << 1$ and so the quantity $k|\vec{n}\cdot\vec{r}\,'|$ appearing in the exponential in the integrand in (8.18) is much smaller than 1. This means that it is useful to expand the exponential in a Taylor series, giving

$$\vec{A}(\vec{r},t) \approx \frac{1}{r}\, e^{\mathrm{i}\,(kr-\omega t)} \sum_{m\ge 0} \frac{(-\mathrm{i}\,k)^m}{m!} \int \vec{J}(\vec{r}\,')\,(\vec{n}\cdot\vec{r}\,')^m\, d^3\vec{r}\,' \,, \tag{8.23}$$

where the terms in the sum fall off rapidly with m.

8.1.3 The induction zone

This is the intermediate zone, where $d << \lambda \sim r$, which means that the fields are being observed from a distance that is comparable with the wavelength, and so $kr \sim 1$. In this case, we need to consider the exact expansion of $e^{\mathrm{i}\,k|\vec{r}-\vec{r}\,'|}\,|\vec{r}-\vec{r}\,'|^{-1}$. It turns out that this can be written as

$$\frac{e^{\mathrm{i}\,k|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = 4\pi \mathrm{i}\,k \sum_{\ell\ge 0} j_\ell(kr')\, h_\ell^{(1)}(kr) \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)\,, \tag{8.24}$$

where $j_\ell(x)$ are spherical Bessel functions and $h_\ell^{(1)}(x)$ are spherical Hankel functions of the first kind. We shall not pursue the investigation of the induction zone further.

8.2 Electric dipole radiation

In the radiation zone, we obtained (8.23)

$$\vec{A}(\vec{r},t) = \frac{1}{r}\, e^{\mathrm{i}\,(kr-\omega t)} \sum_{m\ge 0} \frac{(-\mathrm{i}\,k)^m}{m!} \int \vec{J}(\vec{r}\,')\, (\vec{n}\cdot\vec{r}\,')^m\, d^3\vec{r}\,'\,, \tag{8.25}$$

The terms in this expansion correspond to the terms in a multipole expansion.

Consider first the $m=0$ term, for which

$$\vec{A}(\vec{r},t) = \frac{1}{r}\, e^{\mathrm{i}\,(kr-\omega t)} \int \vec{J}(\vec{r}\,')\, d^3\vec{r}\,'\,, \tag{8.26}$$

This actually corresponds to an electric dipole term. To see this, consider the identity

$$\partial_i'(x_j'\, J_i(\vec{r}\,')) = \delta_{ij}\, J_i(\vec{r}\,') + x_j' \partial_i' J_i(\vec{r}\,') = J_j(\vec{r}\,') + x_j'\, \vec{\nabla}'\cdot \vec{J}(\vec{r}\,')\,. \tag{8.27}$$

The integral of the left-hand side over all space gives zero, since it can be turned into a boundary integral over the sphere at infinity (where the localised sources must vanish):

$$\int \partial_i'(x_j'\, J_i(\vec{r}\,'))\, d^3\vec{r}\,' = \int_S x_j'\, J_i(\vec{r}\,')\, dS_i' = 0\,. \tag{8.28}$$

We also have the charge conservation equation

$$\vec{\nabla}'\cdot\vec{J}(\vec{r}\,',t) + \frac{\partial\rho(\vec{r}\,',t)}{\partial t} = 0\,, \tag{8.29}$$

and so with the time dependence $e^{-\mathrm{i}\,\omega t}$ that we are assuming, this gives

$$\vec{\nabla}'\cdot\vec{J}(\vec{r}\,') = \mathrm{i}\,\omega\,\rho(\vec{r}\,') = \mathrm{i}\,k\,\rho(\vec{r}\,')\,. \tag{8.30}$$

Thus we conclude that

$$\int \vec{J}(\vec{r}\,')\, d^3\vec{r}\,' = -\mathrm{i}\,k \int \vec{r}\,'\,\rho(\vec{r}\,')\, d^3\vec{r}\,'\,, \tag{8.31}$$

and so

$$\vec{A}(\vec{r},t) = -\frac{\mathrm{i}\,k}{r}\, e^{\mathrm{i}\,(kr-\omega t)} \int \vec{r}\,'\,\rho(\vec{r}\,')\, d^3\vec{r}\,'\,, \tag{8.32}$$

The integrand here is just the electric dipole moment,

$$\vec{p} = \int \vec{r}\,'\,\rho(\vec{r}\,')\, d^3\vec{r}\,'\,, \tag{8.33}$$

and so we have

$$\vec{A}(\vec{r},t) = -\frac{\mathrm{i}\,k\,\vec{p}}{r}\, e^{\mathrm{i}\,(kr-\omega t)}\,. \tag{8.34}$$

Note that this leading-order term in the expansion of the radiation field corresponds to an electric dipole, and not an electric monopole. The reason for this is that a monopole term would require that the total electric charge in the source region should oscillate in time. This would be impossible, because the total charge in this isolated system must remain constant, by charge conservation.

It is convenient to factor out the time-dependence factor $e^{-\mathrm{i}\,\omega t}$ that accompanies all the expressions we shall be working with, and to write

$$\vec{A}(\vec{r},t) = \vec{A}(\vec{r})\, e^{-\mathrm{i}\,\omega t}\,, \quad \vec{B}(\vec{r},t) = \vec{B}(\vec{r})\, e^{-\mathrm{i}\,\omega t}\,, \quad \vec{E}(\vec{r},t) = \vec{E}(\vec{r})\, e^{-\mathrm{i}\,\omega t}\,. \tag{8.35}$$

Thus for the electric dipole field we shall have

$$\vec{A}(\vec{r}) = -\frac{\mathrm{i}\,k\vec{p}}{r}\, e^{\mathrm{i}\,kr}\,. \tag{8.36}$$

Then from $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ we find

$$\begin{aligned} B_i &= \epsilon_{ijk}\,\partial_j A_k = -\mathrm{i}\,k\epsilon_{ijk}\,p_k\partial_j\Big(\frac{1}{r}\,e^{\mathrm{i}\,kr}\Big)\,,\\ &= -\mathrm{i}\,k\epsilon_{ijk}p_k\,\Big(-\frac{x_j}{r^3} + \mathrm{i}\,k\,\frac{x_j}{r^2}\Big)\, e^{\mathrm{i}\,kr}\,, \end{aligned} \tag{8.37}$$

and so

$$\vec{B} = k^2\,(\vec{n}\times\vec{p})\,\frac{e^{\mathrm{i}\,kr}}{r}\,\Big(1 + \frac{\mathrm{i}}{kr}\Big)\,. \tag{8.38}$$

From (8.5) we then have

$$\begin{aligned} E_i &= \frac{\mathrm{i}}{k}\,\epsilon_{ijk}\,p_m\partial_j\Big[\epsilon_{k\ell m}k^2 x_\ell\Big(\frac{1}{r^2}+\frac{\mathrm{i}}{kr^3}\Big)e^{\mathrm{i}\,kr}\Big]\,,\\ &= \mathrm{i}\,k\,(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\,p_m\,\Big[\delta_{j\ell}\Big(\frac{1}{r^2}+\frac{\mathrm{i}}{kr^3}\Big) - \frac{2x_jx_\ell}{r^4} - \frac{3\mathrm{i}\,x_jx_\ell}{kr^5} + \mathrm{i}\,k\frac{x_jx_\ell}{r}\,\Big(\frac{1}{r^2}+\frac{\mathrm{i}}{kr^3}\Big)\Big]e^{\mathrm{i}\,kr}\,,\\ &= \frac{k^2}{r}\,(p_i - n_i\,\vec{n}\cdot\vec{p})\,e^{\mathrm{i}\,kr} + \frac{\mathrm{i}\,k}{r^2}\,(p_i - 3\vec{n}\cdot\vec{p}\,n_i)\,e^{\mathrm{i}\,kr} - \frac{1}{r^3}\,(p_i - 3\vec{n}\cdot\vec{p}\,n_i)\,e^{\mathrm{i}\,kr}\,. \end{aligned} \tag{8.39}$$

In 3-vector language, this gives

$$\vec{E} = -k^2\vec{n}\times(\vec{n}\times\vec{p})\,\frac{e^{\mathrm{i}\,kr}}{r} + [3(\vec{n}\cdot\vec{p})\,\vec{n} - \vec{p}]\,\Big(\frac{1}{r^3} - \frac{\mathrm{i}\,k}{r^2}\Big)\,e^{\mathrm{i}\,kr}\,. \tag{8.40}$$

The reason why we have kept all terms in these expressions for $\vec{B}$ and $\vec{E}$, rather than just the leading-order $1/r$ terms, is that if one makes a multipole expansion, in which the electric dipole contribution we have obtained here is the first term in the series, the expressions are in fact exact to all orders in $1/r$. We shall discuss this in more detail later.

Note that we have $\vec{n}\cdot\vec{B}=0$ everywhere, but that $\vec{n}\cdot\vec{E}=0$ only in the radiation zone (i.e. at order $1/r$). In the radiation zone we have

$$\vec{B}=k^2(\vec{n}\times\vec{p})\,\frac{e^{\mathrm{i}\,kr}}{r}\,,\qquad \vec{E}=-k^2\vec{n}\times(\vec{n}\times\vec{p})\,\frac{e^{\mathrm{i}\,kr}}{r}=-\vec{n}\times\vec{B}\,. \tag{8.41}$$

Note that we have $|\vec{B}|=|\vec{E}|$, as usual for radiation fields.

Since (8.38) and (8.40) are in fact valid everywhere, we can also use these expressions in the static zone (i.e. the near zone). Here, in the regime $kr<<1$, we therefore have

$$\vec{B}=\mathrm{i}\,k\,(\vec{n}\times\vec{p})\,\frac{1}{r^2}\,,\qquad \vec{E}=[3(\vec{n}\cdot\vec{p})\,\vec{n}-\vec{p}]\,\frac{1}{r^3}\,. \tag{8.42}$$

The electric field here is precisely like that of a static electric dipole, except that it is oscillating in time. Note that in the near zone we have $|\vec{B}|\sim(kr)\,|\vec{E}|$, which means $|\vec{B}|<<|\vec{E}|$.

