Geometry and Topology in Physics II: Applications

ABSTRACT

Part II of Geometry and Topology in Physics

Contents

1 Introduction

This course follows on from Geometry and Topology in Physics I, in which the basic notions and formalism of differential geometry and topology were introduced. The aim of the second part of this course is to go on to apply the formalism in a number of contexts of physics interest, also developing the basic ideas further as and when the need arises during the course. To begin, we present a brief overview of the essential aspects of differential forms, which provide the basic tools we shall be using in the course. This is essentially material covered in depth in Part I, and reference can be made to the course notes for that course.

1.1 Vectors and tensors

In physics we encounter vectors and tensors in a variety of contexts; for example the notion of the *position vector* in three-dimensional vector analysis and its four-dimensional spacetime analogue; the 4-vector potential in Maxwell theory; the metric tensor in general relativity, and so on. The language in which all of these can be described is the language of differential geometry. The first examples listed were rather special ones, in that the *position vector* is a concept that is applicable only in the restricted case of a flat Euclidean space or Minkowskian spactime. In general, the line joining one point to another in the space or spacetime is not a vector. Rather, one must pass to the limit where one considers two points that are *infinitesimally separated*. Now, in the limit where the separation tends to zero, the line joining the two points can be viewed as a vector. The reason for this need to use a limiting procedure is easily understood if one thinks of a familar non-Euclidean space, the surface of the Earth. For example, the line joing New York to London is not a vector, from the point of view of transformations on the surface of the Earth (i.e. on the 2-sphere). But in the limit where one considers a line joining two nearby points on a street in New York, one approaches more and more closely to a genuine vector on the 2-sphere. We shall make this precise below.

With the observation that a vector is defined in terms of an arrow joining two points that are infinitesimally separated, it is not surprising that the natural mathematical quantity that describes the vector is the *derivative*. Thus we define a vector V as the tangent vector to some curve in the manifold. Suppose that the manifold M has coordinates x^μ in some patch, and that we have a curve described by $x^\mu = x^\mu(t)$, where t is some parameter along the path. Then we may define the tangent vector

$$V = \frac{\partial}{\partial t}\ . \tag{1.1}$$

Note that V is defined here in a coordinate-independent fashion. However, using the chain rule we may express V as a linear combination of the basis vectors $\partial/\partial x^\mu$:

$$V = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \partial_\mu \ , \tag{1.2}$$

where $V^\mu = dx^\mu/dt$. Note that in the last expression in (1.2), we are using the shorthand notation of ∂_μ to mean $\partial/\partial x^\mu$. Einstein summation convention is always understood, so the index μ in (1.2) is understood to be summed over the n index values labelling the coordinates on M. The components V^μ, unlike the vector V itself, *are* coordinate dependent, and we can calculate their transformation rule under general coordinate transformations $x^i \longrightarrow x'^\mu = x'^\mu(x^\nu)$ by using the chain rule again:

$$V = V^\nu \partial_\nu = V^\nu \frac{\partial x'^\mu}{\partial x^\nu} \partial'_\mu \ , \tag{1.3}$$

where ∂'_μ means $\partial/\partial x'^\mu$. By definition, the coefficients of the ∂'_μ in (1.3) are the components of V with respect to the primed coordinate system, and so we read off

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \ . \tag{1.4}$$

This is the standard way that the components of a vector transform. Straightforward generalisation to multiple indices gives the transformation rule for tensors. A p-index tensor T will have components $T^{\mu_1 \cdots \mu_p}$, defined by

$$T = T^{\mu_1 \cdots \mu_p} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_p} \ . \tag{1.5}$$

From this, it follows by analogous calculations to those described above that the components will transform as

$$T'^{\mu_1 \cdots \mu_p} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\nu_p}} T^{\nu_1 \cdots \nu_p} \ , \tag{1.6}$$

under a change of coordinate frame.

1.2 Covectors and cotensors

We may also define quantities whose components carry downstairs indices. The idea here is best introduced by considering a function f on the manifold. Using the chain rule, we see that its differential df can be written as

$$df = \partial_\mu f \, dx^\mu \ . \tag{1.7}$$

We may think of df as a geometrical, coordinate-independent quantity, whose components in a given coordinate basis are the derivatives $\partial_\mu f$. In fact df is a special case of a *covector*.

More generally, we can consider a covector U, with components U_μ, and define

$$U = U_\mu \, dx^\mu \ . \tag{1.8}$$

With U itself being a coordinate-independent construct, we may deduce how its components U_μ transform under general coordinate transformations by following steps analogous to those that we used above for vectors:

$$U = U_\nu \, \frac{\partial x^\nu}{\partial x'^\mu} \, dx'^\mu \ . \tag{1.9}$$

By definition, the coefficients of dx'^μ are the components U'_μ in the primed coordinate frame, and so we read off the transformation rule for 1-form components:

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \, U_\nu \ . \tag{1.10}$$

One may again generalise to multiple-index objects, or cotensors. Thus, for example, we can consider an object U with p-index components,

$$U = U_{\mu_1 \cdots \mu_p} \, dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p} \ . \tag{1.11}$$

The transformation rule for the components $U_{\mu_1 \cdots \mu_p}$ under general coordinate transformations is again easily read off:

$$U'_{\mu_1 \cdots \mu_p} = \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\nu_p}}{\partial x'^{\mu_p}} \, U_{\nu_1 \cdots \nu_p} \ . \tag{1.12}$$

It is easy to see that because ω_μ transforms "inversely" to the way V^μ transforms (compare (1.4) and (1.10)), the quantity $\omega_\mu \, V^\mu$ will be *invariant* under general coordinate transformations:

$$\begin{aligned} \omega'_\mu \, V'^\mu &= \frac{\partial x^\nu}{\partial x'^\mu} \, \frac{\partial x'^\mu}{\partial x^\sigma} \, \omega_\nu \, V^\sigma \\ &= \frac{\partial x^\nu}{\partial x^\sigma} \, \omega_\nu \, V^\sigma \\ &= \delta^\nu_\sigma \, \omega_\nu \, V^\sigma = \omega_\nu \, V^\nu \ . \end{aligned} \tag{1.13}$$

This is the scalar product, or *inner product*, of V with ω. It can be expressed more "geometrically," without reference to specific coordinates, as $\langle \omega, V \rangle$. The coordinate bases ∂_μ and dx^μ for objects with upstairs and downstairs indices are defined to be orthonormal, so that

$$\langle dx^\mu, \partial_\nu \rangle = \delta^\mu_\nu \ . \tag{1.14}$$

It follows from this that

$$\langle \omega, V \rangle = \omega_\mu \, V^\nu \, \langle dx^\mu, \partial_\nu \rangle = \omega_\mu \, V^\nu \, \delta^\mu_\nu = \omega_\mu \, V^\mu \ , \tag{1.15}$$

and so indeed this gives the hoped-for inner product. Note that if we apply this inner product to the differential df, we get

$$\langle df, V\rangle = \partial_\mu f\, V^\nu\, \langle dx^\mu, \partial_\nu\rangle = V^\mu\, \partial_\mu f = V(f)\ . \tag{1.16}$$

In other words, recalling the original definition of V as a differential operator (1.1), we see that in this case the inner product of df and V is nothing but the directional derivative of the function f along the curve parameterised by t; *i.e.* $\langle df, V\rangle = V(f) = \partial f/\partial t$.

1.3 Differential forms

A particularly important class of cotensors are those whose components are totally antisymmetric;

$$U_{\mu_1\cdots\mu_p} = U_{[\mu_1\cdots\mu_p]}\ . \tag{1.17}$$

Here, we are using the notation that square brackets enclosing a set of indices mean that they should be totally antisymmetrised. Thus we have

$$\begin{aligned} U_{[\mu\nu]} &= \frac{1}{2!}\,(U_{\mu\nu} - U_{\nu\mu})\ ,\\ U_{[\mu\nu\sigma]} &= \frac{1}{3!}\,(U_{\mu\nu\sigma} + U_{\nu\sigma\mu} + U_{\sigma\mu\nu} - U_{\mu\sigma\nu} - U_{\sigma\nu\mu} - U_{\nu\mu\sigma})\ , \end{aligned} \tag{1.18}$$

etc. Generally, for p indices, there will be $p!$ terms, comprising the $\frac{1}{2}p!$ even permutations of the indices, which enter with plus signs, and the $\frac{1}{2}p!$ odd permutations, which enter with minus signs. The $1/p!$ prefactor ensures that the antisymmetrisation is of strength one. In particular, this means that antisymmetrising twice leaves the tensor the same: $U_{[[\mu_1\cdots\mu_p]]} = U_{[\mu_1\cdots\mu_p]}$.

Clearly, if the cotensor is antisymmetric in its indices it will make an antisymmetric projection on the tensor product of basis 1-forms dx^μ. Since antisymmetric cotensors are so important in differential geometry, a special symbol is introduced to denote an antisymmetrised product of basis 1-forms. This symbol is the wedge product, $\wedge$. Thus we define

$$\begin{aligned} dx^\mu\wedge dx^\nu &= dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu\ ,\\ dx^\mu\wedge dx^\nu\wedge dx^\sigma &= dx^\mu\otimes dx^\nu\otimes dx^\sigma + dx^\nu\otimes dx^\sigma\otimes dx^\mu + dx^\sigma\otimes dx^\mu\otimes dx^\nu\\ &\quad -dx^\mu\otimes dx^\sigma\otimes dx^\nu - dx^\sigma\otimes dx^\nu\otimes dx^\mu - dx^\nu\otimes dx^\sigma\otimes dx^\mu \end{aligned} \tag{1.19}$$

and so on.

Cotensors antisymmetric in p indices are called p-forms. Suppose we have such an object A, with components $A_{\mu_1\cdots\mu_p}$. Then we expand it as

$$A = \frac{1}{p!}\, A_{\mu_1\cdots\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \ . \tag{1.20}$$

It is quite easy to see from the definitions above that if A is a p-form, and B is a q-form, then they satisfy

$$A \wedge B = (-1)^{pq}\, B \wedge A \ . \tag{1.21}$$

1.4 Exterior derivative

The exterior derivative d acts on a p-form field, and produces a $(p+1)$-form. It is defined as follows. On functions (i.e. 0-forms), it is just the operation of taking the differential; we met this earlier:

$$df = \partial_\mu f\, dx^\mu \ . \tag{1.22}$$

More generally, on a p-form $\omega = 1/p!\, \omega_{\mu_1\cdots\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$, it is defined by

$$d\omega = \frac{1}{p!}\, (\partial_\nu \omega_{\mu_1\cdots\mu_p})\, dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \ . \tag{1.23}$$

Note that from our definition of p-forms, it follows that the components of the $(p+1)$-form $d\omega$ are given by

$$(d\omega)_{\nu\mu_1\cdots\mu_p} = (p+1)\, \partial_{[\nu}\, \omega_{\mu_1\cdots\mu_p]} \ . \tag{1.24}$$

It is easily seen from the definitions that if A is a p-form and B is a q-form, then the following Leibnitz rule holds:

$$d(A \wedge B) = dA \wedge B + (-1)^p\, A \wedge dB \ . \tag{1.25}$$

It is also easy to see from the definition of d that if it acts twice, it automatically gives zero, i.e. $d^2 \equiv 0$. This just follows from (1.23), which shows that d is an *antisymmetric* derivative, while on the other hand partial derivatives *commute*.

A simple, and important, example of differential forms and the use of the exterior derivative can be seen in Maxwell theory. The vector potential is a 1-form, $A = A_\mu\, dx^\mu$. The Maxwell field strength is a 2-form, $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, and we can construct it from A by taking the exterior derivative:

$$F = dA = \partial_\mu\, A_\nu\, dx^\mu \wedge dx^\nu = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \ , \tag{1.26}$$

from which we read off that $F_{\mu\nu} = 2\,\partial_{[\mu}\,A_{\nu]} = \partial_\mu\,A_\nu - \partial_\nu\,A_\mu$. The fact that $d^2 \equiv 0$ means that $dF = 0$, since $dF = d^2 A$. The equation $dF = 0$ is nothing but the Bianchi identity in Maxwell theory, since from the definition (1.23) we have

$$dF = \tfrac{1}{2}\partial_\mu\,F_{\nu\rho}\,dx^\mu \wedge dx^\nu \wedge dx^\rho \ , \tag{1.27}$$

hence implying that $\partial_{[\mu}\,F_{\nu\rho]} = 0$. We can also express the Maxwell field equation elegantly in terms of differential forms. This requires the introduction of the Hodge dual operator $*$. This was discussed at length in Part I of the course, and we will not revisit all the details again here. See the course notes for Part I for details.

For now, we shall move on to a very brief review of the basic notions of metrics, vielbeins, spin connections and curvatures, which we shall then use extensively in the subsequent chapters.

2 Metrics, Connections and Curvature

A metric tensor provides a rule for measuring distances between neighouring points on a manifold. It is an additional piece of structure that was not needed up until this point in the discussion. The metric is a symmetric 2-index cotensor $g_{\mu\nu}$, and in general it is a field on the manifold M, which depends upon the coordinates x^μ. The distance squared between two infinitesimally-separated points is denoted by ds^2, and thus we have, generalising Pythagoras' theorem,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \ . \tag{2.1}$$

2.1 Spin connection and curvature 2-forms

Here, we gather together some basic results from part I of the course. We begin by "taking the square root" of the metric $g_{\mu\nu}$ in (2.1), by introducing a vielbein, which is a basis of 1-forms $e^a = e^a_\mu\, dx^\mu$, with components e^a_μ, having the property

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu \ . \tag{2.2}$$

Here the indices a are a new type, different from the coordinate indices μ we have encountered up until now. They are called local-Lorentz indices, or tangent-space indices, and η_{ab} is a "flat" metric, with constant components. The language of "local-Lorentz" indices stems from the situation when the metric $g_{\mu\nu}$ has Minkowskian signature (which is $(-,+,+,\ldots,+)$ in sensible conventions). The signature of η_{ab} must be the same as that of $g_{\mu\nu}$, so if we are working in general relativity with Minkowskian signature we will have

$$\eta_{ab} = \mathrm{diag}\,(-1,1,1,\ldots,1) \ . \tag{2.3}$$

If, on the other hand, we are working in a space with Euclidean signature $(+,+,\ldots,+)$, then η_{ab} will just equal the Kronecker delta, $\eta_{ab} = \delta_{ab}$, or in other words

$$\eta_{ab} = \mathrm{diag}\,(1,1,1,\ldots,1) \ . \tag{2.4}$$

Of course the choice of vielbeins e^a as the square root of the metric in (2.2) is to some extent arbitrary. Specifically, we could, given a particular choice of vielbein e^a, perform an orthogonal-type transformation to get another equally-valid vielbein e'^a, given by

$$e'^a = \Lambda^a{}_b\, e^b \ , \tag{2.5}$$

where $\Lambda^a{}_b$ is a matrix satisfying the (pseudo)orthogonality condition

$$\eta_{ab}\, \Lambda^a{}_c\, \Lambda^b{}_d = \eta_{cd} \ . \tag{2.6}$$

Note that $\Lambda^a{}_b$ can be coordinate dependent. If the n-dimensional manifold has a Euclidean-signature metric then $\eta = \mathbb{1}$ and (2.6) is literally the orthogonality condition $\Lambda^T\,\Lambda = \mathbb{1}$. Thus in this case the arbitrariness in the choice of vielbein is precisely the freedom to make local $O(n)$ rotations in the tangent space. If the metric signature is Minkowskian, then instead (2.6) is the condition for Λ to be an $O(1, n-1)$ matrix; in other words, one then has the freedom to perform local Lorentz transformations in the tangent space. We shall typically use the words "local Lorentz transformation" regardless of whether we are working with metrics of Minkowskian or Euclidean signature.