Returning now to the radiation zone, we may calculate the radiated power in the usual way, using the Poynting vector. In particular, we saw previously that with the electric and magnetic fields written in the complex notation, the time average of the Poynting flux is given by

$$\langle\vec{S}\rangle=\frac{1}{8\pi}\,\vec{E}\times\vec{B}^{\,*}\,. \tag{8.43}$$

Then the power radiated into the solid angle $d\Omega$ is given by

$$\begin{aligned} dP &= \langle\vec{S}\rangle\cdot\vec{n}\,r^2 d\Omega\,,\\ &= \frac{1}{8\pi}\,[(-\vec{n}\times\vec{B})\times\vec{B}^{\,*}\,]\cdot\vec{n}\,r^2 d\Omega\,,\\ &= \frac{1}{8\pi}\,|\vec{B}|^2\,r^2 d\Omega\,. \end{aligned} \tag{8.44}$$

From (8.41) we therefore have

$$\frac{dP}{d\Omega}=\frac{k^4}{8\pi}\,|\vec{n}\times\vec{p}\,|^2=\frac{k^4}{8\pi}\,([|\vec{p}|^2-(\vec{n}\cdot\vec{p}\,)^2]\,. \tag{8.45}$$

If we take θ to be the angle between $\vec{p}$ and $\vec{n}$, so that $\vec{n}\cdot\vec{p}=p\cos\theta$, then this gives

$$\frac{dP}{d\Omega}=\frac{k^4}{8\pi}\,|\vec{p}\,|^2\sin^2\theta\,. \tag{8.46}$$

Since $d\Omega=\sin\theta d\theta d\varphi$, the total power radiated by the oscillating dipole is then given by

$$P=\int\frac{dP}{d\Omega}d\Omega=2\pi\,\frac{k^4}{8\pi}\,|\vec{p}\,|^2\int_0^\pi\sin^3\theta d\theta=\tfrac{1}{3}k^4\,|\vec{p}\,|^2\,. \tag{8.47}$$

As a concrete example, consider a dipole antenna comprising two thin conducting rods running along the z axis, meeting (but not touching) at the origin, and extending to $z = \pm\frac{1}{2}d$ respectively. The antenna is driven at the centre ($z = 0$) by an alternating current source with angular frequency ω. The current will fall off as a function of z, becoming zero at the tips of the antenna at $z = \pm\frac{1}{2}d$. A reasonable approximation, in the regime we are considering here where $kd << 1$, is that this fall-off is linear in z. Thus we may assume

$$I(z,t) = I(z)e^{-\mathrm{i}\,\omega t} = I_0\,\Big(1 - \frac{2|z|}{d}\Big)\,e^{-\mathrm{i}\,\omega t}\,. \tag{8.48}$$

The equation of charge conservation, $\vec\nabla\cdot\vec J + \partial\rho/\partial t = 0$ then allows us to solve for the charge density. The current (8.48) is essentially confined to the line $x = y = 0$, since we are assuming the conducting rods that form the antenna are thin. Thus really, we have

$$\vec J(\vec r, t) = I(z,t)\,\delta(x)\delta(y)\,. \tag{8.49}$$

Similarly, the charge density will be given by

$$\rho(\vec r, t) = \lambda(z,t)\,\delta(x)\delta(y)\,, \tag{8.50}$$

where $\lambda(z,t)$ is the charge per unit length in the rods. The charge conservation equation therefore becomes

$$\frac{\partial I(z,t)}{\partial z} + \frac{\partial\lambda(z,t)}{\partial t} = 0\,, \tag{8.51}$$

and so, in view of the time dependence, which implies also $\lambda(z,t) = \lambda(z)e^{-\mathrm{i}\,\omega t}$, we have

$$\frac{\partial I(z)}{\partial z} - \mathrm{i}\,\omega\lambda(z) = 0\,. \tag{8.52}$$

Thus we shall have

$$\lambda(z) = -\frac{\mathrm{i}}{\omega}\,\frac{\partial I(z)}{\partial z} = -\frac{\mathrm{i}}{\omega}\,I_0\,\frac{\partial}{\partial z}\Big(1 - \frac{2|z|}{d}\Big)\,. \tag{8.53}$$

This implies

$$\begin{aligned}\lambda(z) &= \frac{2\mathrm{i}\,I_0}{\omega d}\,, \qquad z > 0\,,\\ \lambda(z) &= -\frac{2\mathrm{i}\,I_0}{\omega d}\,, \qquad z < 0\,.\end{aligned} \tag{8.54}$$

The dipole moment $\vec p$ is directed along the z axis, $\vec p = (0,0,p)$, and is given by

$$p = \int_{-d/2}^{d/2} z\lambda(z)dz = \frac{2\mathrm{i}\,I_0}{\omega d}\int_0^{d/2} zdz - \frac{2\mathrm{i}\,I_0}{\omega d}\int_{-d/2}^{0} zdz = \frac{\mathrm{i}\,I_0 d}{2\omega}\,. \tag{8.55}$$

From (8.46), we therefore find that the power per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{k^4|\vec p|^2}{8\pi}\sin^2\theta = \frac{I_0^2\,(kd)^2}{32\pi}\,\sin^2\theta\,, \tag{8.56}$$

where θ is the angle between $\vec{n} = \vec{r}/r$ and the z axis. The total radiated power is therefore given by

$$P = \tfrac{1}{12} I_0^2\,(kd)^2\,. \tag{8.57}$$

8.3 Higher multipoles

As mentioned previously, in a multipole expansion we can obtain exact expressions, term by term, for the electric and magnetic fields. To do this, we go back to the general integral expression

$$\vec{A}(\vec{r}) = \int \vec{J}(\vec{r}\,')\,\frac{e^{\mathrm{i}\,k|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'\,, \tag{8.58}$$

giving $\vec{A}(\vec{r},t) = \vec{A}(\vec{r})\,e^{-\mathrm{i}\,\omega t}$. Let

$$\frac{1}{r}\,e^{\mathrm{i}\,kr} = f(\vec{r}) = f(r)\,. \tag{8.59}$$

(Note that $f(\vec{r}) = f(r)$, i.e. it depends only on the magnitude of $\vec{r}$.) It follows that

$$\frac{e^{\mathrm{i}\,k|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = f(\vec{r}-\vec{r}\,')\,, \tag{8.60}$$

which we can therefore express as the Taylor series

$$\begin{aligned}
f(\vec{r}-\vec{r}\,') &= f(\vec{r}) - x_i'\partial_i f(\vec{r}) + \frac{1}{2!}x_i'x_j'\partial_i\partial_j f(\vec{r}) + \cdots \\
&= f(r) - x_i'\partial_i f(r) + \frac{1}{2!}x_i'x_j'\partial_i\partial_j f(r) + \cdots\,, \\
&= f(r) - x_i'(\partial_i r)\,f'(r) + \tfrac{1}{2}x_i'x_j'[(\partial_i\partial_j r)f'(r) + (\partial_i r)(\partial_j r)\,f''(r)] + \cdots
\end{aligned} \tag{8.61}$$

Thus we find

$$\frac{e^{\mathrm{i}\,k|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\,e^{\mathrm{i}\,kr} + \Big(\frac{1}{r^2} - \frac{\mathrm{i}\,k}{r}\Big)\,(\vec{n}\cdot\vec{r}\,')\,e^{\mathrm{i}\,kr} + \cdots\,. \tag{8.62}$$

The first term in this series gives the electric dipole contribution that we found in the previous section, in (8.26). The second term gives contributions from an electric quadrupole term and a magnetic dipole term. This gives

$$\vec{A}(\vec{r}) = e^{\mathrm{i}\,kr}\,\Big(\frac{1}{r^2} - \frac{\mathrm{i}\,k}{r}\Big)\,\int (\vec{n}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d^3\vec{r}\,'\,. \tag{8.63}$$

In order to interpret this expression, we need to manipulate the integrand a bit. Its i'th component is given by

$$\begin{aligned}
n_j x_j' J_i &= \tfrac{1}{2}(J_i x_j' - J_j x_i')n_j + \tfrac{1}{2}(J_i x_j' + J_j x_i')n_j\,, \\
&= \tfrac{1}{2}\epsilon_{ijk}\,\epsilon_{\ell mk}\,J_\ell x_m' n_j + \tfrac{1}{2}(J_i x_j' + J_j x_i')n_j\,, \\
&= -\epsilon_{ijk}\,n_j\mathcal{M}_k + \tfrac{1}{2}(J_i x_j' + J_j x_i')n_j\,,
\end{aligned} \tag{8.64}$$

where

$$\mathcal{M}_i = \tfrac{1}{2}\epsilon_{ijk}\, x'_j J_k\,, \qquad \text{i.e.} \qquad \vec{\mathcal{M}} = \tfrac{1}{2}\vec{r}\,' \times \vec{J}(\vec{r}\,') \tag{8.65}$$

is the *magnetisation* resulting from the current density $\vec{J}$.

The remaining term in (8.64), i.e. the symmetric term $\frac{1}{2}(J_i x'_j + J_j x'_i)n_j$, can be analysed as follows. Consider

$$\begin{aligned} \partial'_k(x'_i x'_j n_j J_k) &= \delta_{ik} x'_j n_j J_k + \delta_{jk} x'_i n_j J_k + x'_i x'_j n_j \partial'_k J_k\,, \\ &= (x'_i J_j + x'_j J_i) n_j + \mathrm{i}\, x'_i x'_j n_j \omega\rho\,. \end{aligned} \tag{8.66}$$

Integrating this over all space, the left-hand side can be turned into a surface integral over the sphere at infinity, which therefore gives zero. Thus we conclude that

$$\int (x'_i J_j + x'_j J_i) n_j d^3\vec{r}\,' = -\mathrm{i}\,\omega \int x'_i x'_j n_j \rho d^3\vec{r}\,'\,. \tag{8.67}$$

The upshot is that

$$\int (\vec{n}\cdot\vec{r}\,')\, \vec{J}(\vec{r}\,')\, d^3\vec{r}\,' = -\vec{n} \times \int \vec{\mathcal{M}}\, d^3\vec{r}\,' - \frac{\mathrm{i}\,\omega}{2} \int \vec{r}\,'\, (\vec{n}\cdot\vec{r}\,')\, \rho(\vec{r}\,')\, d^3\vec{r}\,'\,. \tag{8.68}$$

Defining the magnetic dipole moment $\vec{m}$ by

$$\vec{m} = \int \vec{\mathcal{M}}\, d^3\vec{r}\,' = \tfrac{1}{2} \int \vec{r}\,' \times \vec{J}(\vec{r}\,')\, d^3\vec{r}\,'\,, \tag{8.69}$$

we conclude that

$$\vec{A}(\vec{r}) = e^{\mathrm{i}\,kr} \Big(\frac{\mathrm{i}\,k}{r} - \frac{1}{r^2}\Big)\, \vec{n} \times \vec{m} + \frac{\mathrm{i}\,k}{2}\, e^{\mathrm{i}\,kr} \Big(\frac{\mathrm{i}\,k}{r} - \frac{1}{r^2}\Big) \int \vec{r}\,'\, (\vec{n}\cdot\vec{r}\,')\, \rho(\vec{r}\,')\, d^3\vec{r}\,'\,. \tag{8.70}$$

8.3.1 Magnetic dipole term

Consider the magnetic dipole term in (8.70) first:

$$\vec{A}(\vec{r}) = e^{\mathrm{i}\,kr} \Big(\frac{\mathrm{i}\,k}{r} - \frac{1}{r^2}\Big)\, \vec{n} \times \vec{m}\,. \tag{8.71}$$

Let

$$f \equiv e^{\mathrm{i}\,kr} \Big(\frac{\mathrm{i}\,k}{r^2} - \frac{1}{r^3}\Big)\,, \tag{8.72}$$

so $\vec{A} = r f \vec{n} \times \vec{m} = f \vec{r} \times \vec{m}$. Then from $\vec{B} = \vec{\nabla} \times \vec{A}$ we shall have

$$\begin{aligned} B_i &= \epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\, \epsilon_{k\ell m}\, \partial_j (f x_\ell) m_m\,, \\ &= \epsilon_{ijk}\, \epsilon_{k\ell m}\, (f' x_\ell \frac{x_j}{r} + f \delta_{j\ell})\,, \\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})(f' x_\ell \frac{x_j}{r} + f\delta_{j\ell})\,, \\ &= r f' n_i \vec{n}\cdot\vec{m} - r f' m_i - 2 f m_i\,. \end{aligned} \tag{8.73}$$

From (8.72) we have

$$f' = e^{\mathrm{i}\,kr}\Big(\frac{3}{r^3} - \frac{3\mathrm{i}\,k}{r^2} - \frac{k^2}{f}\Big) = -\frac{k^2}{r}\,e^{\mathrm{i}\,kr} - 3f\,, \tag{8.74}$$

and so we find

$$\vec B = -k^2\vec n\times(\vec n\times\vec m)\,\frac{e^{\mathrm{i}\,kr}}{r} + [3\vec n\,(\vec n\cdot\vec m) - \vec m\,]\Big(\frac{1}{r^3} - \frac{\mathrm{i}\,k}{r^2}\Big)\,e^{\mathrm{i}\,kr}\,. \tag{8.75}$$

Note that this is identical to the expression (8.40) for the electric field of an electric dipole source, with the electric dipole $\vec p$ replaced by the magnetic dipole, and the electric field replaced by the magnetic field:

$$\vec p \longrightarrow \vec m\,,\qquad \vec E\longrightarrow \vec B\,. \tag{8.76}$$

The electric field of the magnetic dipole can be obtained from (8.5). However, a simpler way to find it here is to note that from the Maxwell equation $\vec\nabla\times\vec E = -\partial\vec B/\partial t$ we have

$$\vec\nabla\times\vec E = \mathrm{i}\,\omega\,\vec B = \mathrm{i}\,k\,\vec B\,, \tag{8.77}$$

and so

$$\vec B = -\frac{\mathrm{i}}{k}\,\vec\nabla\times\vec E\,. \tag{8.78}$$

Now, we already saw from the caclautions for the electric dipole that when the $\vec B$ field (8.38) is substituted into (8.5), it gives rise to the $\vec E$ field given in (8.40). As we just noted, in the present magnetic dipole case, the expression for the $\vec B$ field is just like the expression for the $\vec E$ field in the electric dipole case, and we already know that in the electric case, the $\vec B$ field is given by (8.38). Therefore, we can conclude that in the present magnetic case, the $\vec E$ field that would yield, using (8.78), the result (8.75) for the $\vec B$ field will be just the negative of the expression for $\vec B$ in the electric case (with $\vec p$ replaced by $\vec m$). (The reason for the minus sign is that (8.78) has a minus sign, as compared with (8.5), under the exchange of $\vec E$ and $\vec B$.) Thus the upshot is that the electric field for the magnetic dipole radiation will be given by

$$\vec E = -k^2(\vec n\times\vec m)\,\frac{e^{\mathrm{i}\,kr}}{r}\,\Big(1+\frac{\mathrm{i}}{kr}\Big)\,. \tag{8.79}$$

This result can alternatively be verified (after a rather involved calculation) by directlty substituting (8.75) into (8.5).[21]

[21]The only "gap" in the simple argument we just presented is that any other vector $\vec E\,' = \vec E + \vec\nabla h$ would also give the same $\vec B$ field when plugged into (8.78), where h was an arbitrary function. However, we know that $\vec\nabla\cdot\vec E$ should vanish (we are in a region away from sources), and it is obvious almost by inspection that

An observation from the calculations of the electric and magnetic fields for electric dipole radiation and magnetic dipole radiation is that there is a discrete symmetry under which the two situations interchange:

$$
\begin{aligned}
\vec{p} &\longrightarrow \vec{m} \\
\vec{E} &\longrightarrow \vec{B} \\
\vec{B} &\longrightarrow -\vec{E}
\end{aligned}
\tag{8.80}
$$

This is an example of what is known as "electric/magnetic duality" in Maxwell's equations.

8.3.2 Electric quadrupole term

We now return to the electric quadrupole term in (8.70), namely

$$\vec{A}(\vec{r}) = \frac{\mathrm{i}\,k}{2}\, e^{\mathrm{i}\,kr} \Big(\frac{\mathrm{i}\,k}{r} - \frac{1}{r^2}\Big) \int \vec{r}\,'\,(\vec{n}\cdot\vec{r}\,')\,\rho(\vec{r}\,')\,d^3\vec{r}\,'\,. \tag{8.81}$$

For simplicity, we shall keep only the leading-order radiation term in this expression,

$$\vec{A}(\vec{r}) = -\tfrac{1}{2}k^2 \frac{e^{\mathrm{i}\,kr}}{r} \int \vec{r}\,'\,(\vec{n}\cdot\vec{r}\,')\,\rho(\vec{r}\,')\,d^3\vec{r}\,'\,. \tag{8.82}$$

and furthermore when calculating the $\vec{B}$ and $\vec{E}$ fields, we shall keep only the leading-order $1/r$ terms that come from the derivatives hitting $e^{\mathrm{i}\,kr}$. Thus, from $\vec{B} = \vec{\nabla}\times\vec{A}$ we shall have

$$
\begin{aligned}
\vec{B} &= -\tfrac{1}{2}k^2\,(\mathrm{i}\,k)\,\vec{n}\times\frac{e^{\mathrm{i}\,kr}}{r}\int \vec{r}\,'\,(\vec{n}\cdot\vec{r}\,')\,\rho(\vec{r}\,')\,d^3\vec{r}\,'\,, \\
&= -\frac{\mathrm{i}\,k^3}{2}\,\frac{e^{\mathrm{i}\,kr}}{r}\int (\vec{n}\times\vec{r}\,')(\vec{n}\cdot\vec{r}\,')\,\rho(\vec{r}\,')\,d^3\vec{r}\,'\,.
\end{aligned}
\tag{8.83}
$$

This radiation field can therefore be written simply as

$$\vec{B} = \mathrm{i}\,k\,\vec{n}\times\vec{A}\,. \tag{8.84}$$

In fact, in any expression where we keep only the leading-order term in which the derivative lands on $e^{\mathrm{i}\,kr}$, we shall have the rule

$$\vec{\nabla} \longrightarrow \mathrm{i}\,k\,\vec{n}\,. \tag{8.85}$$

For the electric field, we have, using (8.5) and (8.85),

$$\vec{E} = \frac{\mathrm{i}}{k}\,\vec{\nabla}\times\vec{B} = -\vec{n}\times\vec{B} = -\mathrm{i}\,k\vec{n}\times(\vec{n}\times\vec{A})\,. \tag{8.86}$$

the answer given in (8.79) satisfies this condition. Thus if we had arrived at the wrong answer for $\vec{E}$, it could be wrong only by a term $\vec{\nabla}h$ where $\nabla^2 h = 0$. There is no such function with an exponential factor $e^{\mathrm{i}\,kr}$, and so there is no possibility of our answer (8.79) being wrong. If any doubts remain, the reader is invited to substitute (8.75) into (8.5) to verify (8.79) directly.

The electric quadrupole moment tensor Q_{ij} is defined by

$$Q_{ij} = \int (3x_i x_j - r^2 \delta_{ij})\, \rho(\vec{r})\, d^3\vec{r}\,. \tag{8.87}$$

Define the vector $\vec{Q}(\vec{n})$, whose components $Q(\vec{n})_i$ are given by

$$Q(\vec{n})_i \equiv Q_{ij}\, n_j\,. \tag{8.88}$$

Consider the expression $\frac{1}{3}\vec{n} \times \vec{Q}(\vec{n})$. We shall have

$$\begin{aligned}
[\tfrac{1}{3}\vec{n} \times \vec{Q}(\vec{n})]_i &= \tfrac{1}{3}\epsilon_{ijk}\, n_j Q_{k\ell}\, n_\ell\,,\\
&= \tfrac{1}{3}\epsilon_{ijk}\, n_j n_\ell \int (3x_k x_\ell - r^2 \delta_{k\ell})\, \rho(\vec{r})\, d^3\vec{r}\,,\\
&= \int (\vec{n} \times \vec{r})_i\, (\vec{n}\cdot\vec{r})\, \rho(\vec{r})\, d^3\vec{r}\,,
\end{aligned} \tag{8.89}$$

(since the trace term gives zero). This implies that the expression (8.83) for the electric-quadrupole $\vec{B}$ field can be written as

$$\vec{B} = -\frac{\mathrm{i}\, k^3}{6r}\, e^{\mathrm{i}\, kr}\, \vec{n} \times \vec{Q}(\vec{n})\,. \tag{8.90}$$

Since we have $\vec{E} = \vec{B} \times \vec{n}$ (see (8.86)), it follows that the time-averaged power per unit solid angle will be given by

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{8\pi}\, (\vec{E} \times \vec{B}^{\,*}) \cdot \vec{n}\, r^2\,,\\
&= \frac{k^6}{288\pi}\, |(\vec{n} \times \vec{Q}(\vec{n})) \times \vec{n}|^2 = \frac{k^6}{288\pi}\, (|\vec{Q}(\vec{n})|^2 - |\vec{n}\cdot\vec{Q}(\vec{n})|^2)\,,
\end{aligned} \tag{8.91}$$

and so

$$\frac{dP}{d\Omega} = \frac{k^6}{288\pi}\, |\vec{n} \times \vec{Q}(\vec{n})|^2\,. \tag{8.92}$$

Written using indices, this is therefore

$$\frac{dP}{d\Omega} = \frac{k^6}{288\pi}\, (Q_{ki} Q^*_{kj} n_i n_j - Q_{ij} Q^*_{k\ell} n_i n_j n_k n_\ell)\,. \tag{8.93}$$

As always, having obtained an expression for the power radiated per unit solid angle, it is natural to integrate this up over the sphere, in order to obtain the total radiated power. In this case, we shall need to evaluate

$$\int n_i n_j\, d\Omega\,, \qquad \text{and} \qquad \int n_i n_j n_k n_\ell\, d\Omega\,. \tag{8.94}$$

One way to do this is to parameterise the unit vector $\vec{n}$ in terms of spherical polar angles (θ, φ) in the usual way,

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,, \tag{8.95}$$

and slog out the integrals with $d\Omega = \sin\theta d\theta d\varphi$.

A more elegant way to evaluate the integrals in (8.94) is as follows. For the first integral, we note that the answer, whatever it is, must be a symmetric 2-index tensor. It must also be completely isotropic, since by the time we have integrated over all solid angles it is not possible for the result to be "biased" so that it favours any direction in space over any other. There is only one possibility for the symmetric isotropic tensor; it must be a constant multiple of the Krönecker delta,

$$\int n_i n_j \, d\Omega = c\delta_{ij} \,. \tag{8.96}$$

The constant c can be determined by taking the trace, and using $n_i n_i = 1$:

$$4\pi = \int d\Omega = 3c \,, \tag{8.97}$$

and so we have

$$\int n_i n_j \, d\Omega = \frac{4\pi}{3}\, \delta_{ij} \,. \tag{8.98}$$

In case one doubted this result, it is not too hard in this case to confirm the result by evaluating all the integrals explicitly using (8.95).

Turning now to the secon integral in (8.94), we can use a similar argument. The answer must be a 4-index totally symmetric isotropic tensor. In fact the *only* symmetric isotropic tensors are those that can be made by taking products of Krönecker deltas, and so in this case it must be that

$$\int n_i n_j n_k n_\ell \, d\Omega = b\,(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \,, \tag{8.99}$$

for some constant b. We can determine the constant by multiplying both sides by $\delta_{ij}\delta_{k\ell}$, giving

$$4\pi = \int d\Omega = (9+3+3)b = 15b \,, \tag{8.100}$$

and so

$$\int n_i n_j n_k n_\ell \, d\Omega = \frac{4\pi}{15}\,(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \,. \tag{8.101}$$

With these results we shall have from (8.93) that

$$\begin{aligned}
P &= \int \frac{dP}{d\Omega}\, d\Omega = \frac{k^6}{288\pi}\Big[Q_{ki}Q^*_{kj}\int n_i n_j \, d\Omega - Q_{ij}Q^*_{k\ell}\int n_i n_j n_k n_\ell \, d\Omega\Big] \,, \\
&= \frac{k^6}{288\pi}\Big[\frac{4\pi}{3} Q_{ki}Q^*_{kj}\delta_{ij} - \frac{4\pi}{15}\, Q_{ij}Q^*_{k\ell}(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\Big] \,, \\
&= \frac{k^6}{216}\Big[Q_{ij}Q^*_{ij} - \tfrac{2}{5} Q_{ij}Q^*_{ij} - \tfrac{1}{5} Q_{ii}Q^*_{jj}\Big] \,, \\
&= \frac{k^6}{360}\, Q_{ij}Q^*_{ij} \,.
\end{aligned} \tag{8.102}$$

(Recall that Q_{ij} is symmetric and traceless.)