Briefly reviewing the next steps, we introduce the spin connection, or connection 1-forms, $\omega^a{}_b = \omega^a{}_{b\mu}\, dx^\mu$, and the torsion 2-forms $T^a = \frac{1}{2}T^a_{\mu\nu}\, dx^\mu \wedge dx^\nu$, defining

$$T^a = de^a + \omega^a{}_b \wedge e^b\ . \tag{2.7}$$

Next, we define the curvature 2-forms $\Theta^a{}_b$, *via* the equation

$$\Theta^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\ . \tag{2.8}$$

Note that if we adopt the obvious matrix notation where the local Lorentz transformation (2.5) is written as $e' = \Lambda\, e$, then we have the property that $\omega^a{}_b$, T^a and $\Theta^a{}_b$ transform as follows:

$$\begin{aligned} \omega' &= \Lambda\,\omega\,\Lambda^{-1} + \Lambda\, d\Lambda^{-1}\ , \\ T' &= \Lambda\, T\ , \qquad \Theta' = \Lambda\,\Theta\,\Lambda^{-1}\ . \end{aligned} \tag{2.9}$$

Thus the torsion 2-forms T^a and the curvature 2-forms $\Theta^a{}_b$ both transform nicely, in a covariant way, under local Lorentz transformations, while the spin connection does not; it has an extra inhomogeneous term in its transformation rule. This is the characteristic way in which connections transform. Because of this, we can define a Lorentz-covariant exterior derivative D as follows:

$$DV^a{}_b \equiv dV^a{}_b + \omega^a{}_c \wedge V^c{}_b - \omega^c{}_b \wedge V^a{}_c\ , \tag{2.10}$$

where $V^a{}_b$ is some set of p-forms carrying tangent-space indices a and b. One can easily check that if $V^a{}_b$ itself transforms covariantly under local Lorentz transformations, then so does $DV^a{}_b$. In other words, the potentially-troublesome terms where the exterior derivative lands on the transformation matrix Λ are cancelled out by the contributions from the inhomgeneous second term in the transformation rule for $\omega^a{}_b$ in (2.9). We have taken the example of $V^a{}_b$ with one upstairs and one downstairs tangent space index for simplicity, but the generalisation to arbitrary numbers of indices is immediate. There is one term like the second term on the right-hand side of (2.10) for each upstairs index, and a term like the third term on the right-hand side of (2.10) for each downstairs index.

The covariant exterior derivative D will commute nicely with the process of contracting tangent-space indices with η_{ab}, provided that we require

$$D\,\eta_{ab} \equiv d\eta_{ab} - \omega^c{}_a\,\eta_{cb} - \omega^c{}_b\,\eta_{ac} = 0\ . \tag{2.11}$$

Since we are taking the components of η_{ab} to be literally constants, it follows from this equation, which is known as the equation of *metric compatibility*, that

$$\omega_{ab} = -\omega_{ba}\ , \tag{2.12}$$

where ω_{ab} is, by definition, $\omega^a{}_b$ with the upper index lowered using η_{ab}: $\omega_{ab} \equiv \eta_{ac}\,\omega^c{}_b$. With this imposed, it is now the case that we can take covariant exterior derivatives of products, and freely move the local-Lorentz metric tensor η_{ab} through the derivative. This means that we get the same answer if we differentiate the product and then contract some indices, or if instead we contract the indices and then differentiate.

In addition to the requirement of metric compatibiilty we usually also choose a *torsion-free* spin-connection, meaning that we demand that the torsion 2-forms T^a defined by (2.7) vanish. In practice, we shall now assume this in everything that follows. In fact equation (2.7), together with the metric-compatibility condition (2.12), now determine $\omega^a{}_b$ uniquely. In other words, the two conditions

$$de^a = -\omega^a{}_b \wedge e^b\ , \qquad \omega_{ab} = -\omega_{ba} \tag{2.13}$$

have a unique solution. It can be given as follows. Let us say that, by definition, the exterior derivatives of the vielbeins e^a are given by

$$de^a = -\tfrac{1}{2} c_{bc}{}^a\, e^b \wedge e^c\ , \tag{2.14}$$

where the structure functions $c_{bc}{}^a$ are, by definition, antisymmetric in bc. Then the solution for ω_{ab} is given by

$$\omega_{ab} = \tfrac{1}{2}(c_{abc} + c_{acb} - c_{bca})\, e^c\ , \tag{2.15}$$

where $c_{abc} \equiv \eta_{cd}\, c_{ab}{}^d$. It is easy to check by direct substitution that this indeed solves the two conditions (2.13).

The procedure, then, for calculating the curvature 2-forms for a metric $g_{\mu\nu}$ with vielebeins e^a is the following. We write down a choice of vielbein, and by taking the exterior derivative we read off the coefficients $c_{bc}{}^a$ in (2.14). Using these, we calculate the spin connection using (2.15). Then, we substitute into (2.8), to calculate the curvature 2-forms.

Each curvature 2-form $\Theta^a{}_b$ has, as its components, a tensor that is antisymmetric in two coordinate indices. This is the Riemann tensor, defined by

$$\Theta^a{}_b = \tfrac{1}{2} R^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu \ . \tag{2.16}$$

We may always use the vielbein e^a_μ, which is a non-degenerate $n \times n$ matrix in n dimensions, to convert between coordinate indices μ and tangent-space indices a. For this purpose we also need the inverse of the vielbein, denoted by E^μ_a, and satisfying the defining properties

$$E^\mu_a\, e^a a_\nu = \delta^\mu_\nu \ , \qquad E^\mu_a\, e^b_\mu = \delta^a_b \ . \tag{2.17}$$

Then we may define Riemann tensor components entirely within the tangent-frame basis, as follows:

$$R^a{}_{bcd} \equiv E^\mu_c\, E^\nu_d\, R^a{}_{b\mu\nu} \ . \tag{2.18}$$

Note that we use the same symbol for the tensors, and distinguish them simply by the kinds of indices that they carry. (This requires that one pay careful attention to establishing unambiguous notations, which keep track of which are coordinate indices, and which are tangent-spave indices!) In terms of $R^a{}_{bcd}$, it is easily seen from the various definitions that we have

$$\Theta^a{}_b = \tfrac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d \ . \tag{2.19}$$

From the Riemann tensor two further quantities can be defined; the Ricci tensor R_{ab} and the Ricci scalar R:

$$R_{ab} = R^c{}_{acb} \ , \qquad R = \eta^{ab}\, R_{ab} \ . \tag{2.20}$$

Note that the Riemann tensor and Ricci tensor have the following symmetries, which can be proved straightforwardly from the definitions:

$$\begin{aligned} &R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \ , \\ &R_{abcd} + R_{acdb} + R_{adbc} = 0 \ , \\ &R_{ab} = R_{ba} \ . \end{aligned} \tag{2.21}$$

2.2 Curvature in coordinate basis

For those more familiar with the "traditional" treatment of Riemannian geometry, we can give a "dictionary" for translating between the two formalisms. In the traditional approach, we construct the Christoffel connection, $\Gamma^\mu_{\nu\rho}$ from the metric tensor, using the expression

$$\Gamma^\mu{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}\, \Big(\partial_\nu\, g_{\sigma\rho} + \partial_\rho\, g_{\nu\sigma} - \partial_\sigma\, g_{\nu\rho}\Big)\ . \tag{2.22}$$

This is used in order to construct the covariant derivative, ∇_μ. Its action on tensors with upstairs indices is defined by

$$\nabla_\mu\, V^\nu = \partial_\mu\, V^\nu + \Gamma^\nu{}_{\mu\rho}\, V^\rho\ , \tag{2.23}$$

while for downstairs indices it is

$$\nabla_\mu\, V_\nu = \partial_\mu\, V_\nu - \Gamma^\rho{}_{\mu\nu}\, V_\rho\ . \tag{2.24}$$

Acting on tensors with multiple indices, there will be one Γ term of the appropriate type for each upstairs or downstairs index. The expression (2.22) for the Christoffel connection is in fact determined by the requirement of metric compatibility, namely $\nabla_\mu\, g_{\nu\rho} = 0$. The covariant derivative has the property that acting on any tensor, it gives another tensor. In other words, the object constructed by acting with the covariant derivative will transform under general coordinate transformations according to the rule given in (1.6) and (1.12) for upstairs and downstairs indices.

From the Christoffel connection we construct the Riemman tensor, given by

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\, \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma\, \Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\lambda}\, \Gamma^\lambda{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\lambda}\, \Gamma^\lambda{}_{\rho\nu}\ . \tag{2.25}$$

Although it is not immediately obvious, in view of the fact that $\Gamma^\mu_{\nu\rho}$ is *not* itself a tensor, the quantity $R^\mu_{\nu\rho\sigma}$ does in fact transform tensorially. This can be shown from the previous definitions by a straightforward calculation.

To make contact with the curvature computations using differential forms in the previous section, we note that the Riemman tensor calculated here is the same as the one in the previous section, after converting the indices using the vielbein or inverse vielbein:

$$R^\mu{}_{\nu\rho\sigma} = E^\mu_a\, e^b_\nu\, R^a{}_{b\rho\sigma}\ . \tag{2.26}$$

The coordinate components of the Ricci tensor, and the Ricci scalar, are given by

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}\ , \qquad R = g^{\mu\nu}\, R_{\mu\nu}. \tag{2.27}$$

As usual, we can relate the tensors with tangent-space and coordinate indices by means of the vielbein, so that we have $R_{\mu\nu} = e^a_\mu\, e^b_\nu\, R_{ab}$.

One further identity, easily proven from the definitions in this section, is that

$$\nabla_{[\lambda}\, R^{\mu}{}_{|\nu|\rho\sigma]} = 0 \ , \tag{2.28}$$

where the vertical lines enclosing an index or set of indices indicate that they are excluded from the antisymmetrisation. An appropriate contraction of indices in the Bianchi identity (2.28) leads to the result that

$$\nabla^{\mu}\, R_{\mu\nu} = \tfrac{1}{2}\partial_{\nu}\, R \ . \tag{2.29}$$

A consequence of this is that if we define the so-called *Einstein tensor*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} \ , \tag{2.30}$$

then it is conserved, i.e. $\nabla^{\mu}\, G_{\mu\nu} = 0$.

3 Schwarzschild Black holes

Having reviewed some basic preliminaries, let us now turn to the first of the applications that we shall be considering, namely black holes. We shall begin by deriving the simplest of the black holes, which is a static, spherically-symmetric solution of the pure vacumm Einstein equations. It was found in 1916 by Karl Schwarzschild, much to Einstein's surprise because he never expected that such a complicated and non-linear system as the Einstein equations would admit analytically-solvable non-trivial solutions. Having obtained the local solution, we shall then embark on a study of its global properties, inparticular studying its structure at infinity, on the event horizon, and at the curvature singularity. Having done so we shall then move on to a more complicated example, which introduces new features and subtleties, namely the Reissner-Nordström solution. This is a static, spherically-symmetric solution of the coupled Einstein-Maxwell equations, and it represents a black hole carrying electric (or magnetic) charge as well as mass. The global structure is significantly moree subtle in this case than for the Schwarzschild solution. Another generalisation is to the case of a rotating black hole; this is the Kerr solution. Finally, we can add charge as well and consider the Kerr-Newman family of rotating, charged black holes.

The Einstein equations determine the way in which the geometry of spacetime is influenced by the presence of matter. They take the form, in general,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}\ , \tag{3.1}$$

where G is Newton's gravitational constant, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor of whatever matter might be present in the system. $G_{\mu\nu}$ is called the Einstein tensor, and, as we observed in the previous section, it is conserved, $\nabla^\mu G_{\mu\nu} = 0$. This is important since, as is well known, the energy-momentum tensor for any isolated matter system is also conserved. In fact one of the problems that held Einstein up for a while, more or less until the last minute when he published his major paper on general relativity in 1915, was that he had not included the $-\frac{1}{2} R\, g_{\mu\nu}$ term on the left-hand side of (3.1), and as a result he was running into inconsistencies.

This equation is analogous to the field equation of Maxwell electrodynamics, i.e. $\partial_\mu F^{\mu\nu} = -4\pi\, J^\nu$, where J^μ is the current density 4-tensor, associated with whatever charged particles or fields are present in the system. One can study source-free solutions of the equation $\partial_\mu F^{\mu\nu} = 0$, as well as solutions where sources are present. Examples of source-free solutions include electromagnetic waves. There are also other examples that are "almost" source-free, such as point electric charges. These satisfy the source-free Maxwell equations almost everywhere, except at the actual point where the charge is located, where there is a delta-function charge density term.

The source-free Einstein equations likewise have a variety of solutions. There are gravitational wave solutions, analogous to the electromagnetic waves of Maxwell theory. There are also solutions that are "almost" source free, analogous to the notion of the point charge in electrodynamics. In this gravitational case these have mass singularities rather than charge singularities; they are black holes. Of course the non-linear nature of the Einstein equations means that the various solutions are more complicated, and more subtle, than their electrodynamical cousins. Also, the very essence of general relativity is that one is using a description that is covariant with respect to arbitrary changes of coordinate system. This means that one has to be very careful to distinguish between genuine physics on the one hand, and mere artefacts of particular coordinate systems on the other. This is the beauty and the subtlety of the subject. But, as Sidney Coleman has remarked, "In General Relativity you don't know where you are, and you don't know what time it is." The profundity of this observation should become apparent as we proceed.

3.1 The Schwarzschild solution

It can be argued on general symmetry grounds that a four-dimensional metric that is static (time independent and non-rotating), and spatially spherically-symmetric, can be written in the form

$$ds^2 = -e^{2A}\, dt^2 + e^{2B}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta d\varphi^2)\ , \tag{3.2}$$

where A and B are functions of the radial coordinate r. The way to show this is to begin by writing down the most general possibility for a static metric with rotational symmetry, and then to exploit to the full the freedom to make coordinate transformations, in order to reduce the metric to its simplest possible form. The essence of solution-hunting in general relativity is to start from an ansatz for a symmetry-restricted class of metrics that have been expressed in as simple a form as possible, and then grind through the calculation of the curvature in order to plug into the left-hand side of the Einstein equations (3.1). In our case we are looking for a vacuum solution where $T_{\mu\nu} = 0$, and so we just need to impose that this left-hand side should be zero. Note that by taking the trace of $R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0$ in D dimensions we get

$$R - \tfrac{1}{2} D\, R = 0\ , \tag{3.3}$$

and so, provided we are not in $D = 2$ dimensions we thus deduce that the source-free Einstein equations imply $R = 0$, and hence

$$R_{\mu\nu} = 0 \ . \tag{3.4}$$

Since $R = 0$ follows from this equation by tracing, this means that the entire content of the vacuum Einstein equations in $D \geq 3$ dimensions is contained in the *Ricci flatness* condition (3.4).

We shall calculate the curvature for the metric (3.2) using the method of differential forms. We begin by choosing the natural orthonormal vielbein basis,

$$e^0 = e^A\, dt \ , \quad e^1 = e^B\, dr \ , \quad e^2 = r\, d\theta \quad e^3 = r\, \sin\theta\, d\varphi \ . \tag{3.5}$$

Note that one has to be careful not to confuse indexed vielbeins such as e^a (or even sometimes e^A!) with exponentials of functions, such as e^A. It should always be clear from context, and in practice one usually has little trouble from this potential pitfall.

We now calculate the torsion-free spin connection $\omega^a{}_b$, using $de^a = -\omega^a{}_b \wedge e^b$, together with antisymmetry $\omega_{ab} = -\omega_{ba}$. Calculating the exterior derivatives of the vielbeins, we get

$$\begin{aligned}
de^0 &= A'\, e^A\, dr \wedge dt = -A'\, e^{-B}\, e^0 \wedge e^1 \ , \\
de^1 &= 0 \ , \\
de^2 &= dr{\wedge}d\theta = \frac{1}{r}\, e^{-B}\, e^1 \wedge e^2 \ , \\
de^3 &= \sin\theta\, dr \wedge d\varphi + r\, \cos\theta\, d\theta \wedge d\varphi = \frac{1}{r}\, e^{-B}\, e^1 \wedge e^3 + \frac{1}{r}\, \cot\theta\, e^2 \wedge e^3 \ .
\end{aligned} \tag{3.6}$$

Notice that having acted with the exterior derivative d we have then re-expressed the results back in terms of the vielbeins, using (3.5). This is so that we are ready to take the next step, where we read off the coefficients $c_{ab}{}^c$ in (2.14). By inspection, we see that the non-vanishing components are given by

$$c_{01}{}^0 = A'\, e^{-B} \ , \quad c_{12}{}^2 = -\frac{1}{r}\, e^{-B} \ , \quad c_{13}{}^3 = -\frac{1}{r}\, e^{-B} \ , \quad c_{23}{}^3 = -\frac{1}{r}\, \cot\theta \ . \tag{3.7}$$

Note that since $c_{ab}{}^c = -c_{ba}{}^c$, we do not need to list components related to those given in (3.7) by this symmetry, but it must be recalled that they are also non-zero.