Since the quadrupole moment tensor Q_{ij} is symmetric, it is always possible to choose an orientation for the Cartesian coordinate system such that Q_{ij} becomes diagonal. (This is because the matrix U that diagonalises Q, $Q \to Q_{\rm diag} = U^T Q U$ is itself orthogonal, $U^T U = 1\!\!1$, and therefore the diagonalisation is achieved by an orthogonal transformation of the coordinates.) Thus, having chosen an appropriate orientation for the Cartesian axes, we can assume that

$$Q_{ij} = \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix}, \qquad \text{where} \qquad Q_1 + Q_2 + Q_3 = 0\,. \tag{8.103}$$

The expression (8.93) for the angular power distribution will give

$$\frac{dP}{d\Omega} = \frac{k^6}{288\pi} \Big(Q_1^2 n_1^2 + Q_2^2 n_2^2 + Q_3^2 n_3^2 - (Q_1 n_1^2 + Q_2 n_2^2 + Q_3 n_3^2)^2 \Big)\,. \tag{8.104}$$

One can substitute (8.95) into this in order to obtain an explicit expression for the $dP/d\Omega$ in terms of spherical polar angles (θ, φ).

Consider for simplicity the special case where $Q_1 = Q_2$. This means that there is an axial symmetry around the z axis, and also we shall have

$$Q_1 = Q_2 = -\tfrac{1}{2} Q_3\,. \tag{8.105}$$

Substituting (8.95) and (8.105) into (8.104), we obtain

$$\frac{dP}{d\Omega} = \frac{k^6\, Q_3^2}{128\pi} \sin^2\theta\, \cos^2\theta = \frac{k^6\, Q_3^2}{512\pi} \sin^2 2\theta\,. \tag{8.106}$$

This is indeed, as expected, azimuthally symmetric (it does not depend on φ). It describes a quadrafoil-like power distribution, with four lobes, unlike the figure-of-eight power distribution of the electric dipole radiation. Note also that its frequency dependence is proportional to ω^6 $(= k^6)$, unlike the electric dipole radiation that is proportional to ω^4. A plot of the power distribution for quadrupole radiation is given in Figure 3 below.

8.4 Linear antenna

In the later part of section 8.2, we considered a centre-fed dipole antenna. In that section we made the assumption that the wavelength of the electromagnetic radiation was very large compared with the length of the dipole, i.e. that $kd << 1$. In that limit, one could

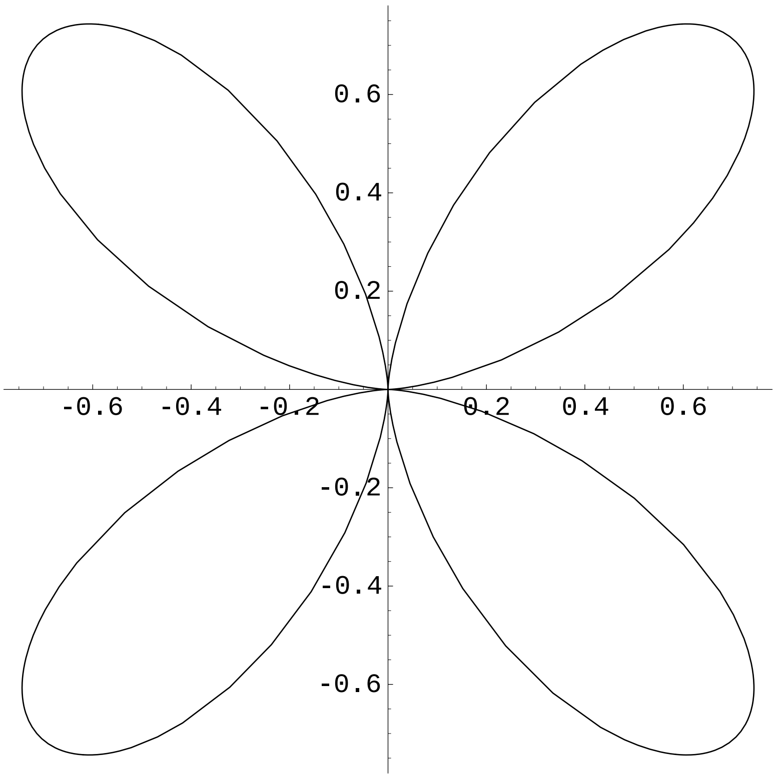

Figure 4: The angular power distribution for electric quadrupole radiation

assume to a good approximation that the current in each arm of the dipole antenna fell off in a linear fashion as a function of z (the axis along which the dipole is located). Thus, with the dipole arms extanding over the intervals

$$-\tfrac{1}{2}d \le z < 0 \qquad \text{and} \qquad 0 < z \le \tfrac{1}{2}d\,, \tag{8.107}$$

we assumed there that the current in each arm was proportional to $(d/2 - |z|)$.

In this section, we shall consider the case where the dipole arms of not assumed to be short campared to the wavelength. Under these circumstances, it can be shown that the current distribution in the dipole arms takes the form

$$\vec{J}(\vec{r},t) = I_0\, \sin k(d/2 - |z|)\, e^{-\mathrm{i}\,\omega t}\, \delta(x)\delta(y)\, \vec{Z}\,, \qquad |z| \le \tfrac{1}{2}d\,, \tag{8.108}$$

where $\vec{Z} = (0,0,1)$ is the unit vector along the z-axis, which is the axis along which the dipole is located.

We then have $\vec{A}(\vec{r},t) = \vec{A}(\vec{r})\, e^{-\mathrm{i}\,\omega t}$, where

$$\vec{A}(\vec{r}) = \int \frac{\vec{J}(\vec{r}\,', t - |\vec{r} - \vec{r}\,|)\, d^3\vec{r}\,'}{|\vec{r} - \vec{r}\,'|}\,. \tag{8.109}$$

Thus in the radiation zone, with $|\vec{r} - \vec{r}\,'| \approx r - \vec{n} \cdot \vec{r}\,'$ as usual, we therefore have

$$\begin{aligned} \vec{A}(\vec{r}) &\approx \vec{Z}\, \frac{I_0\, e^{\mathrm{i}\, kr}}{r} \int_{-d/2}^{d/2} \sin k(d/2 - |z|)\, e^{-\mathrm{i}\, kz\cos\theta}\, dz\,, \\ &= \vec{Z}\, \frac{2I_0\, e^{\mathrm{i}\, kr}}{r}\, \frac{\cos(\frac{1}{2}kd\cos\theta) - \cos(\frac{1}{2}kd)}{\sin^2\theta} \end{aligned} \tag{8.110}$$

As we saw earlier, the magnetic field is given by $\mathrm{i}\, k\, \vec{n} \times \vec{A}$ in the radiation zone, and $\vec{E} = -\vec{n} \times \vec{B}$. Therefore the radiated power per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{r^2}{8\pi}\, |\vec{E} \times \vec{B}^{\,*}|^2 = \frac{r^2}{8\pi}\, |(\vec{B} \cdot \vec{B}^{\,*})\, \vec{n}|^2 = \frac{r^2}{8\pi}\, |\vec{B}|^2\,. \tag{8.111}$$

Here we have

$$|\vec{B}|^2 = |\mathrm{i}\, k\vec{n} \times \vec{A}|^2 = k^2(|\vec{A}|^2 - (\vec{n} \cdot \vec{A})^2) = k^2|\vec{A}|^2 \sin^2\theta\,, \tag{8.112}$$

since $\vec{n} \cdot \vec{Z} = \cos\theta$, and so the radiated power per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi}\, \Big[\frac{\cos(\frac{1}{2}kd\cos\theta) - \cos(\frac{1}{2}kd)}{\sin\theta}\Big]^2\,. \tag{8.113}$$

We can now consider various special cases:

8.4.1 $kd << 1$:

In this case, we can make Taylor expansions of the trigonometric functions in the numerator in (8.113), leading to

$$\begin{aligned} \frac{dP}{d\Omega} &\approx \frac{I_0^2}{2\pi}\, \Big[\frac{1 - \frac{1}{2}(\frac{1}{2}kd)^2\cos^2\theta - 1 - \frac{1}{2}(\frac{1}{2}kd)^2}{\sin\theta}\Big]^2\,, \\ &= \frac{I_0^2}{2\pi}\, \Big(\frac{\frac{1}{2}(\frac{1}{2}kd)^2\, \sin^2\theta}{\sin\theta}\Big)^2\,, \\ &= \frac{I_0^2\, (kd)^2\, \sin^2\theta}{128\pi}\,. \end{aligned} \tag{8.114}$$

This agrees with the result (8.56), after making allowance for the fact that the current in the calculation leading to (8.56) was twice as large as the current in the present calculation.

8.4.2 $kd = \pi$:

In this case, each arm of the dipole has a length equal to $\frac{1}{4}$ of the wavelength, and so $I(z) = I_0\, \sin\frac{1}{2}\pi(1 - 2|z|/d)$. In this case, (8.113) becomes

$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi}\, \frac{\cos^2(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\,. \tag{8.115}$$

8.4.3 $kd = 2\pi$:

In this case, each dipole arm has a length equal to $\frac{1}{2}$ of the wavelength, and $I(z) = I_0 \sin\pi(1 - 2|z|/d)$. In this case (8.113) becomes

$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi}\,\frac{\cos^4(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\,. \tag{8.116}$$

9 Electromagnetism and Quantum Mechanics

9.1 The Schrödinger equation and gauge transformations

We saw at the end of chapter 2, in equation (2.102), that in the non-relativistic limit the Hamiltonian describing a particle of mass m and charge e in the presence of electromagnetic fields given by potentials ϕ and $\vec{A}$ is

$$H = \frac{1}{2m}\,(\pi_i - eA_i)^2 + e\phi\,, \tag{9.1}$$

where π_i is the canonical 3-momentum. In quantum mechanics, we the standard prescription for writing down the Schrödinger equation for the wavefunction ψ describing the particle is to interpret π_i as an operator, and to write

$$H\,\psi = \mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t}\,. \tag{9.2}$$

In the position representation we shall have

$$\pi_i = -\mathrm{i}\,\hbar\,\partial_i\,, \qquad \text{or} \qquad \vec{\pi} = -\mathrm{i}\,\hbar\,\vec{\nabla}\,. \tag{9.3}$$

Thus the Schrödinger equation for a particle of mass m and charge e in an electromagnetic field is

$$-\frac{\hbar^2}{2m}\Big(\vec{\nabla} - \frac{\mathrm{i}\,e}{\hbar}\,\vec{A}\Big)^2\,\psi + e\phi\,\psi = \mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t}\,. \tag{9.4}$$

The Schrödinger equation (9.4) is written in terms of the scalar and vector potentials ϕ and $\vec{A}$ that describe the electromagnetic field. Thus, if we perform a gauge transformation

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda\,, \qquad \phi \longrightarrow \phi' = \phi - \frac{\partial\lambda}{\partial t}\,, \tag{9.5}$$

the Schrödinger equation will change its form. On the other hand, we expect that the physics should be unaltered by a mere gauge transformation, since this leaves the physically-observable electric and magnetic fields unchanged. It turns out that we should simultaneously perform the very specific phase transformation on the wavefunction ψ,

$$\psi \longrightarrow \psi' = e^{\mathrm{i}\,e\lambda/\hbar}\,\psi \tag{9.6}$$

then the Schrödinger equation expressed entirely in terms of the primed quantities (i.e. wavefunction ψ' and electromagnetic potentials ϕ' and $\vec{A}\,'$) will take the identical form to the original unprimed equation (9.4). Thus, we may say that the Schrödinger equation transforms *covariantly* under gauge transformations.