Substituting (3.7) into (2.15), we obtain the following expressions for the spin connection:

$$\begin{aligned}
&\omega_{01} = -A'\, e^{-B}\, e^0 \ , \qquad \omega_{02} = 0 \ , \qquad \omega_{03} = 0 \ , \\
&\omega_{23} = -\frac{1}{r}\, \cot\theta\, e^3 \ , \qquad \omega_{13} = -\frac{1}{r}\, e^{-B}\, e^3 \ , \qquad \omega_{12} = -\frac{1}{r}\, e^{-B}\, e^2 \ .
\end{aligned} \tag{3.8}$$

(In practice, it is usually a good idea at this stage in the calculation to pause and verify that the calculated spin connection does indeed solve the equations $de^a = -\omega^a{}_b \wedge e^b$, to guard against calculational errors.) Finally, we substitute the spin connection into (2.8), to obtain the curvature 2-forms. After a little algebra, we find

$$
\begin{aligned}
&\Theta_{01} = (A''-A'\,B'+A'^2)\,e^{-2B}\,e^0\wedge e^1\ , \quad \Theta_{02} = \frac{A'}{r}\,e^{-2B}\,e^0\wedge e^2\ , \quad \Theta_{03} = \frac{A'}{r}\,e^{-2B}\,e^0\wedge e^3\ ,\\
&\Theta_{23} = \frac{1}{r^2}\,(1-e^{-2B})\,e^2\wedge e^3\ , \quad \Theta_{13} = \frac{B'}{r}\,e^{-2B}\,e^1\wedge e^3\ , \quad \Theta_{12} = \frac{B'}{r}\,e^{-2B}\,e^1\wedge e^2\ .
\end{aligned} \tag{3.9}
$$

From these, it is easy to read off, using (2.19), that the non-vanishing vielbein components of the Riemman tensor are given by

$$
\begin{aligned}
&R_{0101} = (A''-A'\,B'+A'^2)\,e^{-2B}\ , \quad R_{0202} = \frac{A'}{r}\,e^{-2B}\ , \quad R_{0303} = \frac{A'}{r}\,e^{-2B}\ ,\\
&R_{2323} = \frac{1}{r^2}\,(1-e^{-2B})\ , \quad R_{1313} = \frac{B'}{r}\,e^{-2B}\ , \quad R_{1212} = \frac{B'}{r}\,e^{-2B}\ .
\end{aligned} \tag{3.10}
$$

From the definition (2.20), we can now calculate the components of the Ricci tensor, finding (in tangent-space indices)

$$
\begin{aligned}
R_{00} &= R_{0101} + R_{0202} + R_{0303} = \Big(A'' - A'\,B' + A'^2 + \frac{2}{r}\,A'\Big)\,e^{-2B}\ ,\\
R_{11} &= -R_{0101} + R_{1212} + R_{1313} = \Big(-A'' + A'\,B' - A'^2 + \frac{2}{r}\,B'\Big)\,e^{-2B}\ ,\\
R_{22} &= -R_{0202} + R_{1212} + R_{2323} = \Big(-\frac{A'}{r} + \frac{B'}{r} + \frac{1}{r^2}\,e^{2B} - \frac{1}{r^2}\Big)\,e^{-2B}\ ,\\
R_{33} &= -R_{0303} + R_{1313} + R_{2323} = \Big(-\frac{A'}{r} + \frac{B'}{r} + \frac{1}{r^2}\,e^{2B} - \frac{1}{r^2}\Big)\,e^{-2B}\ .
\end{aligned} \tag{3.11}
$$

Thus the vacuum Einstein equations, which we saw above are simply the Ricci-flat condition $R_{ab} = 0$, imply that the functions A and B in the original metric ansatz (3.2) must satisfy the following equations:

$$
\begin{aligned}
A'' - A'\,B' + A'^2 + \frac{2}{r}\,A' &= 0\ ,\\
A'' - A'\,B' + A'^2 - \frac{2}{r}\,B' &= 0\ ,\\
-\frac{A'}{r} + \frac{B'}{r} + \frac{1}{r^2}\,e^{2B} - \frac{1}{r^2} &= 0\ .
\end{aligned} \tag{3.12}
$$

Subtracting the second from the first, we immediately find that $A' + B' = 0$. Since a constant shift in A can simply be absorbed by a rescaling of the time coordinate in (3.2), we can, without losing any generality, take the solution to be

$$
B = -A\ . \tag{3.13}
$$

The third equation in (3.12) then implies that $2r\,A' = e^{-2A} - 1$, which can be integrated immediatly to give

$$e^{2A} = 1 - \frac{2m}{r}\,, \tag{3.14}$$

where m is an arbitrary constant. It is then easily veified that the remaining unused equation, which can be taken to be the first equation in (3.12) by itself, is satisfied. Thus we have arrived at the Schwarzschild solution of the vacuum Einstein equations,

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)\,dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)\,. \tag{3.15}$$

For future puposes, we can now substitute the solutions for A and B back into the expressions in (3.9) for the curvature 2-forms, to find:

$$\begin{aligned} \Theta_{01} &= -\frac{2m}{r^3}\,e^0\wedge e^1\,, \quad \Theta_{02} = \frac{m}{r^3}\,e^0\wedge e^2\,, \quad \Theta_{03} = \frac{m}{r^3}\,e^0\wedge e^3\,, \\ \Theta_{23} &= \frac{2m}{r^3}\,e^2\wedge e^3\,, \quad \Theta_{13} = -\frac{m}{r^3}\,e^1\wedge e^3\,, \quad \Theta_{12} = -\frac{m}{r^3}\,e^1\wedge e^2\,. \end{aligned} \tag{3.16}$$

Thus the non-vanishing tangent-space components of the Riemann tensor are given by

$$\begin{aligned} R_{0101} &= -\frac{2m}{r^3}\,, \qquad R_{0202} = \frac{m}{r^3}\,, \qquad R_{0303} = \frac{m}{r^3}\,, \\ R_{2323} &= \frac{2m}{r^3}\,, \qquad R_{1313} = -\frac{m}{r^3}\,, \qquad R_{1212} = -\frac{m}{r^3}\,. \end{aligned} \tag{3.17}$$

Finally, we can compute the scalar curavture invariant $|\mathrm{Riem}|^2 \equiv R^{abcd}\,R_{abcd}$. From the symmetries of the Riemann tensor, and the particular pattern of non-vanishing components in this case, we can see that

$$|\mathrm{Riem}|^2 = 4\Big((R_{0101})^2 + (R_{0202})^2 + (R_{0303})^2 + (R_{2323})^2 + (R_{1313})^2 + (R_{1212})^2\Big)\,, \tag{3.18}$$

and hence, from (3.17), we find that the Schwarzschild metric has

$$|\mathrm{Riem}|^2 = \frac{48m^2}{r^6}\,. \tag{3.19}$$

3.2 Global structure of the Schwarzschild solution

So far, we have been concerned here only with *local* considerations; writing down the metric ansatz (3.2), calculating the curvature, and then solving the vacuum Einstein equations (3.4). Now, the time has come to interpret the Schwarzschild solution (3.15), and to study its global structure.

3.2.1 Asymptotic structure of the Schwarzschild solution

Firstly, we may note that it looks very reasonable out near infinity. As $r \to \infty$, the metric tends to flat Minkowksi spacetime, where the spatial 3-metric is written in spherical polar coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\ . \tag{3.20}$$

What is more, the *way* in which it approaches Minkowski spacetime is very reasonable. If we consider a weak-field and low-velocity approximation, any metric can, by suitable choice of coordinates, be approximated by

$$ds^2 = -(1+2\phi)\,dt^2 + (\delta_{ij} + h_{ij})\,dx^i\,dx^j\ , \tag{3.21}$$

where we split 4-dimensional indices as $\mu = (0, i)$, and ϕ and h_{ij} are functions of x^i, with $|\phi| << 1$ and $|h_{ij} << 1$. It is rather easy to calculate the Christoffel connection (2.22) and Riemann tensor (2.25) in this approximation, leading to the following dominant terms in $\Gamma^i{}_{00}$ and in the Ricci tensor component R_{00}:

$$\begin{aligned} \Gamma^i{}_{00} &= \partial_i\,\phi\ , & (3.22)\\ R_{00} &= \nabla^2\,\phi\ , & (3.23) \end{aligned}$$

where $\nabla^2 = \partial_i\,\partial_i$ is the spatial Laplacian. Likewise, in this approximation the dominant term in the energy-momentum tensor for any matter is $T_{00} = \rho$, where ρ is the mass density. The the Einstein equation (3.1) can be written, by taking the trace and substituting for R in terms of $T \equiv T_\mu{}^\mu$, as $R_{\mu\nu} = 8\pi\,G\,T_{\mu\nu} - 4\pi\,G\,T\,g_{\mu\nu}$, and hence, in the linearised approximation, we find by looking at the R_{00} component of the Einstein equation that

$$\nabla^2\phi = 4\pi\,G\,\rho\ , \tag{3.24}$$

showing that Einstein's equations reduce to the equations of Newtonian gravity in the weak-field low-velocity limit, and that the function ϕ in the linearised metric (3.21) is nothing but the Newtonian gravitational potential.

Thus we see, looking at the form of the Schwarzschild metric (3.15) when r is large, that m, which until now was simply a constant of integration that arose in solving the Einstein equations, has precisely the interpretation of the *mass* of the object described by the solution, and that the metric has the proper asymptotic form appropriate to the field far from a gravitating object of mass m.

One could of course simply choose to interpret (3.15) in a restricted way, as the metric outside a spherically-symmetric object composed of relatively "normal" material. For example, at the surface of the Earth the quantity m/r is of order 10^{-9} (we are using units where Newton's constant G and the speed of light c are both set equal to 1, so length, mass and time have the same dimensions). Of course inside the Earth one would have to match the exterior solution (3.15) onto another solution that takes into account the distribution of matter within the Earth, but (3.15) is does properly describe the metric outside a non-rotating spherically-symmetric object.

3.2.2 A first look at singularities

Of much greater interest to us here is to take the Schwarzschild metric seriously even at small values of r, to see where that leads us. The first thing one notices about (3.15) is that it becomes singular at $r = 2m$. This is in some sense unexpected, since when we started out we looked for a spherically-symmetric solution that would be expected to describe the geometry outside a "point mass" located at $r = 0$. There is indeed a singularity at $r = 0$, of a rather severe nature. One can see that the metric becomes singular also at $r = 0$, but, as we shall see below, one cannot judge a solution in general relativity just by looking at singularities in the metric, because these can change drastically in different coordinate systems. There is, however, a reliable indicator as to when there is a genuine singularity in the spacetime, namely by looking at scalar invariants built from the Riemann tensor. The point about looking at scalar invariants is that they are, by definition, invariant under changes of coordinate system, and so they provide a coordinate-independent indication of whether or not there are genuine singularities.

The simplest scalar invariant built from the Riemann tensor is the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\rho{}_{\mu\rho\nu}$. However, this is not much help to us here since the Schwarzschild metric satisfies the vacumm Einstein equations and hence, in particular, $R = 0$. Likewise, the quadratic invariant $R^{\mu\nu} R_{\mu\nu}$ would be no help to us either, since it too vanishes. We need an invariant built from the full Riemann tensor, not from its contraction to the Ricci tensor. Indeed, at the end of the previous section we calculated the sinmplest such quantity in the Schwarzschild metric, where we found (3.19) that $|\mathrm{Riem}|^2 = 48m^2/r^6$. This tells us instantly that $r = 0$ is a genuine singularity in the spacetime manfold, at which the curvature diverges. This is in some sense analogous to the diverging of the electric field at the location of a point electric charge. Note that we were fortunate here in finding that $|\mathrm{Riem}|^2$ was divergent; this means that we can be sure that there is a genuine spacetime singularity. The converse is not necessarily true; one can encounter circumstances where the curvature is actually divergent, but $|\mathrm{Riem}|^2$ is not. This might seem odd, since $|\mathrm{Riem}|^2$ in (3.18) is obtained by squaring and summing all the components of the Riemann tensor.

The point is that in our example of the Schwarzschild metric there are always an even number of "0" indices on the non-vanishing components of the Riemann tensor. In more general cases, there might be components with an odd number of "0" components, and the squares of these would enter with minus signs in the calculation of $|\mathrm{Riem}|^2$, because of the indefinite metric signature. Thus one could encounter circumstances where singular behaviour cancelled out between different components of the Riemann tensor. There is a rather intricate theory concerning the question of what set of scalar invariants built from the curvature is sufficient to characterise all potential singularities in the curvature, but we shall not dwell further on this here.

Let us now turn our attention to the singular behaviour of the Schwarzschild metric (3.15) at $r = 2m$. It was decades after the original discovery of the Schwarzschild solution before this was properly understood, in in the early days people would speak of the "Schwarzschild singularity" at $r = 2m$ as if it were a genuine singularity in the spacetime. In fact, as we shall see, there is physically nothing singular at $r = 2m$; the apparent singularity in (3.15) is simply a consequence of the fact that the (t, r, θ, φ) coordinate system breaks down there. There are many physically interesting phenomena associated with this region in the spacetime, but there is no singularity. It is known, for reasons that will become clear, as an "event horizon."

The notion of a coordinate system breaking down at an otherwise perfectly regular point or region in a space is a perfectly familiar one. We can consider polar coordinates on the plane as an example, where the metric is

$$ds^2 = dr^2 + r^2\, d\theta^2 \ . \tag{3.25}$$

This metric is singular at the origin; the metric component g_{rr} vanishes there, and the determinant of the metric vanishes too. But, as we well know, a transformation to Cartesian coordinates (x, y), related to (r, θ) by $x = r \cos\theta$ and $y = r \sin\theta$, puts the metric (3.25) into the standard Cartesian form $ds^2 = dx^2 + dy^2$, and now we see that indeed $r = 0$, which is now described by $x = y = 0$, is perfectly regular.

Before getting down to a detailed study of the global structure of the Schwarzschild metric, let us pause to make sure that the discussion is not going to be purely academic. If it were the case that an observer out at large distance could never reach the region $r = 2m$, then one might question why it would be so important to study the global structure there. On the other hand, if an observer can reach it in a finite time, then it is clearly of great importance (especially to the observer!) to understand what he will find there. This is actually already a slightly subtle issue because, as we shall see, an observer who stays safely out near infinity will never see the infalling observer pass through the event horizon at $r = 2m$. What about the infalling observer? Is he actually inside the event horizon? In the words of President Clinton, it all depends on what you mean by "is."

3.2.3 An interlude on geodesics

A way to determine whether an observer can reach some region in a spacetime is to study the so-called *geodesic motion* in the metric. Geodesics are the paths followed by freely-falling particles, and they can be shown to be given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\,\frac{dx^\nu}{d\tau}\,\frac{dx^\rho}{d\tau} = 0\ , \qquad (3.26)$$

where τ is the proper time and $x^\mu(\tau)$ is the path followed by the particle. The easiest way to see why this must be the right equation is to invoke the *equivalence principle*. Namely, we know that in flat Minkowski spacetime the correct equation of motion for a free particle is $\frac{d^2x^\mu}{d\tau^2} = 0$, and so it must be that in an arbitrary spacetime with gravitational fields, the equation of motion is the covariantised form of the Minkowski spacetime equation. This must be true because no matter what spacetime we consider, it is always possible, by a suitable general coordinate transformation, to change to a coordinate system where $g_{\mu\nu} = \eta_{mu\nu}$ and $\Gamma^\mu{}_{\nu\rho} = 0$ at a point. Nearby to that point, the equalities will still be good approximations.

Notice, by the way, that in the low-velocity weak-field approximation that we talked about earlier, geodesic motion in the linearised metric (3.21), for which the Christoffel connection components $\Gamma^i{}_{00}$ are given by (3.22), is therefore described by the equation

$$\frac{d^2x^i}{dt^2} = -\partial_i\,\phi\ , \qquad (3.27)$$

since $\tau \approx t$ in this limit. Equation (3.27) is otherwise known as Newton's second law of motion.

It is worth remarking here that, notwithstanding the 18'th century teachings of the 218 undergraduate physics class, the "gravitational force" is a frame-dependent concept, and in Einstein's theory it is the Christoffel connection $\Gamma^\mu{}_{\nu\rho}$ that describes it. In other words, the second term in (3.26) describes the tendency of an otherwise free particle to deviate from straight-line motion, and it is this second term that therefore describes what we call the "gravitational forces" that act on the particle. It is just as much a gravitational force whether it is caused by an acceleration relative to Minkowski spacetime (i.e. centrifugal forces are real gravitational forces), or whether it is caused by the presence of matter.

There *are* differences between these situations, which are measured by the vanishing or non-vanishing of the Riemann tensor, but this has nothing to do with the existence or non-existence of gravitational forces.

It is not hard to see that the equation of geodesic motion (3.26) can be derived from the Lagrangian

$$L = \tfrac{1}{2} g_{\mu\nu}\, \dot x^\mu\, \dot x^\nu \ , \tag{3.28}$$

where we are using $\dot x^\mu$ as a shorthand notation for $dx^\mu/d\tau$. To see this, just use the Euler-Lagrange equations coming from varying x^μ, i.e.

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot x^\mu}\Big) = 0 \ , \tag{3.29}$$

remembering that $g_{\mu\nu}$ is a function of the coordinates x^μ, which are now themselves viewed as functions of τ. From the definition (2.22), the result (3.26) now follows.

A further point to note is that for a physical particle the Lagrangian (3.28) actually takes the value $-1/2$. This is because the proper time interval $d\tau$ is related to the proper distance ds appearing in the metric interval by $d\tau^2 = -ds^2$, and $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{\mu\nu}\, \dot x^\mu\, \dot x^\nu\, d\tau^2$. The fact that $L = -1/2$ for the physical particle immediately gives us a constant of the motion, which helps considerably when solving the geodesic equation.

Let us now calculate the motion of radially-infalling geodesics in the Schwarzschild metric. (We could consider more general geodesic motion with angular dependence too, which would be relevant for considering planetary orbits, *etc.* From the point of view of testing whether an observer crosses the event horizon, however, any non-radial component to the motion would merely be a "time-wasting" manoeuvre, counter-productive from the point of view of getting there as quickly as possible.) For radial motion, therefore, it follows from (3.15) and (3.28) that we should consider the Lagrangian

$$L = -\tfrac{1}{2}\Big(1 - \frac{2m}{r}\Big)\, \dot t^2 + \tfrac{1}{2}\Big(1 - \frac{2m}{r}\Big)^{-1}\, \dot r^2 \ . \tag{3.30}$$

The Euler-Lagrange equation for t gives

$$\Big(1 - \frac{2m}{r}\Big)\, \dot t = E \ , \tag{3.31}$$

where E is a constant. The constant of the motion $L = -1/2$ then gives us the equation for infalling radial motion:

$$\dot r = -\Big(E^2 - 1 + \frac{2m}{r}\Big)^{1/2} \ , \tag{3.32}$$

where the choice of sign is determined by the fact that we are looking for the *ingoing* solution. Note that for a particle coming in from infinity the constant E must be such that $E^2 > 1$.

Suppose that at proper time τ_0 the particle is at radius r_0. It follows, by integrating (3.32), that the further elapse of proper time for it to reach $r = 2m$ is given by

$$
\begin{aligned}
\tau_{2m} - \tau_0 &= \int d\tau = \int_{r_0}^{2m} \frac{dr}{\dot r} \,, \\
&= \int_{2m}^{r_0} \frac{dr}{\sqrt{E^2 - 1 + \frac{2m}{r}}} \,.
\end{aligned}
\tag{3.33}
$$

This is perfectly finite, and so the ingoing particle does indeed fall through the event horizon in a finite proper time.

Notice, however, that an observer who watches from infinity will never see the particle reach the horizon. Such an observer measures time using the coordinate t itself, and so his calculation of the elapsed time will be

$$
\begin{aligned}
t_{2m} - t_0 &= \int dt = \int_{r_0}^{2m} \frac{\dot t \, dr}{\dot r} \,, \\
&= \int_{2m}^{r_0} \frac{E\, dr}{\left(1 - \frac{2m}{r}\right)\sqrt{E^2 - 1 + \frac{2m}{r}}} \,,
\end{aligned}
\tag{3.34}
$$

which diverges. In fact as the particle gets nearer and nearer the horizon the time measured in the t cooordinate gets more and more "stretched out," and radiation, or signals, from the particle get more and more red-shifted, but it is never seen to reach, or cross, the horizon. Seen from infinity, infalling observers never die; they just fade away.

3.2.4 The event horizon

In order to test the suspicion that $r = 2m$ is non-singular, and just not well-described by the (t, r, θ, φ) coordinate system, let us try changing variables to a different coordinate system. Of course it is not the (θ, φ) part that is at issue here, and in fact we can effectively suppress this in all of the subsequent discussion. We really need only concern ourselves with what is happening in the (t, r) plane, with the understanding that each point in this plane really represents a 2-sphere of radius r in the original spacetime. To abbreviate the writing, we can define the metric $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ on the unit-radius 2-sphere. To establish notation, let us denote by $\mathbf{g}$ the original Schwarzschild metric (3.15), and denote by $\mathcal{M}$ the manifold on which it is valid, namely,

$$
\mathcal{M}: \qquad r > 2m \,. \tag{3.35}
$$

(Actually, there are two disjoint regions where the metric is valid, namely $0 < r < 2m$, and $r > 2m$. Since we want to include the description of the asymptotic external region far from the mass, it is natural to choose $\mathcal{M}$ as the $r > 2m$ region.) Together, we may refer to the pair $(\mathcal{M}, \mathbf{g})$ as the original Schwarzschild spacetime.

The best starting point for the sequence of coordinate transformations that we shall be using is to consider a *null* ingoing geodesic, rather than the timelike ones followed by massive particles that we considered in the previous section. A null geodesic has the property that

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{3.36}$$

where λ parameterises points along its path, $x^\mu = x^\mu(\lambda)$. Note that we can't use the proper time τ as the parameter now, since $d\tau = 0$ along the path of a null geodesic (such as a light beam), and so we choose some other parameterisation in terms of λ instead. From the Schwarzschild metric (3.15) we can see that a radial null geodesic (for which $ds^2 = 0$) must satisfy

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2} \,. \tag{3.37}$$

It is natural to introduce a new radial coordinate r^*, defined by

$$r^* \equiv \int^r \frac{dr}{1 - \frac{2m}{r}} = r + 2m \log\left(\frac{r - 2m}{2m}\right) \,. \tag{3.38}$$

This is known as the Regger-Wheeler radial coordinate, and it has the effect of stretching out the distance to horizon, pushing it to infinity. Sometimes r^* is called the "tortoise coordinate," although this is a bit of a misnomer since the fabled tortoise gets there in the end.

We now define advanced and retarded *null* coordinates v and u, known as "Eddington-Finkelstein coordinates:"

$$v = t + r^* \,, \qquad -\infty < v < \infty \,, \tag{3.39}$$

$$u = t - r^* \,, \qquad -\infty < v < \infty \,. \tag{3.40}$$

Radially-infalling null geodesics are described by $v = \text{constant}$, while radially-outgoing null geodesics are described by $u = \text{constant}$. If we plot the lines of constant u and constant v in the (t, r) plane, we can begin to see what is going on. (See Figure 1.) Out near infinity, we have $v \approx t + r$ and $u \approx t - r$, and the lines $v = \text{constant}$ and $u =$ constant just asymptote to 45-degree straight lines of gradient -1 and $+1$ respectively. Light-cones look normal out near infinity, with 45-degree edges defined by $v =$ constant and $u =$ constant. As we get nearer the horizon, these light cones become more and more acute-angled, until on the horizon itself they have become squeezed into cones of zero vertex-angle. Inside the horizon they have tipped over, and lie on their sides.

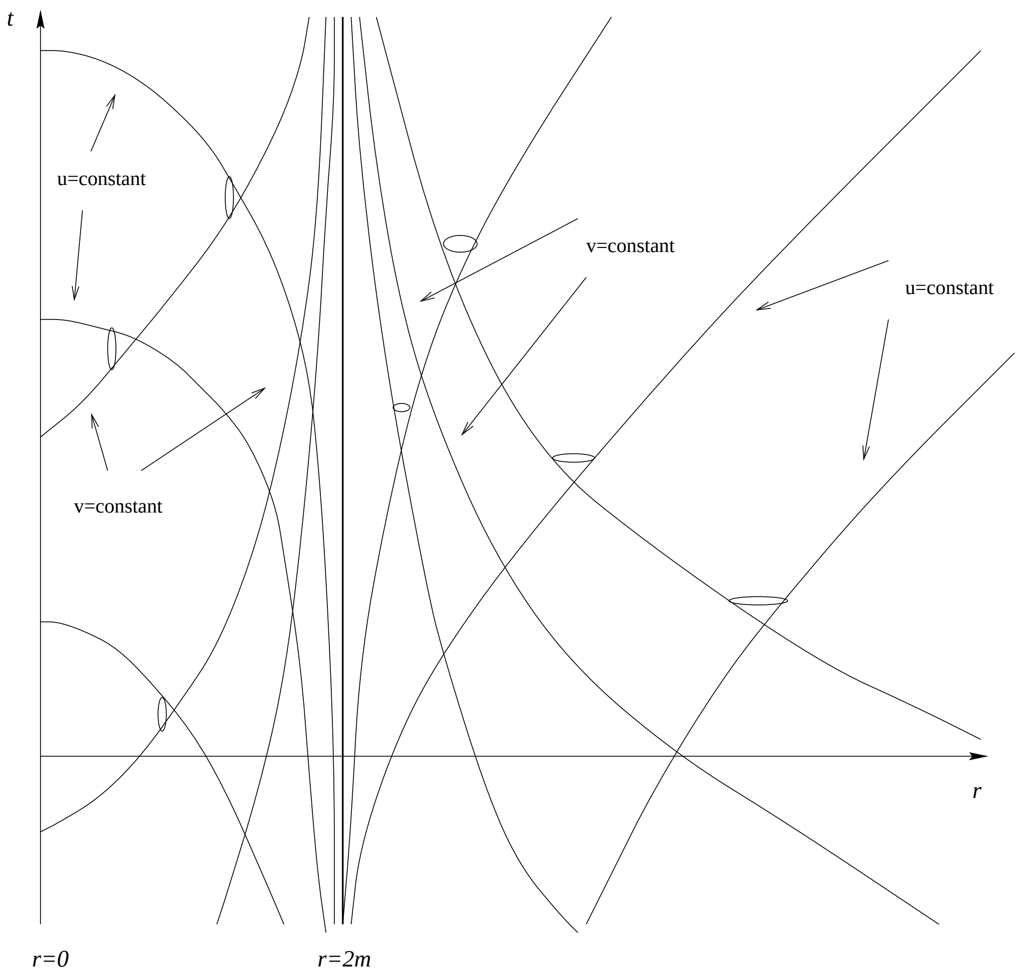

Figure 1: Schwarzschild spacetime $(\mathcal{M}, \mathbf{g})$.

The light cones are getting squeezed like this because we are trying to describe things near the horizon using the time coordinate t which is really appropriate only for an observer out at large distances. We have already seen that the use of the coordinate t to describe an infalling particle leads to the misleading impression that it never actually reaches $r = 2m$, let alone passes through it.

Guided by the behaviour of the light-cones, we are therefore led to try replacing the coordinate t in the original Schwarzschild metric (3.15) by v, using (3.39) to set $t = v - r^*$. Thus we find that the metric becomes

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)\, dv^2 + 2dr\, dv + r^2\, d\Omega^2 \ . \tag{3.41}$$

This now has no divergence at $r = 2m$, and, because of the constant cross-term $2dr\, dv$, its inverse is perfectly finite there too; in other words, the metric is non-singular at $r = 2m$. We can now plot another spacetime diagram, where we use v and r as the coordinates on the plane. Since we know that out near infinity the $v =$constant lines are well thought-of as being at 45-degrees with slope -1, it is natural to choose this as our plotting scheme everywhere. This gives us the picture shown in Figure 2. We see now that the light-cones do not degenerate on the horizon. They do, however, tilt over more and more as one approaches the horizon, until at $r = 2m$ itself they have tipped so that the future light-cone lies entirely within the direction of decreasing r. In fact $r = 2m$ is a null surface, and the spacetime is not time symmetric. The surface $r = 2m$ acts as a one-way membrane; future-directed timelike and null paths can cross only in one direction, from $r > 2m$ to $r < 2m$. They reach the singularity at $r = 0$ in a finite proper time or affine distance. Past-directed timelike or null curves in the region $0 < r < 2m$, on the other hand, cannot reach the singularity at $r = 0$. In other words a future-directed null ray has only one way to go; inwards. The fate of a massive particle, whose path must lie inside the null cone, is the same.

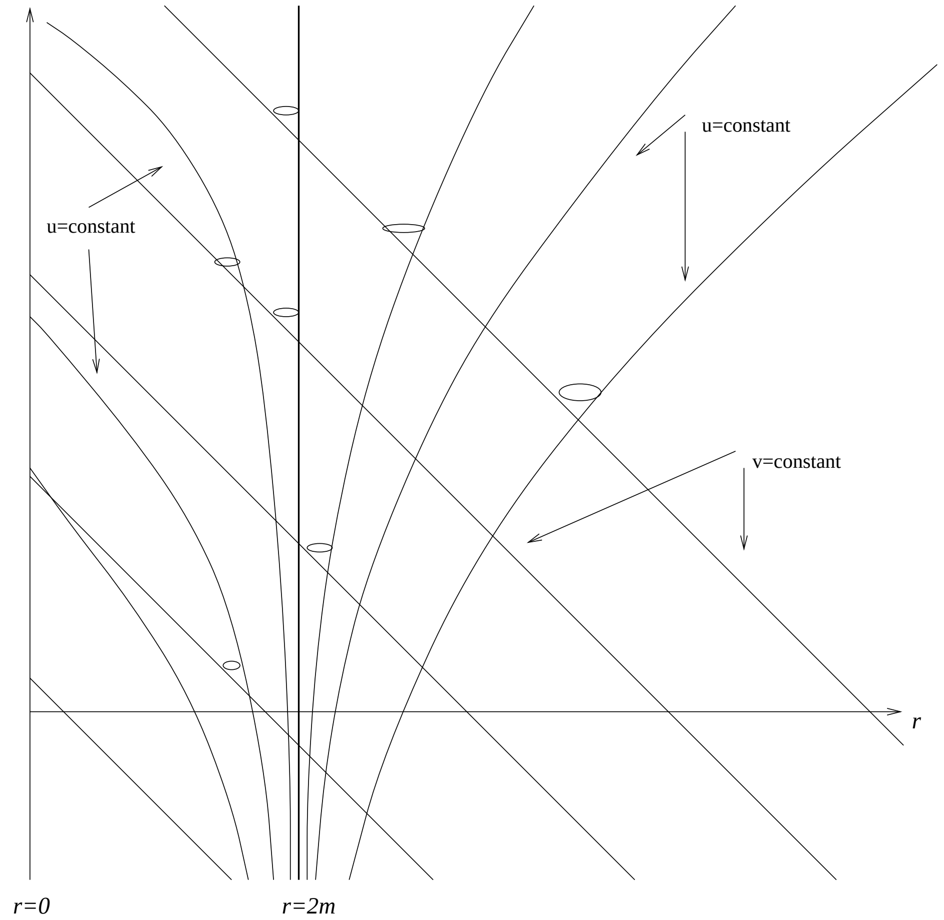

Figure 2: Schwarzschild spacetime $(\mathcal{M}', \mathbf{g}')$.

Let us denote by $\mathbf{g}'$ the metric (3.41). Since there is no metric singularity at $r = 2m$, we see that the range of the radial coordinate r, which was restricted to the region $r > 2m$ in the original spacetime $(\mathcal{M}, \mathbf{g})$ with metric $\mathbf{g}$ given by (3.15), can now be extended to cover the entire region $r > 0$. Thus we have an analyic extension $(\mathcal{M}', \mathbf{g}')$ of the Schwarzschild spacetime, where

$$\mathcal{M}' : \qquad r > 0 \; . \tag{3.42}$$

There is an alternative analytic extension of $(\mathcal{M}, \mathbf{g})$ that we can consider, where we substitute for the time coordinate using the retarded Eddington-Finkelstein coordinate u defined in (3.40), rather than the advanced coordinate v. This gives another form for the Schwarzschild metric, which we shall call $\mathbf{g}''$:

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)\, du^2 - 2du\, dr + r^2\, d\Omega^2 \; . \tag{3.43}$$

This is again nonsingular at $r = 2m$, and is analytic on a manifold $\mathcal{M}''$ with

$$\mathcal{M}'' : \qquad r > 0 \; . \tag{3.44}$$

However, although the region of analyticity here is the same as for the extension $\mathcal{M}'$, the two analytic extensions $\mathcal{M}'$ and $\mathcal{M}''$ are quite different. The time asymmetry in the $\mathcal{M}''$ manifold is the opposite of that in $\mathcal{M}'$. The surface $r = 2m$ is again null, but this time it is a one-way membrane acting in the opposite direction; it is now only past-directed timelike or null curves that can cross from $r > 2m$ to $r < 2m$. This is depicted in Figure 3.