To see the details of how this works, it is useful first to define what are called *covariant derivatives.* We this both for the three spatial derivatives, and also for the time derivative. Thus we define

$$D_i \equiv \partial_i - \frac{\mathrm{i}\,e}{\hbar}\,A_i\,, \qquad D_0 \equiv \frac{\partial}{\partial t} + \frac{\mathrm{i}\,e}{\hbar}\,\phi\,. \tag{9.7}$$

Note that the original Schrödinger equation (9.4) is now written simply as

$$-\frac{\hbar^2}{2m}\,D_i D_i\psi - \mathrm{i}\,\hbar\,D_0\psi = 0\,. \tag{9.8}$$

Next, perform the transformations

$$\begin{aligned} \vec{A} \quad &\longrightarrow \quad \vec{A}\,' = \vec{A} + \vec{\nabla}\lambda\,, \qquad \phi \longrightarrow \phi' = \phi - \frac{\partial\lambda}{\partial t}\,,\\ \psi \quad &\longrightarrow \quad \psi' = e^{\mathrm{i}\,e\lambda/\hbar}\,\psi \end{aligned} \tag{9.9}$$

The crucial point about this is that we have the following:

$$\begin{aligned} D_i'\,\psi' \quad &\equiv \quad \Big(\partial_i - \frac{\mathrm{i}\,e}{\hbar}\,A_i'\Big)\,\psi' = \Big(\partial_i - \frac{\mathrm{i}\,e}{\hbar}\,A_i - \frac{\mathrm{i}\,e}{\hbar}\,(\partial_i\lambda)\Big)\,\Big(e^{\mathrm{i}\,e\lambda/\hbar}\,\psi\Big)\,,\\ &= \quad e^{\mathrm{i}\,e\lambda/\hbar}\,\Big(\partial_i - \frac{\mathrm{i}\,e}{\hbar}\,A_i - \frac{\mathrm{i}\,e}{\hbar}\,(\partial_i\lambda) + \frac{\mathrm{i}\,e}{\hbar}\,(\partial_i\lambda)\Big)\,\psi\,,\\ &= \quad e^{\mathrm{i}\,e\lambda/\hbar}\,\Big(\partial_i - \frac{\mathrm{i}\,e}{\hbar}\,A_i\Big)\,\psi\,, \end{aligned} \tag{9.10}$$

and

$$\begin{aligned} D_0'\,\psi' \quad &\equiv \quad \Big(\frac{\partial}{\partial t} + \frac{\mathrm{i}\,e}{\hbar}\,\phi'\Big)\,\psi' = \Big(\frac{\partial}{\partial t} + \frac{\mathrm{i}\,e}{\hbar}\,\phi - \frac{\mathrm{i}\,e}{\hbar}\,\frac{\partial\lambda}{\partial t}\Big)\,\Big(e^{\mathrm{i}\,e\lambda/\hbar}\,\psi\Big)\,,\\ &= \quad e^{\mathrm{i}\,e\lambda/\hbar}\,\Big(\frac{\partial}{\partial t} + \frac{\mathrm{i}\,e}{\hbar}\,\phi - \frac{\mathrm{i}\,e}{\hbar}\,\frac{\partial\lambda}{\partial t} + \frac{\mathrm{i}\,e}{\hbar}\,\frac{\partial\lambda}{\partial t}\Big)\,\psi\,,\\ &= \quad e^{\mathrm{i}\,e\lambda/\hbar}\,\Big(\frac{\partial}{\partial t} + \frac{\mathrm{i}\,e}{\hbar}\,\phi\Big)\,\psi\,. \end{aligned} \tag{9.11}$$

In other words, we have

$$D_i'\psi' = e^{\mathrm{i}\,e\lambda/\hbar}\,D_i\psi\,, \qquad D_0'\psi' = e^{\mathrm{i}\,e\lambda/\hbar}\,D_0\psi\,, \tag{9.12}$$

which means that $D_i\psi$ and $D_0\psi$ transform the same way as ψ itself under a gauge transformation, namely just with a homogeneous phase transformation factor $e^{\mathrm{i}\,e\lambda/\hbar}$. This would not, of course, be the case for the "ordinary" derivatives $\partial_i\psi$ and $\partial_0\psi$, because for these, there would be an extra additive term, where the derivative lands on the space-time dependent gauge parameter λ.

This means that $D_i\psi$ and $D_0\psi$ transform the same way as ψ itself under the gauge transformations (9.9), namely just by acquiring the phase factor $e^{\mathrm{i}\,e\lambda/\hbar}$. This is a non-trivial statement, since the gauge parameter λ is an arbitrary function of space and time. Had we been considering standard partial derivatives ∂_i and $\partial/\partial t$ rather than the covariant deriavtives defined in (9.7), it would most certainly not have been true. For example,

$$\partial_i\psi' = \partial_i\Big(e^{\mathrm{i}\,e\lambda/\hbar}\,\psi\Big) = e^{\mathrm{i}\,e\lambda/\hbar}\,\partial_i\psi + e^{\mathrm{i}\,e\lambda/\hbar}\,\frac{\mathrm{i}\,e}{\hbar}\,(\partial_i\lambda)\,\psi \ne e^{\mathrm{i}\,e\lambda/\hbar}\,\partial_i\psi\,, \tag{9.13}$$

precisely because the derivative can land on the space-time dependent gauge-transformation parameter λ and thus give the second term, which spoils the covariance of the transformation. The point about the covariant derivatives is that the contributions from the gauge transformation of the gauge potentials precisely cancels the "unwanted" second term in (9.13).

By iterating the calculation, it also follows that $D_i'D_i'\psi' = e^{\mathrm{i}\,e\lambda/\hbar}\,D_iD_i\psi$, and so we see that the Schrödinger equation (9.8) written in terms of the primed fields, i.e.

$$-\frac{\hbar^2}{2m}\,D_i'D_i'\psi' - \mathrm{i}\,\hbar\,D_0'\psi' = 0\,, \tag{9.14}$$

just implies the Schrödinger equation in terms of unprimed fields, since

$$\begin{aligned} 0 &= -\frac{\hbar^2}{2m}\,D_i'D_i'\psi' - \mathrm{i}\,\hbar\,D_0'\psi'\,,\\ &= e^{\mathrm{i}\,e\lambda/\hbar}\,\Big(-\frac{\hbar^2}{2m}\,D_iD_i\psi - \mathrm{i}\,\hbar\,D_0\psi\Big)\,. \end{aligned} \tag{9.15}$$

What we have proved above is that the Schrödinger equation transforms covariantly under electromagnetic gauge transformations, provided that at the same time the wave function is scaled by a space-time dependent phase factor, as in (9.9). Note that we use the term "covariant transformation" here in the same sense as we used it earlier in the course when discussing the behaviour of the Maxwell equations under Lorentz transformations. The actual transformation is totally different in the two contexts; here we are discussing the behaviour of the Schrödinger equation under *gauge transformations* rather than Lorentz transformations, but in each case the essential point, which is characteristic of a covariance of any equation under a symmetry transformation, is that the equation expressed in terms of the symmetry-transformed (primed) variables is identical in form to the original equation for the unprimed variables, but with a prime placed on every field.

Note that the two definitions of the spatial and time covariant derivatives in (9.7) can be unified into the single 4-dimensional definition

$$D_\mu = \partial_\mu - \frac{\mathrm{i}\, e}{\hbar}\, A_\mu \tag{9.16}$$

since we have $A^\mu = (\phi, \vec{A})$, and hence $A_\mu = (-\phi, \vec{A})$.

The Schrödinger equation itself provides only an approximate description of the quantum theory of matter. This is obvious from the fact that it is obtained by starting from a *non-relativistic* classical Hamiltonian, and then applying the quantisation procedure. And indeed, clearly the Schrödinger equation (9.8) does not transform covariantly under Lorentz transformations. (The fact that time is treated on a completely different footing from space makes this obvious.) The non-relativistic Schrödinger equation is therefore inconsistent with causality (essentially, the notion that nothing can travel faster than light). At the very least, one should therefore be taking a *relativistic* classical theory as the the starting point for applying the quantisation procedure. It turns out that this is not sufficient. If one constructs a relativistic generalisation of the Schrödinger equation, one then encounters difficulties in giving a probabilistic interpretation of the wave function, related to the fact that the probability density current is not positive definite. The resolution of this problem requires a further process of quantisation, known as *second quantisation*, in which the wave-function itself becomes an operator that creates and annihilates particles. Theories of this type are known as *quantum field theories*, and they lie at the heart of all the fundamental theories of matter and forces.

An example is *quantum electrodynamics*, which is a quantum field theory describing electromagnetism together with electrons (and necessarily positrons, which are the antiparticles of electrons). In this theory the 4-vector gauge potential A_μ becomes a quantum field, which creates and annihilates photons, and the electron is described by a quantum field that satisfies a relativistic equation known as the Dirac equation. It is one of the most spectacularly successful theories known, leading to predictions that have been experimentally verified to quite remarkable accuracy.

9.2 Magnetic monopoles

The Maxwell equations

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -4\pi\, J^\nu\,, \\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= 0 \end{aligned} \tag{9.17}$$

take on a more symmetrical-looking form if we introduce the dual of the field-strength tensor, defined by

$$\widetilde{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma}\,. \tag{9.18}$$

In terms of $\widetilde{F}_{\mu\nu}$, the second equation in (9.17) (i.e. the Bianchi identity) becomes

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0\,. \tag{9.19}$$

From $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk} B_k$, it is easy to see that

$$\widetilde{F}_{0i} = B_i\,, \qquad \widetilde{F}_{ij} = \epsilon_{ijk}\, E_k\,. \tag{9.20}$$

It follows that $\widetilde{F}_{\mu\nu}$ is obtained from $F_{\mu\nu}$ by making the replacements

$$\vec{E} \longrightarrow -\vec{B}\,, \qquad \vec{B} \longrightarrow \vec{E}\,. \tag{9.21}$$

The symmetry between the two Maxwell equations would become even more striking if there were a current on the right-hand side of (9.19), analogous to the electric 4-current density on the right-hand-side of the first Maxwell equation in (9.17). Since the rôles of $\vec{E}$ and $\vec{B}$ are exchanged when passing from $F_{\mu\nu}$ to $\widetilde{F}_{\mu\nu}$, it is evident that the 4-current needed on the right-hand side of (9.19) must be a *magnetic* 4-current density, J_M^μ. Let us now attach a subscript E to the standard electric 4-current density, in order to emphasise which is which in the following. The generalised Maxwell equations will now be written as

$$\partial_\mu F^{\mu\nu} = -4\pi\, J_E^\nu\,, \qquad \partial_\mu \widetilde{F}^{\mu\nu} = -4\pi\, J_M^\nu\,. \tag{9.22}$$

Particles with magnetic charge, known as magnetic monopoles, have never been seen in nature. However, there seems to be no reason in principle why they should not exist, and it is of interest to explore their properties in a little more detail. A point electric charge e has an electric field given by

$$\vec{E} = \frac{e\,\vec{r}}{r^3}\,. \tag{9.23}$$

Thus by analogy, a point magnetic monopole, with magnetic charge g, will have a magnetic field given by

$$\vec{B} = \frac{g\,\vec{r}}{r^3}\,. \tag{9.24}$$

This satisfies

$$\vec{\nabla}\cdot\vec{B} = 4\pi\,\rho_M\,, \qquad \rho_M = g\,\delta^3(\vec{r})\,, \tag{9.25}$$

where $\rho_M = J_M^0$ is the magnetic charge density.

We shall be interested in studying the quantum mechanics of electrically-charged particles in the background of a magnetic monopole. Since the Schrödinger equation is written in terms of the potentials ϕ and $\vec{A}$, we shall therefore need to write down the 3-vector potential $\vec{A}$ for the magnetic monopole. To do this, we introduce Cartesian coordinates (x, y, z), related to spherical polar coordinates (r, θ, φ) in the standard way,

$$x = r\sin\theta\,\cos\varphi\,, \quad y = r\sin\theta\,\sin\varphi\,, \quad x = r\cos\theta\,, \tag{9.26}$$

and we also define

$$\rho^2 = x^2 + y^2\,. \tag{9.27}$$

Consider the 3-vector potential

$$\vec{A} = g\,\Big(\frac{zy}{r\rho^2}, -\frac{zx}{r\rho^2}, 0\Big)\,. \tag{9.28}$$

Using

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r}\,, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}\,, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}\,, \\ \frac{\partial \rho}{\partial x} &= \frac{x}{\rho}\,, \qquad \frac{\partial \rho}{\partial y} = \frac{y}{\rho}\,, \qquad \frac{\partial \rho}{\partial z} = 0\,, \end{aligned} \tag{9.29}$$

it is easily seen that

$$B_x = \partial_y A_z - \partial_z A_y = g\partial_z\Big(\frac{zx}{r\rho^2}\Big) = \frac{gx}{r\rho^2} - \frac{gxz^2}{r^3\rho^2} = \frac{gx}{r^3}\,, \tag{9.30}$$

and similarly

$$B_y = \frac{gy}{r^3}\,, \qquad B_z = \frac{gz}{r^3}\,. \tag{9.31}$$

Thus indeed we find that

$$\vec{\nabla}\times\vec{A} = \frac{g\vec{r}}{r^3}\,, \tag{9.32}$$

and so the 3-vector potential (9.28) describes the magnetic monopole field (9.24).

In terms of spherical polar coordinates we have $\rho^2 = x^2 + y^2 = r^2\sin^2\theta$, and so (9.28) can be written as

$$\vec{A} = \frac{g\cot\theta}{r}\,(\sin\varphi, -\cos\varphi, 0)\,. \tag{9.33}$$

Not surprisingly, this potential is singular at $r = 0$, since we are describing an idealised point magnetic charge. In exactly the same way, the potential $\phi = e/r$ describing a point electric charge diverges at $r = 0$ also. However, the potential (9.33) also diverges everywhere along the z axis, i.e. at $\theta = 0$ and $\theta = \pi$. It turns out that these latter singularities are "unphysical," in the sense that they can be removed by making gauge transformations.

This is not too surprising, when we note that the magnetic field itself, given by (9.24) has no singularity along the z axis. It is, of course, genuinely divergent at $r = 0$, so that is a real physical singularity.

To see the unphysical nature of the singularities in (9.33) along $\theta = 0$ and $\theta = \pi$, we need to make gauge transformations, under which

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\lambda\,. \tag{9.34}$$

Consider first taking

$$\lambda = g\,\varphi = g\,\arctan\frac{y}{x}\,. \tag{9.35}$$

From this, we find

$$\vec{\nabla}\lambda = -\frac{g}{r}\,\mathrm{cosec}\theta\,(\sin\varphi, -\cos\varphi, 0)\,. \tag{9.36}$$

Letting the gauge-transformed potential be $\vec{A}\,'$, we therefore find

$$\vec{A}\,' = \vec{A} + \vec{\nabla}\lambda = \frac{g}{r}\,\frac{\cos\theta - 1}{\sin\theta}\,(\sin\varphi, -\cos\varphi, 0) = -\frac{g}{r}\,\tan\tfrac{1}{2}\theta\,(\sin\varphi, -\cos\varphi, 0)\,. \tag{9.37}$$

It can be seen that $\vec{A}$ is completely non-singular along $\theta = 0$ (i.e. along the positive z axis). It is, however, singular along $\theta = \pi$ (i.e. along the negative z axis).

We could, on the other hand, perform a gauge transformation with λ given by

$$\lambda = -g\,\varphi = -g\,\arctan\frac{y}{x} \tag{9.38}$$

instead of (9.35). Defining the gauge-transformed potential as $\vec{A}\,''$ in this case, we find

$$\vec{A}\,'' = \frac{g}{r}\,\cot\tfrac{1}{2}\theta\,(\sin\varphi, -\cos\varphi, 0)\,. \tag{9.39}$$

This time, we have obtained a gauge potential that is non-singular along $\theta = \pi$ (i.e. the negative z axis), but it is singular along $\theta = 0$ (the positive z axis).

There is no single choice of gauge in which the 3-vector potential for the magnetic monopole is completely free of singularities away from the origin $r = 0$. We have obtained two expressions for the vector potential, one of which, $\vec{A}\,'$, is non-singular along the positove z axis, and the other, $\vec{A}\,''$, is non-singular along the negative z axis. The singularity that each has is known as a "string singularity," since it lies along a line, or string. By making gauge transformations the location of the string can be moved around, but it can never be removed altogether.

In the discussion above, the z axis appears to have played a preferred rôle, but this is, of course, just an artefact of our gauge choices. We could equally well have chosen a different expression for $\vec{A}$, related by a gauge transformation, for which the string singularity ran along any desired line, or curve, emanating from the origin.

9.3 Dirac quantisation condition

We have seen that gauge potentials for the magnetic monopole, free of singularities on the positive and negative z axes resepctively, are given by

$$\begin{aligned} \vec{A}' &= -\frac{g}{r}\,\tan\tfrac{1}{2}\theta\,(\sin\varphi, -\cos\varphi, 0)\,, \\ \vec{A}'' &= \frac{g}{r}\,\cot\tfrac{1}{2}\theta\,(\sin\varphi, -\cos\varphi, 0)\,. \end{aligned} \tag{9.40}$$

The two are themselves related by a gauge transformation, namely

$$\vec{A}'' = \vec{A}' + \vec{\nabla}(-2g\varphi)\,. \tag{9.41}$$

Now let us consider the quantum mechanics of an electron in the background of the magnetic monopole. As we discussed in section 9.1, the Schrödinger equation for the electron is given by (9.4), where e is its charge, and m is its mass. We shall consider the Schrödinger equation in two different gauges, related as in (9.41). Denoting the corresponding electron wave-functions by ψ' and ψ'', we see from (9.9) (9.41) that we shall have

$$\psi'' = e^{-2\mathrm{i}\, eg\varphi/\hbar}\,\psi'\,. \tag{9.42}$$

However, we have seen that the gauge transformation is not physical, but merely corresponds to shifting the string singularity of the magnetic monopole from the negative z axis to the positive z axis. Quantum mechanically, the physics will only be unchanged if the electron wave-function remains single valued under a complete 2π rotation around the z axis. This means that the phase factor in the relation (9.42) must be equal to unity, and so it must be that

$$\frac{2eg}{\hbar}\,2\pi = 2\pi\, n\,, \tag{9.43}$$

where n is an integer. Thus it must be that the product of the electric charge e on the electron, and the magnetic charge g on the magnetic monopole, must satisfy the so-called *Dirac quantisation condition*,

$$2e\, g = n\, \hbar\,. \tag{9.44}$$

It is interesting to note that although a magnetic monopole has never been observed, it would only take the existence of a single monopole, maybe somewhere in another galaxy, to imply that electric charges everywhere in the universe must quantised in units of

$$\frac{\hbar}{2g}\,, \tag{9.45}$$

where g is the magnetic charge of the lonely magnetic monopole. In fact all observed electric charges are indeed quantised; in integer multiples of the charge e on the electron, in everyday life, and in units of $\frac{1}{3}e$ in the quarks of the theory of strong interactions. It is tempting to speculate that the reason for this may be the existence of a magnetic monopole somewhare out in the vastness of space, in a galaxy far far away.

10 Local Gauge Invariance and Yang-Mills Theory

10.1 Relativistic quantum mechanics

We saw in the previous section that the ordinary non-relativistic quantum mechanics of a charged particle in an electromagnetic field has the feature that it is *covariant* under electromagnetic gauge transformations, provided that the usual gauge trnasformation of the 4-vector potential is combined with a phase transformation of the wavefunction for the charged particle:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda\,, \qquad \psi \longrightarrow e^{\mathrm{i}\, e\lambda/\hbar}\, \psi\,. \tag{10.1}$$

The essential point here is that the gauge transformation parameter λ can be an arbitrary function of the spacetime coordinates, and so the phase transformation of the wavefunction is a spacetime-dependent one. Such spacetime-dependent transformations are known as *local* transformations.

One could turn this around, and view the introduction of the electromagnetic field as the necessary addition to quantum mechanics in order to allow the theory to be covariant under local phase transformations of the wavefunction. If we started with quantum mechanics in the absence of electromagnetism, so that for a free particle of mass m we have

$$-\frac{\hbar^2}{2m}\,\vec\nabla^2\,\psi = \mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t}\,, \tag{10.2}$$

then the Schrödinger equation is obviously covariant under *constant* phase transformations of the wavefunction,

$$\psi \longrightarrow e^{\mathrm{i}\,c}\,\psi\,, \tag{10.3}$$

where c is an arbitrary constant. And, indeed, the physics described by the wavefunction is invariant under this phase transformation, since all physical quantities are constructed using a product of ψ and its complex conjugate $\bar\psi$ (for example, the probability density $|\psi|^2$), for which the phase factors cancel out. Also, clearly, the Schrödinger equation (10.2) does *not* transform nicely under local phase transformations, since the derivatives will now land on the (spacetime dependent) phase factor in (10.3) and give a lot of messy terms.

As we now know, the way to achieve a nice covariant transformation of the Schrödinger equation under local phase transformations is to replace the partial derivatives ∂_i and ∂_0 in

(10.2) by covariant derivatives

$$D_i = \partial_i - \frac{\mathrm{i}\, e}{\hbar} A_i\,, \qquad D_0 = \partial_0 + \frac{\mathrm{i}\, e}{\hbar}\,\phi \tag{10.4}$$

where A_i and ϕ transform in the standard way for the electromagnetic gauge potentials at the same time as the local phase transformation for ψ is performed, as in (10.1). From this point of view, it could be said that we have *derived* electromagnetism as the field needed in order to allow the Schrödinger equation to transform covariantly under local phase transformations of the wavefunction.

The idea now is to extend this idea to more general situations. By again demanding local "phase" transformations of some quantum-mechanical equation, we will now be able to derive a generalisation of electromagnetism known as Yang-Mills theory.

Working with a non-relativistic equation like the Schrödinger equation is rather clumsy, because of the way in which space and time arise on such different footings. It is more elegant (and simpler) to switch at this point to the consideration of relativistic quantum mechanical equation. There are various possible equations one could consider, but they all lead to equivalent conclusions about the generalisation of electromagnetism. Examples one could consider include the Dirac equation, which provides a relativistic description of the electron, or any other fermionic particle with spin $\frac{1}{2}$. A simpler opetion is to consider the Klein-Gordon equation for a relativistic particle of spin 0 (otherwise known as a scalar field). The Klein-Gordon equation for a free scalar field φ with mass m is very simple, namely

$$\Box\,\varphi - m^2\varphi = 0\,, \tag{10.5}$$

where $\Box = \partial^\mu\partial_\mu$ is the usual d'Alembertian operator, which, as we know, is Lorentz invariant.[22] Note that from now on, we shall use units where Planck's constant $\hbar$ is set equal to 1.

In what follows, we shall make the simplifying assumption that the scalar field is massless, and so its Klein-Gordon equation is simly

$$\Box\varphi = 0\,. \tag{10.6}$$

[22]The non-relativistic Schrödinger equation can be derived from the Klein-Gordon equation (10.5) in an appropriate limit. The leading-order time dependence of a field with mass (i.e. energy) m will be $e^{-\mathrm{i}\, mt}$ (in units where we set $\hbar = 1$). Thus the appropriate non-relativistic approximation is where where the wavefunction φ is assumed to be of the form $\varphi \sim e^{-\mathrm{i}\, mt}\,\psi$, with ψ only slowly varying in time, which means that the term $\partial^2\psi/\partial t^2$ can be neglected in comparison to the others. Substituting into (10.5) and dropping the $\partial^2\psi/\partial t^2$ term gives precisely the Schrödinger equation for the free massive particle, namely $-(1/2m)\nabla^2\psi = \mathrm{i}\,\partial\psi/\partial t$.

We shall do this because no essential feature that we wish to explore will be lost, and it will slightly shorten the equations. It is completely straightforward to add it back in if desired.[23]

The Klein-Gordon equation (10.6) can be derived from the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi\,. \tag{10.7}$$

Varying the action $I = \int d^4x\mathcal{L}$, we find

$$\delta I = -\int d^4x\partial^\mu\varphi\,\partial_\mu\delta\varphi = \int d^4x\,(\partial_\mu\partial^\mu\varphi)\,\delta\varphi\,, \tag{10.8}$$

(dropping the boundary term at infinity as usual), and so demanding that the action be stationary under arbitrary variations $\delta\varphi$ implies the Klein-Gordon equation (10.6).