It is clear that neither of the analytic extensions $(\mathcal{M}', \mathbf{g}')$ or $(\mathcal{M}'', \mathbf{g}'')$ by itself captures the entire structure of the full Schwarzschild geometry. We can, however, go one stage further and construct a larger extension of the spacetime by using both the v and u coordinates, in place of t and r. Thus from (3.15), (3.39) and (3.40) we obtain the metric

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)\, dv\, du + r^2\, d\Omega^2 \; . \tag{3.45}$$

Here, we are now using r simply as a shorthand symbol for the quantity defined by

$$\tfrac{1}{2}(v - u) = r + 2m\, \log(r - 2m) \; . \tag{3.46}$$

Now define new coordinates $\tilde{v}$ and $\tilde{u}$, known as Kruskal coordinates, by

$$\tilde{v} = e^{\frac{v}{4m}} \; , \qquad \tilde{u} = -e^{-\frac{u}{4m}} \; . \tag{3.47}$$

In terms of these, we arrive at the metric $\mathbf{g}^*$, given by

$$ds^2 = \frac{16m^2\, e^{-\frac{r}{2m}}}{r}\, d\tilde{v}\, d\tilde{u} + r^2\, d\Omega^2 \; , \tag{3.48}$$

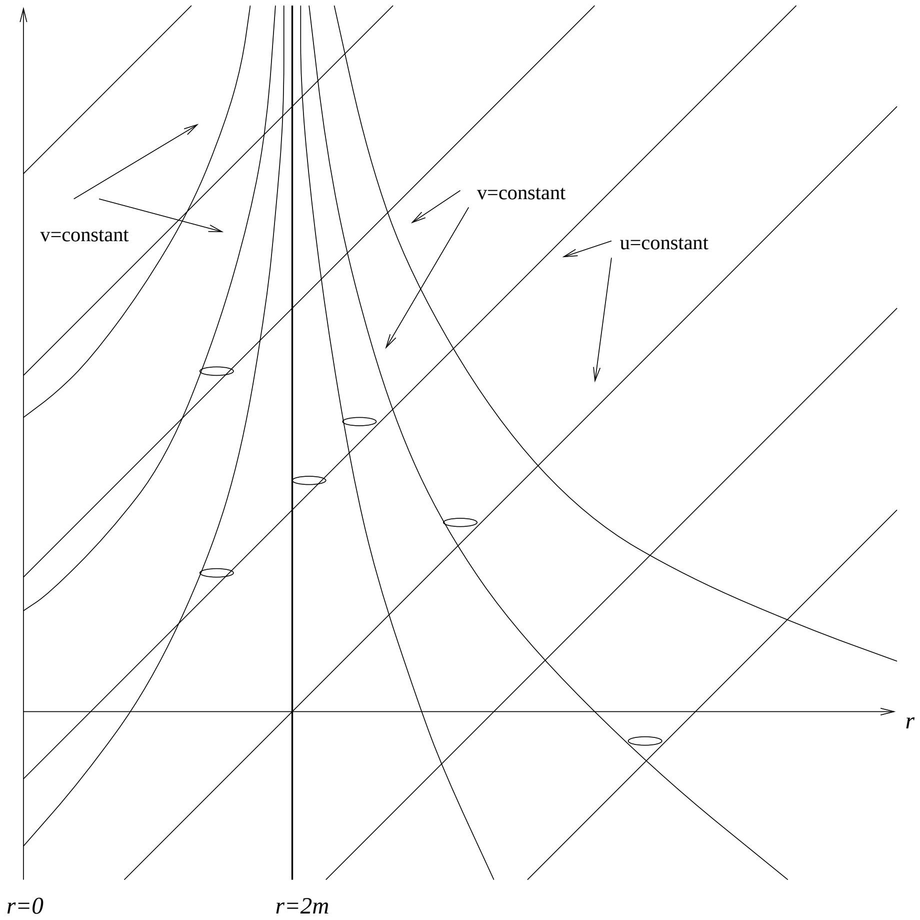

Figure 3: Schwarzschild spacetime $(\mathcal{M}'', \mathbf{g}'')$.

It is useful also to define

$$\tilde{t} = \tfrac{1}{2}(\tilde{v} + \tilde{u}) \ , \qquad \tilde{x} = \tfrac{1}{2}(\tilde{v} - \tilde{u}) \ , \tag{3.49}$$

in terms of which the metric $\mathbf{g}^*$ becomes

$$ds^2 = \frac{16m^2\, e^{-\frac{r}{2m}}}{r}\,(-d\tilde{t}^2 + d\tilde{x}^2) + r^2\, d\Omega^2 \ . \tag{3.50}$$

The quantity r is defined implicitly by

$$\tilde{t}^2 - \tilde{x}^2 = \tilde{v}\,\tilde{u} = -(r - 2m)\, e^{\frac{r}{2m}} \ . \tag{3.51}$$

On the manifold $\mathcal{M}^*$, defined by the coordinates $(\tilde{t}, \tilde{x}, \theta, \varphi)$ for $\tilde{t}^2 - \tilde{x}^2 < 2m$, the metric $\mathbf{g}^*$ given by (3.50) has components that are positive and analytic. We may draw a new spacetime diagram, given in Figure 4, to represent the manifold $\mathcal{M}^*$. The pair $(\mathcal{M}^*, \mathbf{g}^*)$ is the *maximal analytic extension* of the original Schwarzschild solution. The region I, defined by $\tilde{x} > |\tilde{t}|$, is isometric to the original Schwarzschild spacetime $(\mathcal{M}, \mathbf{g})$, for which $r > 2m$. The region $\tilde{x} > -\tilde{t}$, corresponding to regions I and II in Figure 4, is isometric to the advanced analytic extension $(\mathcal{M}', \mathbf{g}')$. Similarly the region $\tilde{x} > \tilde{t}$, corresponding to regions I and II′ in Figure 4, is isometric to the retarded analytic extension $(\mathcal{M}'', \mathbf{g}'')$.

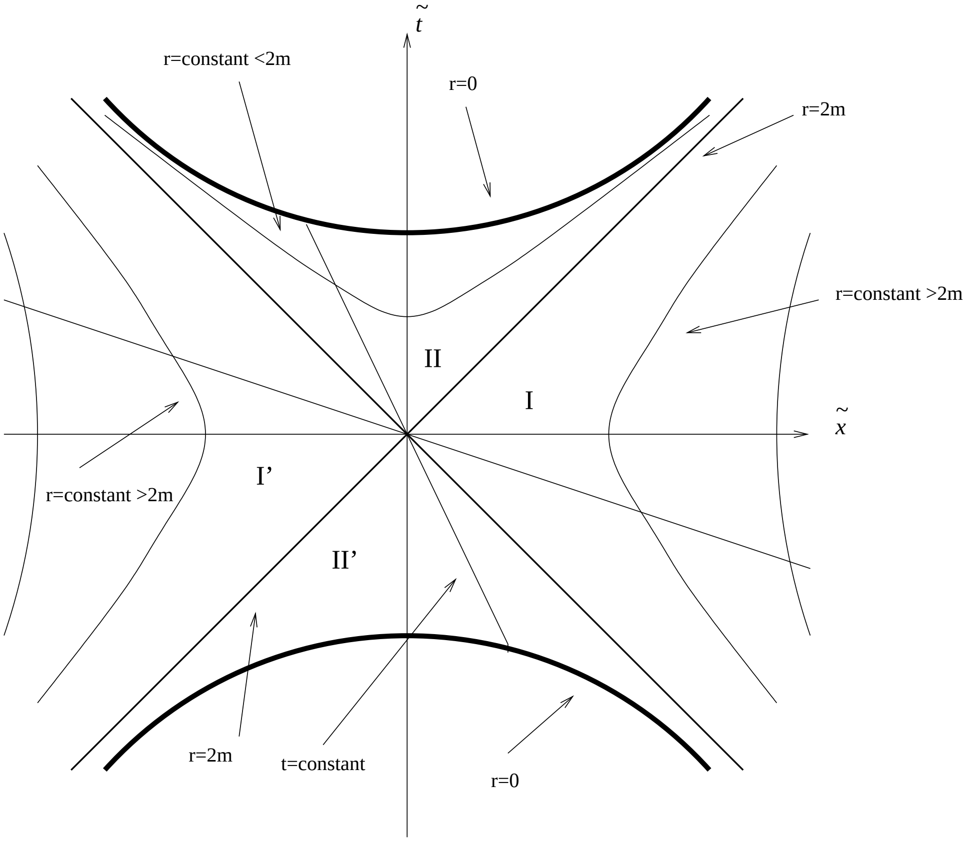

Figure 4: Schwarzschild spacetime $(\mathcal{M}^*, \mathbf{g}^*)$.

There is also a region I′, defined by $\tilde{x} < -|\tilde{t}|$, which again is isometric to the exterior spacetime $(\mathcal{M}, \mathbf{g})$. This is another asymptotically-flat universe, separated from "our" universe by a "throat" where the area $4\pi\, r^2$ of the 2-spheres in the (θ, φ) directions has shrunk down to a minimum value of $16\pi\, m^2$ (i.e. $r = 2m$), and then expanded out again. In fact one can see from Figure 4 that the regions I′ and II are isometric to the advanced Finkelstein extension of region I′, and that the regions I′ and II′ are isometric to the retarded Finkelstein extension of I′. No timelike or null curves can cross from region I to region I′; in fact any such curve that crosses from I′ into the region where $r < 2m$ will necessarily end up at the (upper) singularity at $r = 0$. So neither material objects, nor information, can cross from I′ to I.

Finally, in our analysis of the maximal analytic extension of the Schwarzschild solution we can make one further transformation of the coordinates, which has the effect of bringing infinity in to a finite distance, so that the entire spacetime can be fitted onto the back of a postage stamp (times a 2-sphere sitting over each point, of course). We do this by making use of the arctangent function, which has the property of mapping the entire real line into the interval between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Thus we define new coordinates V and U, in place of $\tilde{v}$ and $\tilde{U}$, where

$$V = \arctan\left(\frac{\tilde{v}}{\sqrt{2m}}\right) , \qquad U = \arctan\left(\frac{\tilde{u}}{\sqrt{2m}}\right) , \tag{3.52}$$

where

$$-\pi < V + U < \pi \ , \qquad \text{and} \qquad -\tfrac{1}{2}\pi < V < \tfrac{1}{2}\pi \ , \quad -\tfrac{1}{2}\pi < U < \tfrac{1}{2}\pi \ . \tag{3.53}$$

With this mapping, the Kruskal maximal extension of Figure 4 turns into the so-called Penrose diagram for the Schwarzschild spacetime, depicted in Figure 5. Note that we can express r in terms of U and V as

$$\tan V \ \tan U = -(r - 2m)\, e^{\frac{r}{2m}} \ . \tag{3.54}$$

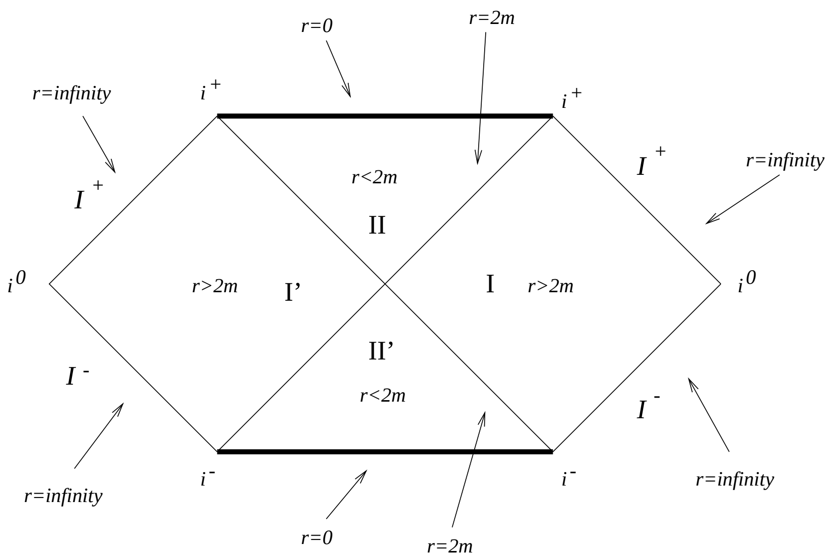

Figure 5: The Penrose diagram for the Schwarzschild spacetime $(\mathcal{M}^*, \mathbf{g}^*)$.

Essentially all that has been done in this last transformation is to bring infinity in to a finite distance. However, by doing so a new feature has come to light, namely that there are a number of different kinds of asymptotic infinity. These can be characterised as the places where the various different kinds of particles come from, and where they end up. Thus we have the places denoted by i^-, which is where massive particles (which follow timelike geodesics) came from at $r = \infty$ in the distant past, and i^+, which is where they end up at $r = \infty$ in the distant future, if they are fortunate enough to have followed paths that keep them away from the event horizon and the singularity of the black hole. The regions denoted by $\mathcal{I}^-$ (and pronounced, regretfully, as "scri") are likewise the places that massless particles (following null geodesics) came from at $r = \infty$ in the distant past, and $\mathcal{I}^+$ is where the lucky ones end up at in the distant future. (Note that in Figure 5 the symbols for scri, appearing on the outer diagonal borders of the diagram, appear just as italic I, owing to the limited xfig skills of the author.) Finally, hypothetical particles of negative mass-squared (tachyons) would follow spacelike geadesics, and these begin and end at i^0. The regions $i^\pm$ are known as future and past timelike infinity, the regions $\mathcal{I}^\pm$ are known as future and past null infinity, and i^0 is known as spacelike infinity. Of course one should remember that the effect of having squeezed the entire universe onto a postage stamp is that one can gain a false impression of distance. In particular, for example, although i^0 looks like a single point in the Penrose diagram, it is actually an entire infinite region. (This is over and above the now-familiar fact that each point in any of our two-dimensional spacetime diagrams really represents a 2-sphere.) Likewise, the "points" labelled i^- and i^+ are infinite in extent. Furthermore, another aspect of the Penrose diagram is that i^+ and i^-, at $r = \infty$, appear to be coincident with the ends of the horizontal $r = 0$ lines, which represent the spacelike curvature singularities. This is again an unfortunate impression created by the foreshortening resulting from the arctangent mapping, and they are in actuality infinitely separated. In the words of Douglas Adams, in *The Hitchhiker's Guide to the Galaxy*, "The universe is a big place."

It should be remarked that the discussion in this section has been somewhat of an idealisation, and the maximal analytic extension of the Schwarzschild solution is not what would arise in a physical situation where a black hole formed as a result of gravitational collapse. In particular, the "south-west" part of the Penrose diagram would be missing in a realistic example where a star collapsed to form a black hole. This is perhaps just as well, because the south-west part of the diagram really describes a "white hole" from our point of view as dwellers in the eastern part of the diagram; particles and null rays can come out of it, but they cannot go in. A Penrose diagram for a star that collapses to form a Schwarzschild black hole is depicted in Figure 6. The shaded area represents the inside of the star.

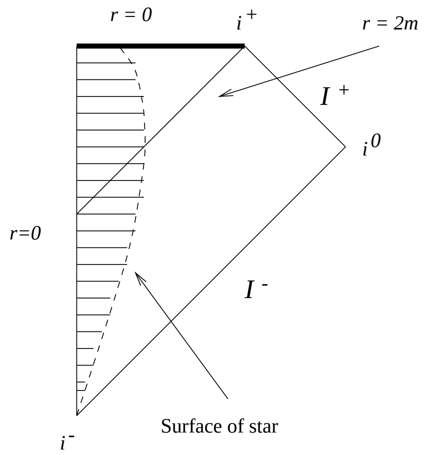

Figure 6: The Penrose diagram for a collapsing spherically-symmetric star.

4 Charged Black Holes

4.1 The Reissner-Norström solution

The black holes that we studied in the previous chapter were the simplest variety, which are characterised by only one parameter, namely their mass m. As was mentioned previously, more complicated black holes can also arise, with additional parameters characterising them. Within the framework of pure general relativity, with no matter fields present, there is just one other possible quantity that can characterise a black hole, namely its angular momentum J. There is an exact solution describing such rotating black holes, known as the Kerr solution. We shall defer further discussion of this case for now, because it is technically quite complicated, although it does have many new and interesting features in its global structure. A simpler generalisation of the Schwarzschild black hole, which exhibits some rather similar new features, is provided by considering spherically-symmetric solutions of the coupled Einstein-Maxwell equations, where the gravitating object carries electric (or magnetic) charge, as well as mass.

The Einstein part of the coupled Einstein-Maxwell equations is obtained by taking the Einstein equations (3.1), with $T_{\mu\nu}$ being the energy-momentum tensor for the Maxwell field strength $F_{\mu\nu}$. The Maxwell equations must now be formulated in curved spacetime. By the principal of equivalence, which essentially says that physics in an arbitrary coordinate system should be described by generally-covariant equations that reduce to the familiar flat-space ones when specialised to a Minkowski-spacetime metric, we can deduce that the source-free Maxwell equation $\partial^\mu F_{\mu\nu} = 0$ must be replaced by the generally-covariant equation $\nabla^\mu F_{\mu\nu} = 0$.

There is an elegant, and useful, way to derive the coupled set of Einstein-Maxwell equations from a Lagrangian. Let us write the Lagrangian

$$\mathcal{L} = \sqrt{-g}\,(R - \tfrac{1}{4}F^2)\ , \tag{4.1}$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$, F^2 means $F_{\mu\nu}\,F^{\mu\nu}$, and by definition, $F_{\mu\nu}$ here is a shorthand for $\partial_\mu\,A_\nu - \partial_\nu\,A_\mu$. If we vary the action $I = \int d^4x\,L$ with respect to the metric, then demanding that this be stationary will give the Einstein equations, while demanding that the variation of I with respect to the Maxwell potential A_μ be stationary will give the Maxwell equations. To carry out the variation with respect to the metric, it is convenient to vary $g^{\mu\nu}$ rather than $g_{\mu\nu}$. We need to note, from the theory of matrices, that we will have

$$\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\ . \tag{4.2}$$

Also, to calculate δR, we note that since $R = g^{\mu\nu}\,R_{\mu\nu}$, we will have $\delta R = g^{\mu\nu}\,\delta R_{\mu\nu} + R_{\mu\nu}\,\delta g^{\mu\nu}$. Now, we also know that $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, and that the Riemann tensor is given by (2.25), with $\Gamma^\mu{}_{\nu\rho}$ given by (2.22). Thus we can deduce that

$$\begin{aligned}
\delta R_{\mu\nu} &= \delta R^\rho{}_{\mu\rho\nu}\ ,\\
\delta R^\mu{}_{\nu\rho\sigma} &= \nabla_\rho\,\delta\Gamma^\mu{}_{\sigma\nu} - \nabla_\sigma\,\delta\Gamma^\mu{}_{\rho\nu}\ ,\\
\delta\Gamma^\mu{}_{\sigma\nu} &= \tfrac{1}{2}g^{\mu\lambda}\,(\nabla_\sigma\,\delta g_{\lambda\nu} + \nabla_\nu\,\delta g_{\lambda\sigma} - \nabla_\lambda\,\delta g_{\sigma\nu})\ .
\end{aligned} \tag{4.3}$$

The last two lines can be verified by direct computation from the various definitions in chapter 2. A way to save a lot of time is to note that although $\Gamma^\mu{}_{\sigma\nu}$ is not a tensor, its variation $\delta\Gamma^\mu{}_{\sigma\nu}$ *is* a tensor (in fact the difference between *any* two Christoffel connections is a tensor, since the inhomogeneous terms in their transformation rules cancel). Consequently, when calculating $\delta\Gamma^\mu{}_{\sigma\nu}$ from (2.22), it follows that the partial derivatives *must* conspire to become covariant derivatives after the variation. Likewise, when calculating $\delta R^\mu{}_{\nu\rho\sigma}$ from (2.25), the partial derivatives and the bare Christoffel connections after the variation *must* conspire to produce covariant derivatives.

The upshot from all this is that we eventually conclude that

$$\delta(\sqrt{-g}\,R) = \sqrt{-g}\,(R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\,\nabla_\rho\,\nabla^\rho)\,\delta g^{\mu\nu}\ . \tag{4.4}$$

In the variation of the action I, the last two terms here will integrate to zero, since they are total derivatives. Thus after including the variation of the metric in $\sqrt{-g}\,F^2 = \sqrt{-g}\,F_{\mu\nu}\,F_{\rho\sigma}\,g^{\mu\rho}\,g^{\nu\sigma}$, we arrive at the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} = T_{\mu\nu} = \tfrac{1}{2}(F_{\mu\rho}\,F_\nu{}^\rho - \tfrac{1}{4}F^2\,g_{\mu\nu})\ . \tag{4.5}$$

This is precisely consistent with the general form (3.1), where we have, for convenience, chosen units where $8\pi\, G = 1$.

Varying the action with respect to the vector potential A_μ instead, we arrive at

$$\partial_\mu\Big(\sqrt{-g}\, F^{\mu\nu}\Big) = 0 \ . \tag{4.6}$$

Although this does not at first sight look covariant, it actually is; it is straightforward to show, using (2.22) and the definitions of the covariant derivative in chapter 2, that it is equivalent to

$$\nabla_\mu\, F^{\mu\nu} = 0 \ . \tag{4.7}$$

We are now in a position to look for our solution of the Einstein-Maxwell equations. We again assume spatial spherical symmetry and time-independence, and so the ansatz for the metric will be the same one (3.2) that we adopted when constructing the Schwarzschild solution. For the potential, the spherically-symmetric time-independent ansatz for the potential will be

$$A = \phi\, dt \ , \tag{4.8}$$

where ϕ, the electrostatic potential, is taken to be dependent only on r. (We shall look for an electrically-charged black hole here.) After straightforward calculations, one finds that the Maxwell equation (4.6) gives

$$\frac{d}{dr}\Big(r^2\,\phi'\, e^{-A-B}\Big) = 0 \ , \tag{4.9}$$

and that the vielbein components of the energy-momentum tensor defined in (4.5) are

$$T_{00} = -T_{11} = T_{22} = T_{33} = \tfrac14 \phi'^2\, e^{-2A-2B} \ . \tag{4.10}$$

Equating these to the vielbein components of the Ricci tensor, which were found in (3.11), we see that again the sum of the first two equations gives $B = -A$. Now (4.9) is easily solved for ϕ, and hence the third of the Einstein equations from (3.11) can then be solved for A. The final result is that the metric and potential are given by

$$\begin{aligned} ds^2 &= -\Big(1-\frac{2m}{r}+\frac{e^2}{r^2}\Big)\, dt^2 + \Big(1-\frac{2m}{r}+\frac{e^2}{r^2}\Big)^{-1}\, dr^2 + r^2\,(d\theta^2+\sin^2\theta\, d\varphi^2)\ , \\ A &= \phi\, dt = \frac{2e}{r}\, dt\ , \end{aligned} \tag{4.11}$$

where m and e are constants. In fact m is again the mass, and e is the electric charge. (The unusual-looking factor of 2 in the expression for the potential is a consequence of our choice of conventions.) This is the Reissner-Nordström solution, describing an electrically-charged, spherically-symmetric, static black hole.

4.2 Global structure of the Reissner-Nordström solution

The Reissner-Nordström solution that we obtained in the previous subsection has some features in common with the Schwarzschild solution (3.15) of the previous chapter. There are also some important differences, and, as we shall see, the global structure of the maximal analytic extension of the Reissner-Nordström spacetime is quite different from that of the Schwarzschild spacetime.

First, we note that the metric is free of curvature singularities everywhere except at $r = 0$. In fact, a straightforward calculation shows that

$$|\mathrm{Riem}|^2 = \frac{48m^2}{r^6} - \frac{96\,e^2\,m}{r^7} + \frac{56e^4}{r^8}\ . \tag{4.12}$$

The function $(1 - \frac{2m}{r} + \frac{e^2}{r^2})$ appearing in the metric has roots, possibly complex, of the form $r = r_\pm$, where

$$r_+ = m + \sqrt{m^2 - e^2} \qquad r_- = m - \sqrt{m^2 - e^2}\ . \tag{4.13}$$

Consequently, we have three different regimes to consider, namely $e^2 < m^2$, $e^2 = m^2$ and $e^2 > m^2$. For $e^2 < m^2$ there are two distinct real, positive, roots; these coalesce to one double root at $r = m$ if $e^2 = m^2$. Finally, if $e^2 > m^2$, the two roots are complex.

Let us first calculate the analogue of the Regge-Wheeler "tortoise" coordinate for the Reissner-Nordström metric. In other words, we solve for radial null geodesics in the Reissner-Nordström geometry, with $0 = ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})\, dt^2 + (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1}\, dr^2$. It follows by integrating this that we will have ingoing and outgoing null geodesics with $r^* = -t$ and $r^* = +t$ respectively, where

$$e^2 < m^2: \qquad r^* = r + \frac{r_+^2}{r_+ - r_-}\,\log(r - r_+) - \frac{r_-^2}{r_+ - r_-}\,\log(r - r_-)\ , \tag{4.14}$$

$$e^2 = m^2: \qquad r^* = m\,\log\left((r-m)^2\right) - \frac{m^2}{r-m}\ , \tag{4.15}$$

$$\begin{aligned} e^2 > m^2: \qquad r^* &= r + m\,\log\left((r-m)^2 + e^2 - m^2\right) \\ &\quad - \frac{2(e^2 - 2m^2)}{\sqrt{e^2-m^2}}\,\arctan\Big[\frac{r-m}{\sqrt{e^2-m^2}}\sqrt{e^2-m^2}\Big]\ . \end{aligned} \tag{4.16}$$

We can dispose of the case $e^2 > m^2$ rather easily. The roots $r_\pm$ are complex, and hence the function $(1 - \frac{2m}{r} + \frac{e^2}{r^2})$ has no zeros for $r > 0$. This means that the curvature singularity at $r = 0$ is not hidden behind an horizon, and it can in fact be seen from infinity. This can be demonstrated by looking at the r^* coordinate given in (4.16). We see that an outgoing null geodesic, which will satisfy $r^* = t$, requires only a finite amount of coordinate time to travel from $r = 0$ to any finite distance r. In other words, one can stand at a safe distance from the singularity and look at it. More technically, we can say that null geodesics can emanate from the singularity and end up at $\mathcal{I}^+$. When this circumstance arises, the singularity is called a *Naked Singularity.* By contrast, in the Schwarzschild solution, we saw that the singularity was "hidden" behind the event horizon at $r = 2m$, and no timelike or null curves could pass from $r = 0$ to the "outside." In the 1960's a conjecture was formulated, known as the "Cosmic Censorship Hypothesis," which asserted that no physically-realistic collapsing matter system could ever end up having naked singularities; they would always be decently clothed behind event horizons. This has subsequently been proven. In particular, it can be shown that no realistic system can evolve to give an $e^2 > m^2$ Reissner-Nordström metric. In the dimensionless natural units which we are using it is sometimes easy to forget what the scales of the various quantities are. It is worth remarking, therefore, that if a macroscopic black hole with $e^2 > m^2$ did exist, it would be a fearsome object carrying a gargantuan amount of charge.

Let us postpone the discussion of the intermediate case $e^2 = m^2$ for now, and look next at the situation when $e^2 < m^2$. The function $(1 - \frac{2m}{r} + \frac{e^2}{r^2})$ now has two distinct, real, positive, roots $r_\pm$, given by (4.13). This means that there are in fact two distinct event horizons; the *outer horizon* at $r = r_+$, and the *inner horizon* at $r = r_-$. These mark the boundaries where the function $(1 - \frac{2m}{r} + \frac{e^2}{r^2})$ passes through zero and changes sign, implying that the time coordinate t is spacelike for $r_- < r < r+$, while it is genuinely timelike for $r > r_+$ and for $0 < r < r_-$. We may short-circuit some of the intermediate steps paralleling our discussion for the Schwarzschild metric, and first go directly to the double-null coordinates

$$v = t + r^* \,, \qquad u = t - r^* \,, \tag{4.17}$$

in terms of which the Reissner-Nordström metric becomes

$$ds^2 = -\Big(1 - \frac{2m}{r} + \frac{e^2}{r^2}\Big)\, dv\, du + r^2\, d\Omega^2 \,. \tag{4.18}$$

At this stage, things start to get a little tricky. First, to simplify the formulae a bit, let us define two constants $\kappa_\pm$, by

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2} \,. \tag{4.19}$$

The expression for the r^* coordinate (4.14) now becomes

$$r^* = r + \frac{1}{2\kappa_+}\, \log(r - r_+) + \frac{1}{2\kappa_-}\, \log(r - r_-) \,. \tag{4.20}$$

Now introduce coordinates $v_\pm$ and $u_\pm$, defined by

$$v_\pm = e^{\kappa_\pm\, v} \,, \qquad u_\pm = -e^{-\kappa_\pm\, u} \,. \tag{4.21}$$

These are analogous to the Kruskal coordinates $(\tilde{v}, \tilde{u})$ that we used in the Schwarzschild maximal analytic extension, only here there we need two different pairs, (v_+, u_+) and (v_-, u_-), which will cover different patches on the spacetime manifold. Note that we have the following identities:

$$v_+\, u_+ = -(r - r_+)\,(r - r_-)^{\kappa_+/\kappa_-}\, e^{2\kappa_+\, r}\ , \qquad dv_+\, du_+ = -\kappa_+^2\, v_+\, u_+\, dv\, du\ , \tag{4.22}$$
$$v_-\, u_- = -(r - r_-)\,(r - r_+)^{\kappa_-/\kappa_+}\, e^{2\kappa_-\, r}\ , \qquad dv_-\, du_- = -\kappa_-^2\, v_-\, u_-\, dv\, du\ , \tag{4.23}$$

Substituting into (4.18), we see that the metric becomes

$$ds^2 = -\frac{(r - r_-)^{1-\kappa_+/\kappa_-}}{\kappa_+^2\, r^2}\, e^{-2\kappa_+\, r}\, dv_+\, du_+ + r^2\, d\Omega^2\ , \tag{4.24}$$

and so it is non-singular for $r > r_-$, with a coordinate singularity at $r = r_-$. In fact these (v_+, u_+) coordinates cover a region looking very like the Kruskal diagram (Figure 4) for Schwarzschild, except that the genuine $r = 0$ singularity in Figure 4 is now relabelled as the $r = r_-$ coordinate singularity, and the $r = 2m$ lines in Figure 4 become $r = r_+$. This is depicted in Figure 7.

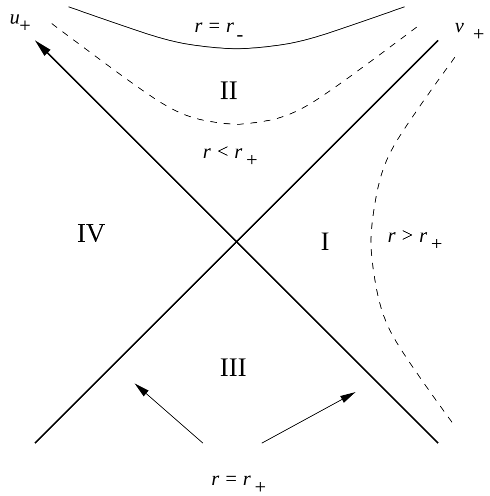

Figure 7: The region $r > r_-$ in Reissner-Nordström.

The shortcomings of the (v_+, u_+) coordinates are overcome by using the (v_-, u_-) coordinates instead, in terms of which the metric (4.18) becomes

$$ds^2 = -\frac{(r - r_+)^{1-\kappa_-/\kappa_+}}{\kappa_-^2\, r^2}\, e^{-2\kappa_-\, r}\, dv_+\, du_+ + r^2\, d\Omega^2\ , \tag{4.25}$$

This is non-singular for $r < r_+$, with a coordinate singularity at $r = r_+$. Since $r_+ > r_-$, this means that the (v_+, u_+) and $v_-, u_-)$ coordinate patches overlap. The Kruskal-type diagram for the (v_-, u_-) coordinates is depicted in Figure 8. Now, the two main diagonals represent $r = r_+$, and the singularity at $r = 0$ corresponds to the two vertical arcs on the left and right hand sides of the diagram. The crucial point is that there is the region of overlap between the validity of the (v_+, u_+) and the (v_-, u_-) coordinates, when $r_- < r < r_+$. This means that region II in Figure 7 is actually the same as region II in Figure 8. On the other hand, region III in Figure 7 is distinct from region III$'$ in Figure 8. However, since region II in Figure 7 connects to an exterior spacetime in the past (namely regions I, III and IV), it follows by time-reversal invariance that region III$'$ in Figure 8 must connect to an exterior spacetime in its future. This argument then repeats indefinitely, so that we must go on stacking up copies of Figure 7, then Figure 8, then Figure 7 again, and so on, into the infinite past and future.