Before moving on to the generalisation to Yang-Mills theory, we shall first review, again, the derivation of electromagnetism as the field needed in order to turn a global phase invariance into a local invariance, this time from the viewpoint of the relativistic Klein-Gordon equation. To do this, we first need to enlarge the system of wavefunctions from one real scalar to two. Suppose, then, we have two real scalars called φ_1 and φ_2, each satisfying a Klein-Gordon equation. These equations can therefore be derived from the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_1\,\partial_\mu\varphi_1 - \tfrac{1}{2}\partial^\mu\varphi_2\,\partial_\mu\varphi_2\,. \tag{10.9}$$

We can conveniently combine the two real fields into a complex scalar field ϕ, defined by

$$\phi = \frac{1}{\sqrt{2}}\,(\varphi_1 + \mathrm{i}\,\varphi_2)\,. \tag{10.10}$$

The Lagrangian density can then be written as

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\bar\phi\,\partial_\mu\phi\,. \tag{10.11}$$

The complex field ϕ therefore satisfies the Klein-Gordon equation

$$\square\,\phi = 0\,. \tag{10.12}$$

[23]Note that we can only discuss a non-relativistic limit for the *massive* Klein-Gordon equation. This is because the non-relativisitic approximation (discussed in the previous footnote) involved assuming that each time derivative of ψ with respect to t was small compared with m times ψ. Clearly this would no longer be true if m were zero. Put another way, a massless particle is inherently relativistic, since it must travel at the speed of light (like the photon). We shall not be concerned with taking the non-relativistic limit in what follows, and so working with a massless field will not be a problem.

It is clear that the complex field ϕ has a global phase invariance, under

$$\phi \longrightarrow e^{\mathrm{i}\,\alpha}\,\phi\,, \tag{10.13}$$

where α is a constant. (The term "global" is used to describe such phase transformations, which are identical at every point in spacetime.) This can be seen at the level of the Klein-Gordon equation (10.12), since the constant phase factor simply passes straight through the d'Alembertian operator. It can also be seen at the level of the Lagrangian density, since again the derivatives do not land on the phase factor, and furthermore, the $e^{\mathrm{i}\,\alpha}$ phase factor from transforming ϕ is cancelled by the $e^{-\mathrm{i}\,\alpha}$ phase factor from tramsforming $\bar\phi$.

It is also clear that the Lagrangian density is *not* invariant under *local* phase transformations, where α is assumed now to be spacetime dependent. This is because we now have

$$\partial_\mu\phi \longrightarrow \partial_\mu(e^{\mathrm{i}\,\alpha}\,\phi) \longrightarrow e^{\mathrm{i}\,\alpha}\,\partial_\mu\phi + \mathrm{i}\,(\partial_\mu\alpha)\,\phi\,. \tag{10.14}$$

It is the second term, where the derivatives land on α, that spoils the invariance.

The remedy, not surprisingly, is to introduce a gauge potential A_μ, and replace the partial derivatives by covariant derivatives

$$D_\mu = \partial_\mu - \mathrm{i}\,e\,A_\mu\,, \tag{10.15}$$

where now ϕ will be interpreted as describing a complex scalar field with electric charge e. As we saw before when discussing the Schrödinger equation, the covariant derivative acting on ϕ has a nice transformation property under the local phase tramsformations of ϕ, provided at the same time we transform A_μ:

$$\phi \longrightarrow e^{\mathrm{i}\,\alpha}\,\phi\,,\qquad A_\mu \longrightarrow A_\mu + \frac{1}{e}\,\partial_\mu\alpha\,. \tag{10.16}$$

This implies that $D_\mu\phi$ transforms nicely as

$$D_\mu\phi \longrightarrow e^{\mathrm{i}\,\alpha}\,D_\mu\phi\,, \tag{10.17}$$

and so the new Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\overline{D^\mu\phi})\,(D_\mu\phi) \tag{10.18}$$

is indeed invariant. This is the "derivation" of ordinary electromagnetism.

In this viewpoint, where we are deriving electromagnetism by requiring the local phase invariance of the theory under (10.13), has not yet given any dynamics to the gauge field A_μ. Indeed, one cannot *derive* a dynamical existence for A_μ because in fact there is no unique answer. What one can do, however, is to introduce a dynamical term "by hand," which has all the natural properties one would like. First of all, we want a dynamical term that respects the gauge invariance we have already achieved in the rest of the Lagrangian. Secondly, we expect that it should give rise to a second-order dynamical equation for A_μ. We are back to the discussion of section 4.2, where we derived Maxwell's equations from an action principal. The steps leading to the answer are as follows.

First, to make a gauge-invariant term we need to use the gauge-invariant field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10.19}$$

as the basic "building block." Then, to make a Lorentz-invariant term, the lowest-order possibility is to form the quadratic invariant $F^{\mu\nu}\, F_{\mu\nu}$. Taking the standard normalisation as discussed in section 4.2, we are therefore lead to propose the total Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\overline{D^\mu\phi})\,(D_\mu\phi) - \frac{1}{16\pi}\, F^{\mu\nu}\, F_{\mu\nu}\,, \tag{10.20}$$

where $D_\mu = \partial_\mu - \mathrm{i}\, e\, A_\mu$. It is easily verified that the Euler-Lagrange equations resulting from this Lagrangian density are as follows. Requiring the stationarity of the action under variations of the wavefunction ϕ implies

$$D^\mu D_\mu\phi = 0\,, \qquad \text{i.e.} \qquad (\partial^\mu - \mathrm{i}\, eA^\mu)(\partial_\mu - \mathrm{i}\, eA_\mu)\phi = 0\,, \tag{10.21}$$

This is the gauge-covariant generalisation of the original uncharged Klein-Gordon equation.

Requiring stationarity under variations of A_μ implies

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu\,, \tag{10.22}$$

where

$$J_\mu = -\mathrm{i}\, e\Big(\bar\phi D_\mu\phi - (\overline{D_\mu\phi})\,\phi\Big)\,. \tag{10.23}$$

Thus A_μ satisfies the Maxwell field equation, with a source current density given by (10.23). This is exactly what one would hope for; the complex field ϕ carries electric charge e, and so it is to be expected that it should act as a source for the electromagnetic field. In the process of giving dynamics to the electromagnetic field we have, as a bonus, derived the current density for the scalar field.

10.2 Yang-Mills theory

At the end of the previous subsection we rederived electromagnetism as the field needed in order to turn the global phase invariance of a complex scalar field that satisfies the Kelin-Gordon equation into a local phase invariance. The phase factor $e^{\mathrm{i}\,\alpha}$ is a unit-modulus complex number. The set of all unit-modulus complex numbers form the group $U(1)$; i.e. 1×1 complex matrices U satisfying $U^\dagger U = 1$. (For 1×1 matrices, which are just numbers, there is of course no distinction between Hermitean conjugation, denoted by a dagger, and complex conjugation.)

In order to derive the generalisation of electromagnetism to Yang-Mills theory we need to start with an extended system of scalar fields, each satisfying the Klein-Gordon equation, whose Lagrangian is invariant under a larger, non-abelian, group.[24] We shall take the example of the group $SU(2)$ in order to illustrate the basic ideas. One can in fact construct a Yang-Mills theory based on any Lie group.

The group $SU(2)$ should be familiar from quantum mechanics, where it arises when one discusses systems with intrinstic spin $\frac{1}{2}$. The group can be defined as the set of 2×2 complex matrices U subject to the conditions

$$U^\dagger U = 1\,, \qquad \det U = 1\,. \tag{10.24}$$

It can therefore be parameterised in the form

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\,, \tag{10.25}$$

where a and b are complex numbers subject to the constraint

$$|a|^2 + |b|^2 = 1\,. \tag{10.26}$$

If we write $a = x_1 + \mathrm{i}\, x_2$, $b = x_3 + \mathrm{i}\, x_4$, the constraint is described by the surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \tag{10.27}$$

in Euclidean 4-space, and so the elements of the group $SU(2)$ are in one-to-one correspondence with the points on a unit 3-dimensional sphere. Clearly $SU(2)$ has three independent parameters.[25]

[24] An abelian group is one where the order of combination of group elements makes no difference. By contrast, for a non-ableian group, if two elements U and V are combined in the two different orderings, the results are, in general, different. Thus, for a group realised by matrices under multiplication, for example, one has in general that $UV \neq VU$.

[25] For comparison, the group $U(1)$, whose elements U can be parameterised as $U = e^{\mathrm{i}\,\alpha}$ with $0 \le \alpha < 2\pi$, has one parameter. Since $e^{\mathrm{i}\,\alpha}$ is periodic in α the elements of $U(1)$ are in one-to-one correspondence with the points on a unit circle, or 1-dimensional sphere. In fact the circle, S^1, and the 3-sphere, S^3, are the only spheres that are isomorphic to groups.

The group $SU(2)$ can be generated by exponentiating the three Pauli matrices τ_a, where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{10.28}$$

They satisfy the commutation relations

$$[\tau_a, \tau_b] = 2\mathrm{i}\,\epsilon_{abc}\,\tau_c\,, \tag{10.29}$$

i.e. $[\tau_1, \tau = 2] = 2\mathrm{i}\,\tau_3$, and cyclic permutations.

Let

$$T_a = \frac{1}{2\mathrm{i}}\,\tau_a\,. \tag{10.30}$$

We shall therefore have

$$[T_a, T_b] = \epsilon_{abc}\,T_c\,. \tag{10.31}$$

Note that the T_a, which are called the generators of the Lie algebra of $SU(2)$, are anti-Hermitean,

$$T_a^\dagger = -T_a\,. \tag{10.32}$$

They are also, of course, traceless.

The $SU(2)$ group elements can be written as

$$U = e^{\alpha_a\,T_a}\,, \tag{10.33}$$

where α_a are three real parameters. (This is the analogue of writing the $U(1)$ elements U as $U = e^{\mathrm{i}\,\alpha}$.) It is easy to check that the unitarity of U, i.e. $U^\dagger U = 1$, follows from the anti-Hermiticity of the generators T_a:

$$U^\dagger U = \Big(e^{\alpha_a T_a}\Big)^\dagger\,e^{\alpha_b T_b} = e^{\alpha_a T_a^\dagger}\,e^{\alpha_b T_b} = e^{-\alpha_a T_a}\,e^{\alpha_b T_b} = 1\,. \tag{10.34}$$

The unit-determinant property follows from the tracelessness of the T_a, bearing in mind that for any matrix X we have $\det X = \exp(\mathrm{tr}\log X)$:

$$\det U = \det(e^{\alpha_a T_a}) = \exp[\mathrm{tr}\log(e^{\alpha_a T_a})] = \exp[\mathrm{tr}(\alpha_a T_a)] = \exp[0] = 1\,. \tag{10.35}$$

Suppose now that we take a pair of *complex* scalar fields, called ϕ_1 and ϕ_2, each of which satisfies the massless Klein-Gordon equation. We may assemble them into a complex 2-vector, which we shall call ϕ:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,. \tag{10.36}$$

This vector-valued field therefore satisfies the Klein-Gordon equation

$$\Box\phi = 0\,, \tag{10.37}$$

which can be derived from the Lagrangian density

$$\mathcal{L} = -(\partial^\mu \phi^\dagger)(\partial_\mu \phi)\,. \tag{10.38}$$

It is obvious that the Lagrangian density (10.38) is invariant under global $SU(2)$ transformations

$$\phi \longrightarrow U\,\phi\,, \tag{10.39}$$

where U is a constant $SU(2)$ matrix. Thus, we have

$$\mathcal{L} \longrightarrow -\partial^\mu(\phi^\dagger\, U^\dagger)\,\partial_\mu(U\phi) = -(\partial^\mu\phi^\dagger)\,U^\dagger\,U\,(\partial_\mu\phi) = -(\partial^\mu\phi^\dagger)\,(\partial_\mu\phi) = \mathcal{L}\,. \tag{10.40}$$

Obviously, $\mathcal{L}$ would not be invariant if we allowed U to be space-time dependent, for the usual reason that we would get extra terms where the derivatives landed on the U transformation matrix. Based on our experience with the local $U(1)$ phase invariance of the theory coupled to electromagnetism, we can expect that again we could achieve a local $SU(2)$ invariance by introducing appropriate analogues of the electromagnetic field. In this case, since the $SU(2)$ group is characterised by 3 parameters rather than the 1 parameter characterising $U(1)$, we can expect that we will need 3 gauge fields rather than 1. We shall called these A_μ^a, where $1 \le a \le 3$. In fact it is convenient to assemble the three gauge fields into a 2×2 matrix, by defining

$$A_\mu = A_\mu^a\,T_a\,, \tag{10.41}$$

where T_a are the generators of the $SU(2)$ algebra that we introduced earlier.