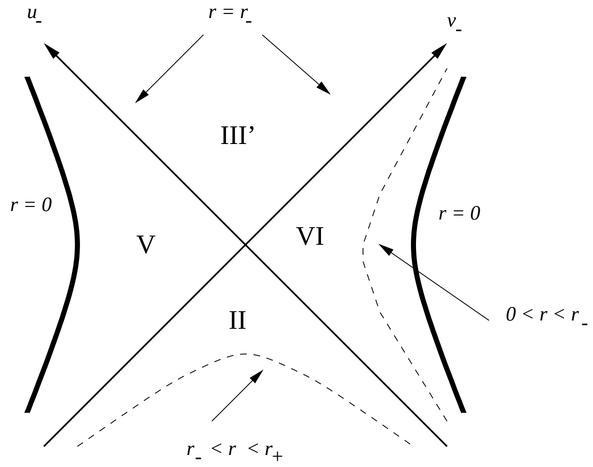

Figure 8: The region $0 < r < r_+$ in Reissner-Nordström.

If we now make arctangent transformations of the kind we used for Schwarzschild, we can make an entire Figure 7 plus Figure 8 pair fit onto a finite-sized piece of paper. However, since we have to stack up an infinite number of such pairs, we will still have a Penrose diagram that streches off to infinity along the vertical axis. We might say that if Schwarzschild spacetime can be fitted onto a postage stamp, then for Reissner-Nordström we need an infinite roll of stamps. This is depicted in Figure 9.

The most striking difference between the Reissner-Nordström and the Schwarzschild maximal analytical extensions is that for Reissner-Nordström, the curvature singularities at $r = 0$ are *timelike*, rather than spacelike. This means that an infalling timelike curve can in fact avoid the singularity, and come out into another asymptotic region. For example, in Figure 9 a particle (or observer) can start in region I, pass through regions II, VI and

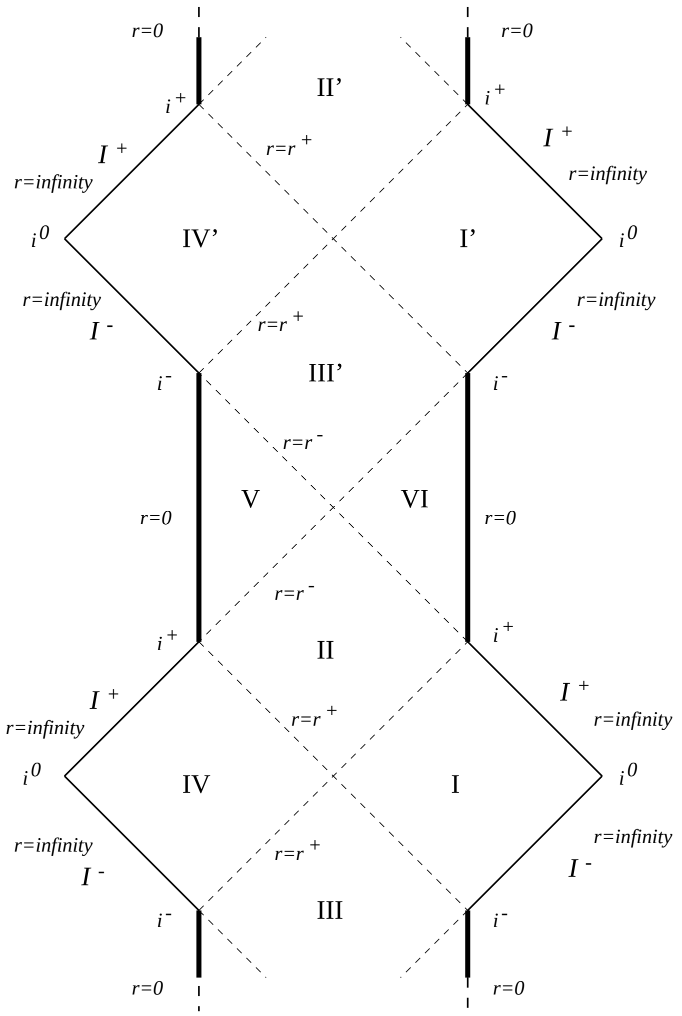

Figure 9: The maximal analytic extension of Reissner-Nordström.

III′, and come out into region I′. There is no possibility of returning, however, so if we inhabited region I we could never receive reports of what was happening in region I′. By the same token, however, it would be possible in principle for an observer to enter our region I from region II, having started out on the next "postage stamp" down on the roll. Such an observer would emerge from the outer horizon of the black hole. One should really view the $r = r_+$ boundary between regions II and I as the outer horizon of a white hole, in fact, since future-directed particles or null rays can only come out of it; they cannot cross inwards. Again, as in the Schwarzschild spacetime of the previous chapter, one should be cautious about taking the entire maximal analytic extension too seriously as a physical spacetime, since a realistic gravitational collapse will not give rise to the entire diagram.

The remaining case to consider is when $e^2 = m^2$. We see from (4.13) that the inner and outer horizons now coalesce, at $r = m$. The metric in this limit is known as the *Extremal Reissner-Nordström solution*, and in terms of the original coordinates it takes the form

$$ds^2 = -\Big(1 - \frac{m}{r}\Big)^2 dt^2 + \Big(1 - \frac{m}{r}\Big)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ . \tag{4.26}$$

This is singular at $r = m$, and so in the now familiar way, we change first to the appropriate ingoing Eddington-Finkelstein type coordinates (v, r), where $v = t + r^*$ and r^* is defined in (4.15). This turns the metric into the form

$$ds^2 = -\Big(1 - \frac{m}{r}\Big)^2 dv^2 + 2dv\, dr + r^2\, d\Omega^2\ , \tag{4.27}$$

where again we use the abbreviated notation $d\Omega^2$ for the metric on the unit 2-sphere. This is non-singular for all $r > 0$, including, in particular, the horizon at $r = m$. As usual, one can easily show that infalling timelike geodesics can reach and cross the horizon in a finite proper time.

The analysis of the maximal analytic extension proceeds in a similar fashion to the previous discussion for $e^2 < m^2$. Essentially all that changes is that region II and its copies II$'$, *etc.* all disappear, since r_- and r_+ are now both equal to m. Thus we arrive at the maximal analytic extension depicted in Figure 10. This spacetime with $e = m$ is known as the extremal Reissner-Nordström solution. Note that the points marked by a "p" on the left-hand vertical axis in Figure 10 are actually at $r = \infty$, and not at $r = 0$. This is again one of the penalties exacted upon those who presume to fit the universe onto a scrap of paper.

Note, incidentally, that the horizon at $r = m$, like all those that we have encountered, has the property of being a null surface. A null surface is defined as follows. Suppose we have a surface, or hypersurface, defined by $f(x) = 0$, where x represents the spacetime coordinates x^μ. It follows that the 1-form df, with components $\partial_\mu f$, will be perpendicular to the surface. If one now calculates the norm of this covector, namely $|df|^2 \equiv g^{\mu\nu}\, \partial_\mu f\, \partial_\nu f$, then the surface is defined to be null, timelike or spacelike according to whether this norm is zero, positive or negative. In all our cases the equation defining the event horizon is of the form $f(r) = 0$ (for example, in the present case of the extremal Reissner-Nordström metric, it is $f(r) \equiv r - m = 0$, and so we have $|df|^2 = |dr|^2 = g^{rr}$. It is easily seen, either in the original diagonal forms for the metrics, or in the Eddington-Finkelstein forms where the metric has off-diagonal components, that g^{rr} vanishes at the horizons. For example, in the present case we have $g^{rr} = (1 - m/r)^2$, demonstrating that the event horizon is a null surface.

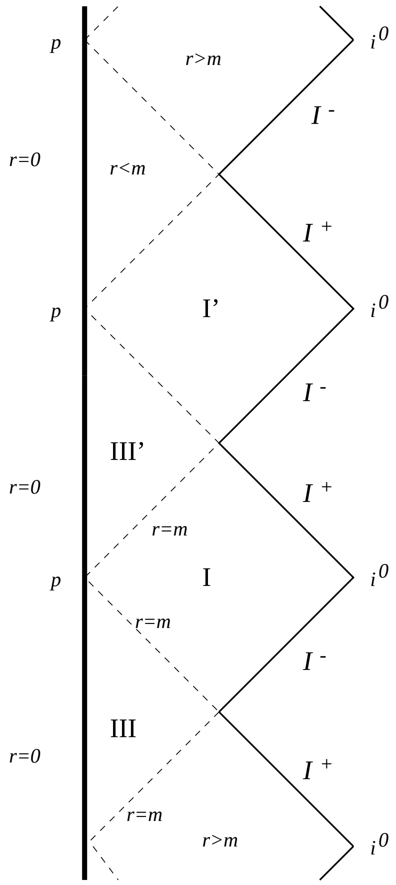

Figure 10: The maximal analytic extension of extremal Reissner-Nordström.

5 Rotating Black Holes

5.1 An interlude on Killing vectors

So far, the black-hole solutions that we have been considering have been spherically-symmetric, and *static*. The term static should now be made a little more precise. First of all, it means that everything (i.e. the metric, and any other fields, such as the Maxwell field in the case of charged solutions) should be independent of time. But it also means more than this; there should be no off-diagonal components in the metric tensor that mix between the asymptotic time coordinate t and any of the other coordinates. In other words, there should be no off-diagonal components of the form g_{tr}, $g_{t\theta}$ or $g_{t\varphi}$. We can actually express this a little more precisely and invariantly, if we first introduce another concept which will prove to be useful later, namely that of a *Killing vector*.

Suppose we have a covector $\omega = \omega_\mu \, dx^\mu$, and we make an infinitesimal coordinate transformation $x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu$; so $\delta x^\mu \equiv x'^\mu - x^\mu = \xi^\mu$ is infinitesimal. Being the infinitesimal difference between neighbouring coordinate values, ξ^μ is actually a vector. Then we may consider the effect of this coordinate transformation on ω_μ, :

$$\begin{aligned}
\delta\,\omega &= \delta\,\omega_\mu\,dx^\mu + \omega_\mu\,d\delta\,x^\mu \\
&= \xi^\nu\,\partial_\nu\,\omega_\mu\,dx^\mu + \omega_\mu\,d\xi^\mu \\
&= \xi^\nu\,\partial_\nu\,\omega_\mu\,dx^\mu + \omega_\mu\,\partial_\nu\,\xi^\mu\,dx^\nu \\
&\equiv \mathcal{L}_\xi\,\omega_\mu\,dx^\mu\ .
\end{aligned} \tag{5.1}$$

where in the last line we have defined the so-called Lie derivative of ω with respect to ξ. Thus we have

$$\mathcal{L}_\xi\,\omega_\mu = \xi^\nu\,\partial_\nu\,\omega_\mu + \omega_\nu\,\partial_\mu\,\xi^\nu\ . \tag{5.2}$$

This generalises in the obvious way to multi-index cotensors, with one term like the last one for each downstairs index:

$$\mathcal{L}_\xi\,T_{\mu_1\cdots\mu_n} = \xi^\nu\,\partial_\nu\,T_{\mu_1\cdots\mu_n} + T_{\nu\cdots\mu_n}\,\partial_{\mu_1}\,\xi^\nu + \cdots + T_{\mu_1\cdots\nu}\,\partial_{\mu_n}\,\xi^\nu\ . \tag{5.3}$$

Similarly, we may look at the infinitesimal transformation of tensors with upstairs indices, making use of the easily-derived result that $\delta\,\partial_\mu = -\partial_\mu\,\xi^\nu\,\partial_\nu$. Hence we extend the definition of the Lie derivative to vectors and tensors with upstairs indices:

$$\begin{aligned}
\mathcal{L}_\xi\,V^\mu &= \xi^\nu\,\partial_\nu\,V^\mu - V^\nu\,\partial_\nu\,\xi^\mu\ , \\
\mathcal{L}_\xi\,T^{\mu_1\cdots\mu_n} &= \xi^\nu\,\partial_\nu\,T^{\mu_1\cdots\mu_n} - T^{\nu\cdots\mu_n}\,\partial_\nu\,\xi^{\mu_1} - \cdots - T^{\mu_1\cdots\nu}\,\partial_\nu\,\xi^{\mu_n}\ .
\end{aligned} \tag{5.4}$$

The extension to tensors with upstairs and downstairs indices follows immediately from (5.3) and (5.4). Note that the Lie derivative takes a tensor into another tensor.

Having introduced the Lie derivative, let us consider applying it to the metric itself. From (5.3), it follows that we will have

$$\mathcal{L}_\xi\,g_{\mu\nu} = \xi^\rho\,\partial_\rho\,g_{\mu\nu} + g_{\rho\nu}\,\partial_\mu\,\xi^\rho + g_{\mu\rho}\,\partial_\nu\,\xi^\rho\ . \tag{5.5}$$

From the definitions (2.22) and (2.24), it is not hard to see that this can be re-expressed as

$$\mathcal{L}_\xi\,g_{\mu\nu} = \nabla_\mu\,\xi_\nu + \nabla_\nu\,\xi_\mu\ . \tag{5.6}$$

Now, it can sometimes happen that the metric ds^2 is actually left unaltered in form by the infinitesimal transformation ξ^μ, implying, from (5.6), that we will have

$$\nabla_\mu\,\xi_\nu + \nabla_\nu\,\xi_\mu\ = 0\ . \tag{5.7}$$

This equation is known as Killing's equation, and a vector ξ^μ that satisfies it is known as a Killing vector. If a Killing vector exists, it corresponds to the existence of a *symmetry* of the metric.

Consider the example of the Schwarzschild metric (3.2). It is evident that sending $t \longrightarrow t + c$ is a symmetry, which leaves the form of the metric unchanged, where c is any constant. Thus we may deduce that the vector K^μ with all components except vanishing except for $K^t = 1$ is a Killing vector, and indeed it is straightforward to see that it satisfies (5.7). (In fact it is much easier to look at (5.5), and to see that it has the property that $\mathcal{L}_K\, g_{\mu\nu} = 0$, since this calculation does not involve deriving the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$.) Thus we have the Killing vector

$$K = \frac{\partial}{\partial t} \tag{5.8}$$

in the Schwarzschild metric, corresponding to the time-translation invariance of the metric.

In fact the Schwarzschild metric has a total of four Killing vectors. The other three are associated with the rotational invariance of the metric, which, it will be recalled, is spherically symmetric. Thus the additional three Killing vectors correspond to the three-parameter rotation group $SO(3)$ that acts on the 2-spheres described by the coordinates (θ, φ). Two of the three are slightly complicated, but the third is very simple, namely

$$L = \frac{\partial}{\partial \varphi}\ . \tag{5.9}$$

This is in fact the generator L_3 of the three rotation-group generators L_i, and it corresponds to the azimuthal symmetry transformation $\varphi \longrightarrow \varphi + c$, where c is a constant. Again, it is manifest by inspection that this is a symmetry of the Schwarzschild metric. For completeness, let us give the other two rotational Killing vectors:

$$\begin{aligned}
L_1 &= \sin\varphi\, \frac{\partial}{\partial\theta} + \cot\theta\, \cos\varphi\, \frac{\partial}{\partial\varphi}\ ,\\
L_2 &= \cos\varphi\, \frac{\partial}{\partial\theta} - \cot\theta\, \sin\varphi\, \frac{\partial}{\partial\varphi}\ .
\end{aligned} \tag{5.10}$$

One easily checks that together with $L_3 = L$, the three Killing vectors L_i generate the $SO(3)$ algebra: $[L_i, L_j] = \epsilon_{ijk}\, L_k$. It is quite easy to show from (5.5) that L_1 and L_2 are indeed Killing vectors. One can also straightforwardly show from the definitions that in any spacetime, the commutator of *any* two Killing vectors must necessarily also be a Killing vector (possibly zero).