We next define the covariant derivative D_μ, whose action on the complex 2-vector of scalar fields ϕ is defined by

$$D_\mu\phi = \partial_\mu\phi + A_\mu\,\phi\,. \tag{10.42}$$

Since we don't, *a priori*, know how A_μ should transform we shall work backwards and demand that its transformation rule should be such that D_μ satisfies the nice property we should expect of a covariant derivative in this case, namely that if we transform ϕ under a *local* $SU(2)$ transformation

$$\phi \longrightarrow \phi' = U\,\phi\,, \tag{10.43}$$

then we should also have that

$$(D_\mu\phi) \longrightarrow (D_\mu\phi)' = U(D_\mu\phi)\,. \tag{10.44}$$

Working this out, we shall have

$$\begin{aligned}(D_\mu\phi)' &= D'_\mu\,\phi' = (\partial_\mu + A'_\mu)(U\,\phi)\,,\\ &= (\partial_\mu U)\,\phi + U\,\partial_\mu\phi + A'_\mu\,U\,\phi\,,\\ &= U\,D_\mu\phi = U\,\partial_\mu\phi + U\,A_\mu\,\phi\,.\end{aligned} \tag{10.45}$$

Equating the last two lines, and noting that we want this to be true for all possible ϕ, we conclude that

$$\partial_\mu U + A'_\mu\,U = U\,A_\mu\,. \tag{10.46}$$

Multiplying on the right with $U^\dagger$ then gives the result that

$$A'_\mu = U\,A_\mu\,U^\dagger - (\partial_\mu U)\,U^\dagger\,. \tag{10.47}$$

This, then, will be the gauge transformation rule for the Yang-Mills potentials A_μ. In other words, the full set of local $SU(2)$ transformations comprise[26]

$$\begin{aligned}A_\mu &\longrightarrow A'_\mu = U\,A_\mu\,U^\dagger - (\partial_\mu U)\,U^\dagger\,,\\ \phi &\longrightarrow \phi' = U\,\phi\,.\end{aligned} \tag{10.48}$$

What we have established is that if we replace the Lagrangian density (10.38) by

$$\mathcal{L} = -(D^\mu\phi)^\dagger\,(D_\mu\phi)\,, \tag{10.49}$$

then it will be invariant under the *local* $SU(2)$ transformations given by (10.48). The proof is now identical to the previous proof of the invariance of (10.38) under global $SU(2)$ transformations. The essential point is that the local transformation matrix U "passes through" the covariant derivative, in the sense that $(D_\mu\phi)' = D'_\mu(U\phi) = U\,D_\mu\phi$.

So far, we have suceeded in constructing a theory with a local $SU(2)$ symmetry, but as yet, the Yang-Mills potentials A^a_μ that we introduced do not have any dynamics of their own. Following the strategy we applied to the case of electromagnetism and local $U(1)$ invariance, we should now look for a suitable term to add to the Lagrangian density (10.49) that will do the job. Guided by the example of electromagnetism, we should first find a a field strength tensor for the Yang-Mills fields, which will be the analogue of the electromagnetic field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{10.50}$$

[26]Note that this non-abelian result, which takes essentially the same form for any group, reduces to the previous case of electromagnetic theory is we specialise to the abelian group $U(1)$. Essentially, we would just write $U = e^{\mathrm{i}\,\alpha}$ and plug into the transformations (10.48). Since left and right multiplication are the same in the abelian case, the previous results for electromagnetic gauge invariance can be recovered.

It is clear that the expression (10.50) is not suitable in the Yang-Mills case. If one were to try adopting (10.50) as a definition for the field strength, then a simple calculation shows that under the $SU(2)$ gauge transformation for A_μ given in (10.48), the field strength would transform into a complete mess. It turns out that the appropriate generalistion that is needed for Yang-Mills is to define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \,. \tag{10.51}$$

Of course this would reduce to the standard electromagnetic field strength in the abelian $U(1)$ case, since the commutator $[A_\mu, A_\nu] \equiv A_\mu A_\nu - A_\nu A_\mu$ would then vanish.

The first task is to check how $F_{\mu\nu}$ defined by (10.51) transforms under the $SU(2)$ gauge transformation of A_μ given in (10.48). We shall have

$$\begin{aligned}
F_{\mu\nu} \quad \longrightarrow \quad & F'_{\mu\nu} = \partial_\mu A'_\nu + A'_\mu\, A'_\nu - (\mu \leftrightarrow \nu) \,, \\
= \quad & \partial_\mu (U A_\nu U^\dagger - (\partial_\nu U) U^\dagger) + (U A_\mu U^\dagger - (\partial_\mu U) U^\dagger)(U A_\nu U^\dagger - (\partial_\nu U) U^\dagger) - (\mu \leftrightarrow \nu) \,, \\
= \quad & (\partial_\mu U) A_\nu U^\dagger + U(\partial_\mu A_\nu) - U A_\nu U^\dagger (\partial_\mu U) U^\dagger - (\partial_\mu \partial_\nu U) U^\dagger + (\partial_\nu U) U^\dagger (\partial_\mu U) U^\dagger \\
& + U A_\mu U^\dagger U A_\nu U^\dagger - U A_\mu U^\dagger (\partial_\nu U) U^\dagger - (\partial_\mu U) U^\dagger U A_\nu U^\dagger \\
& + (\partial_\mu U) U^\dagger (\partial_\nu U) U^\dagger - (\mu \leftrightarrow \nu) \,, \\
= \quad & U(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu\, A_\nu - A_\nu\, A_\mu) U^\dagger \,,
\end{aligned} \tag{10.52}$$

where the notation $-(\mu \leftrightarrow \nu)$ means that one subtracts off from the terms written explicitly the same set of terms with the indices μ and ν exchanged. Comparing with (10.51) we see that the upshot is that under the $SU(2)$ gauge transformation for A_μ given in (10.48), the field strength $F_{\mu\nu}$ defined in (10.51) transforms as

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = U F_{\mu\nu} U^\dagger \,. \tag{10.53}$$

This means that $F_{\mu\nu}$ transforms *covariantly* under $SU(2)$ gauge transformations. It would of course, reduce to the *invariance* of the elctromagnetic field strength transformation ($F'_{\mu\nu} = F_{\mu\nu}$) in the abelian case.

It is now a straightforward matter to write down a suitable term to add to the Lagrangian density (10.49). As for electromagnetism, we want a gauge-invariant and Lorentz-invariant quantity that is quadratic in fields. Thus we shall take

$$\mathcal{L} = -(D^\mu \phi)^\dagger\, (D_\mu \phi) + \frac{1}{8\pi}\, \mathrm{tr}(F^{\mu\nu}\, F_{\mu\nu}) \,, \tag{10.54}$$

The proof that $\mathrm{tr}(F^{\mu\nu}\, F_{\mu\nu})$ is gauge invariant is very simple; under the $SU(2)$ gauge transformation we shall have

$$\mathrm{tr}(F^{\mu\nu}\, F_{\mu\nu}) \quad \longrightarrow \quad \mathrm{tr}(F^{\mu\nu\prime}\, F'_{\mu\nu}) = \mathrm{tr}(U F^{\mu\nu} U^\dagger U F_{\mu\nu} U^\dagger) = \mathrm{tr}(U F^{\mu\nu} F_{\mu\nu} U^\dagger)$$

$$= \quad \mathrm{tr}(F^{\mu\nu} F_{\mu\nu} U^\dagger U) = \mathrm{tr}(F^{\mu\nu}\, F_{\mu\nu})\,. \tag{10.55}$$

The equations of motion for the ϕ and A_μ fields can be derived from (10.54) in the standard way, as the Euler-Lagrange equations that follow from requiring that the action $I = \int d^4x \mathcal{L}$ be stationary under variations of ϕ and A_μ respectively. First, let us just consider the *source-free* Yang-Mills equations that will result if we just consider the Lagrangian density for the Yang-Mills fields alone,

$$\mathcal{L}_{YM} = \frac{1}{8\pi}\, \mathrm{tr}(F^{\mu\nu}\, F_{\mu\nu})\,. \tag{10.56}$$

Wrting $I_{YM} = \int d^4x \mathcal{L}_{YM}$, we shall have

$$\begin{aligned}
\delta I_{YM} &= \frac{1}{8\pi}\, 2\ \mathrm{tr} \int d^4x \delta F_{\mu\nu}\, F^{\mu\nu}\,,\\
&= \frac{1}{4\pi}\, \mathrm{tr} \int d^4x (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu + [\delta A_\mu, A_\nu] + [A_\mu, \delta A_\nu]) F^{\mu\nu}\,,\\
&= \frac{1}{2\pi}\, \mathrm{tr} \int d^4x (\partial_\mu \delta A_\nu + [A_\mu, \delta A_\nu]) F^{\mu\nu}\,,\\
&= \frac{1}{2\pi}\, \mathrm{tr} \int d^4x (-\delta A_\nu\, \partial_\mu F^{\mu\nu} + A_\mu \delta A_\nu F^{\mu\nu} - \delta A_\nu A_\mu F^{\mu\nu})\,,\\
&= -\frac{1}{2\pi}\, \mathrm{tr} \int d^4x \delta A_\nu\, (\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}])\,,
\end{aligned} \tag{10.57}$$

and so requiring that the action be stationary implies

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0\,. \tag{10.58}$$

These are the source-free Yang-Mills equations, which are the generalisation of the source-free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. Obviously the Yang-Mills equations reduce to the Maxwell equations in the abelian $U(1)$ case.

If we now include the $-(D^\mu\phi)^\dagger\, (D_\mu \phi)$ term in the above calculation, we shall find that

$$\delta I = \int d^4x (\phi^\dagger \delta A_\nu D^\nu \phi - (D^\nu\phi)^\dagger \delta A_\nu \phi) - \frac{1}{2\pi}\, \mathrm{tr} \int d^4x \delta A_\nu\, (\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}])\,. \tag{10.59}$$

Requiring stationarity under the variations δA_ν now gives the Yang-Mills equations with sources,

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 2\pi\, J^\nu\,, \tag{10.60}$$

where

$$J_\mu = (D_\mu \phi)\, \phi^\dagger - \phi\, (D_\mu\phi)^\dagger\,. \tag{10.61}$$

Note that is is a 2×2 matrix current, as it should be, since the Hermitean-conjugated 2-vector sits to the right of the unconjugated 2-vector.[27]

[27]It is helpful to introduce an index notation to label the rows and columns of the 2×2 matrices, in order to verify that (10.61) is the correct expression.

This completes this brief introduction to Yang-Mills theory. As far as applications are concerned, it is fair to say that Yang-Mills theory lies at the heart of modern fundamental physics. The weak nuclear force is described by the Weinberg-Salam model, based on the Yang-Mills gauge group $SU(2)$. The W and Z bosons, which have been seen in particle accelerators such as the one at CERN, are the $SU(2)$ gauge fields. The strong nuclear force is described by a Yang-Mills theory with $SU(3)$ gauge group, and the 8 gauge fields associated with this theory are the gluons that mediate the strong interactions. Thus one may say that almost all of modern particle physics relies upon Yang-Mills theory.