The timelike Killing vector K given in (5.8) clearly commutes with the three spatial rotation Killing vectors L_i. Thus we have a factorised 4-parameter symmetry group $\mathbb{R}\, \times$

$SO(3)$ for the Schwarzschild metric, where the $\mathbb{R}$ factor corresponds to time translations. Of couse when we say that the Killing vector K is timelike, this statement is actually true only when $r > 2m$. In fact the Killing vector K is easily seen to have magnitude-squared $|K|^2 \equiv g_{\mu\nu}\, K^\mu\, K^\nu = g_{tt}$, and so it is timelike when $r > 2m$, null when $r = 2m$, and spacelike when $r < 2m$.

The Reissner-Nordström metric has the same 4-parameter symmetry group. Again, we see that the Killing vector K has magnitude-squared $|K|^2$ given by g_{tt}. This means that in the $e^2 < m^2$ case it is timelike when $r > r_+$ and also when $0 < r < r_-$; it is null on both the horizons $r = r_-$ and $r = r_+$, and it is spacelike in the intermediate region $r_- < r < r_+$.

Having introduced the notion of a Killing vector, we can now give a somewhat more precise, if less transparent, definition of what is meant by a static spectime. A spactime is said to be static if its metric admits a timelike Killing vector field that is orthogonal to a family of spacelike surfaces. In particular we see that if the metric is such that $\partial/\partial t$ is a Killing vector, and if all off-diagonal components g_{ti} of the metric vanish, then this condition of staticity is satisfied.

5.2 The Kerr solution

Unlike the Schwarzschild and Reissner-Nordström solutions, the Kerr solution describes a *rotating* black hole, and consequently it is not spherically symmetric, since it has a preferred axis of rotation. In terms of the discussion of the previous subsection, it will still have the Killing vector $L = \partial/\partial\varphi$, corresponding to the azimuthal rotational symmetry, but it will no longer have the other two Killing vectors L_1 and L_2 given in (5.10). The Kerr metric is still time-independent, and it has $K = \partial/\partial t$ as a Killing vector, but it will no longer satisfy the conditions for being static. Rather, it is what is known as *stationary.* An asymptotically flat metric is said to be stationary if and only if it admits a Killing vector field that is timelike near infinity.

We shall not derive the Kerr solution here, since it is actually a rather involved computation. It was found by Kerr in 1963, by solving the vacuum Einstein equations for metrics that are stationary and azimuthally-symmetric. The resulting solution can be expressed in Boyer-Lindquist coordinates in the form

$$\begin{aligned} ds^2 &= -\frac{(\Delta - a^2\,\sin^2\theta)}{\rho^2}\,dt^2 - \frac{4m\,a\,r\,\sin^2\theta}{\rho^2}\,dt\,d\varphi \\ &\quad + \frac{\Big((r^2+a^2)^2 - \Delta\,a^2\,\sin^2\theta\Big)\,\sin^2\theta}{\rho^2}\,d\varphi^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2\,, \end{aligned} \tag{5.11}$$

where

$$\rho^2 \equiv r^2 + a^2 \cos^2\theta \,, \qquad \Delta \equiv r^2 - 2m\,r + a^2 \,. \tag{5.12}$$

One can also write the metric in a slightly different form, as

$$ds^2 = \rho^2 \Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + (r^2 + a^2)\, \sin^2\theta\, d\varphi^2 - dt^2 + \frac{2m\,r}{\rho^2}\,(a\, \sin^2\theta\, d\varphi - dt)^2 \,. \tag{5.13}$$

The Kerr metric has a curvature singularity at $\rho^2 = 0$, i.e. at

$$r = 0 \,, \qquad \theta = \frac{\pi}{2} \,. \tag{5.14}$$

This can be seen from the curvature invariant $|\mathrm{Riem}|^2$, which, after lengthy computations, can be seen to be given by

$$|\mathrm{Riem}|^2 = \frac{48m^2\,(r^2 - a^2\, \cos^2\theta)(\rho^4 - 16a^2\, r^2\, \cos^2\theta)}{\rho^{12}} \tag{5.15}$$

It is evident that when the parameter a is set to zero, the metric reduces precisely to the Schwarzschild solution. In general, it is invariant under the simultaneous replacements $t \longrightarrow -t$ and $\varphi \longrightarrow -\varphi$, which is what one would expect for an object that is rotating. In fact it can be shown, by comparing the asymptotic form of the metric (5.11) at large distance with the form of metrics in the Newtonian limit, that $J = a\,m$ is the angular momentum as measured at infinity. The procedure for showing this is analogous to the weak-field Newtonian analysis that we performed in scetion 3, for showing that the parameter m in the Schwarzschild solution has the interpretation of mass. In fact here too, one finds that m is the mass.

As with the Reissner-Nordström solution, here too we have three different cases to consider, depending on the nature of the roots of the function Δ. Thus if $a^2 < m^2$ there are two real roots, which coalesce if $a^2 = m^2$. If $a^2 > m^2$ the roots are complex, and so Δ is nonvanishing for all real r.

Let us consider the case $a^2 > m^2$ first. The metric (5.11) is singular only at $r = 0$. It is useful to introduce new coordinates $(x, y, z, \bar t)$, known as Kerr-Schild coordinates, and defined by

$$\begin{aligned} &x + \mathrm{i}\, y = (r + \mathrm{i}\, a)\, e^{\mathrm{i}\,\varphi_+}\, \sin\theta \,, \\ &z = r\, \cos\theta \,, \qquad \bar t = -r + \int \Big(dt + (r^2 + a^2)\, \Delta^{-1}\, dr\Big) \,, \end{aligned} \tag{5.16}$$

where

$$\varphi_+ = \int (d\varphi + a\, \Delta^{-1}\, dr) \,. \tag{5.17}$$

In terms of these coordinates, the metric (5.11) becomes

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 - d\bar{t}^2 \\ &\quad + \frac{2m\,r^2}{r^4 + a^2\,z^2}\Big(\frac{r(x\,dx + y\,dy) - a\,(x\,dy - y\,dx)}{r^2 + a^2} + \frac{z\,dz}{r} + d\bar{t}\Big)^2 \,, \end{aligned} \tag{5.18}$$

where r is now determined implicitly by the equation

$$r^4 - (x^2 + y^2 + z^2 - a^2)\,r^2 - a^2\,z^2 = 0 \,. \tag{5.19}$$

As an aside, we can immediately see from this expression that the metric is flat if $m = 0$, even if a is non-zero.

Equation (5.19) can be viewed as defining a surface in (x, y, z) space for each value of r. In fact it describes what are known as confocal ellipsoids. For our purposes, we just need to note that these degenerate at $r = 0$ to the disk

$$x^2 + y^2 \le a^2 \,, \qquad z = 0. \tag{5.20}$$

The boundary of the disk corresponds to $\theta = \pi/2$, and hence we see from (5.14) that the curvature singularity at $r = 0$ actually corresponds to the ring $x^2 + y^2 = a^2$, $z = 0$.

One can actually analytically continue the r coordinate from positive to negative values. This is done by introducing another plane, defined by coordinates (x', y', z'), with the top side of the disk $x^2+y^2 < a^2$, $z = 0$ identified with the bottom side of the disk $x'^2+y'^2 < a^2$, $z' = 0$, and *vice versa*. After extending the metric (5.18) according to this scheme, one finds from (5.16) that the metric in the region covered by (x', y', z') is again of the form (5.11), but with r now negative. It is evident therefore that at large negative values of r the metric is again asymptotically flat, but now with a negative mass $-m$.

The magnitude-squared of the azimuthal-symmetry Killing vector $L = \partial/\partial\varphi$ is given by $|L|^2 = g_{\varphi\varphi}$, and so from (5.11) we see that when $\theta = \pi/2$, we have

$$|L|^2 = r^2 + a^2 + \frac{2m\,a^2}{r} \,. \tag{5.21}$$

Thus at points with small negative r, near to the ring singularity at $r = 0$, $\theta = \pi/2$, we see that L, which is normally spacelike, becomes timelike. But since φ is a periodic angular coordinate, this means that the circles (t, r, θ) =constant become timelike in this region. This means that there are closed timelike curves in the spacetime. In fact, since there are no horizons when $a^2 > m^2$, these curves can be deformed to pass through any point in the spacetime. This represents a violation of causality in the spacetime, opening up all the usual science-fiction possibilities of attempting to bring about a paradox by killing one's own grandmother, *etc.* In fact it is more than just grandmothers who are at risk in this spacetime, since the absence of horizons implies that the ring singularity is visible from the asymptotic region near $r = \infty$. This would be a naked singularity, implying a breakdown of all predictability, and fortunately the formation of this $a^2 > m^2$ spacetime is ruled out by the Cosmic Censor.

Let us turn now to the case when $a^2 < m^2$. The metric (5.11) is now singular not only at $r = 0$, but also at the two roots $r = r_\pm$ of the function Δ:

$$r_+ = m + \sqrt{m^2 - a^2}\ , \qquad r_- = m - \sqrt{m^2 - a^2}\ . \tag{5.22}$$

These surfaces are analogous to the $r_\pm$ surfaces in the Reissner-Nordström solution. To extend the metric across these surfaces, we introduce the Kerr coordinates $(r, \theta, \varphi_+, u_+)$, where

$$u_+ = t + \int^r \frac{(r'^2 + a^2)\, dr'}{\Delta(r')}\ , \qquad \varphi_+ = \varphi + \int^r \frac{a\, dr'}{\Delta(r')}\ , \tag{5.23}$$

where $\Delta(r')$ means $r'^2 - 2m\, r' + a^2$. In terms of these coordinates, the metric (5.11) takes the form

$$\begin{aligned} ds^2 &= \rho^2\, d\theta^2 - 2a\, \sin^2\theta\, dr\, d\varphi_+ + 2dr\, du_+ - \Big(1 - \frac{2m\, r}{\rho^2}\Big)\, du_+^2 \\ &\quad + \frac{(r^2 + a^2)^2 - \Delta\, a^2\, \sin^2\theta}{\rho^2}\, \sin^2\theta\, d\varphi_+^2 - \frac{4a\, m\, r}{\rho^2}\, \sin^2\theta\, d\varphi_+\, du_+\ . \end{aligned} \tag{5.24}$$

It is easily seen that this is regular at $r = r_+$ and $r = r_-$. It again has a singularity at $r = 0$, and a similar analysis to the one we carried out for $a^2 > m^2$ shows that here too, the curvature singularity actually lies on a ring at $r = 0$.

The extended spacetime described by the metric (5.24) covers all values of r in the range $-\infty < r < \infty$, with the exception of $r = 0$. Another extension is provided by introducing instead the coordinates $(r, \theta, \varphi_-, u_-)$, where

$$u_- = t - \int^r \frac{(r'^2 + a^2)\, dr'}{\Delta(r')}\ , \qquad \varphi_- = \varphi - \int^r \frac{a\, dr'}{\Delta(r')}\ . \tag{5.25}$$

This gives a metric simiar to (5.24), but with the replacements $(\varphi_+, u_+) \longrightarrow (-\varphi_-, -u_-)$. This again covers $-\infty < r < \infty$, but it is inequivalent to the analytic extension described by (5.24). The situation is analogous to the two extensions $(\mathcal{M}', \mathbf{g}')$ and $(\mathcal{M}'', \mathbf{g}'')$ in Schwarzschild, corresponding to the use of advanced and retarded Eddington-Finkelstein coordinates respectively. We can again patch together the entire maximal analytic extension, by studying carefully what overlap there is between the regions covered in the two charts. We find a situation similar to that for the Reissner-Nordström solution, with an infinite chain of regions patched together along the time direction. This is depicted in Figure 11.

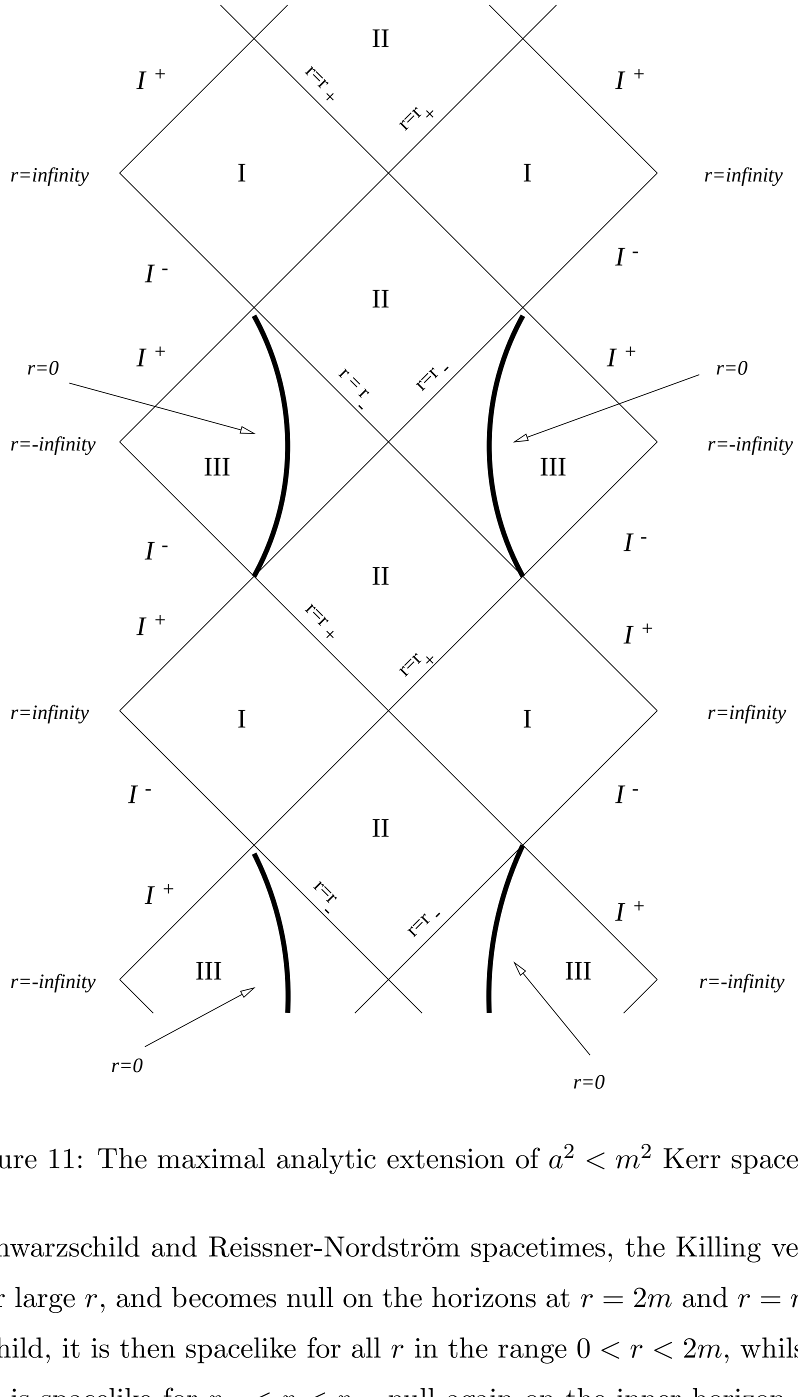

Figure 11: The maximal analytic extension of $a^2 < m^2$ Kerr spacetime.

In the Schwarzschild and Reissner-Nordström spacetimes, the Killing vector $K = \partial/\partial t$ is timelike for large r, and becomes null on the horizons at $r = 2m$ and $r = r_+$ respectively. In Schwarzschild, it is then spacelike for all r in the range $0 < r < 2m$, whilst for Reissner-Nordström it is spacelike for $r_- < r < r_+$, null again on the inner horizon at $r = r_-$, and timelike for $0 < r < r_-$. One might think from these examples that the Killing vector that is timelike at infinity is always null on horizons. However, this is not the case in the Kerr spacetime, as we shall now discuss.

Clearly the magnitude squared of $K = \partial/\partial t = K^\mu \partial_\mu$ is given by g_{tt}, since K^μ has just the single non-vanishing component $K^t = 1$. Thus in the Kerr spacetime we see from (5.11) that

$$|K|^2 = g_{tt} = -(r^2 - 2m\,r + a^2\,\cos^2\theta)\ . \tag{5.26}$$

This means that the Killing vector K becomes null on the surfaces

$$r = m \pm \sqrt{m^2 - a^2\,\cos^2\theta}\ . \tag{5.27}$$

In particular, this means that there is a region between the outer horizon at $r = r_+$ and the so-called *stationary limit surface* at $r = m + \sqrt{m^2 - a^2\,\cos^2\theta}$, within which K is spacelike. This region outside the horizon where K is spacelike is called the *ergosphere*; this is depicted in Figure 12. It is that part of the inside of an oblate ellipsoid that lies outside a sphere which touches the ellipsoid at the north and south poles.

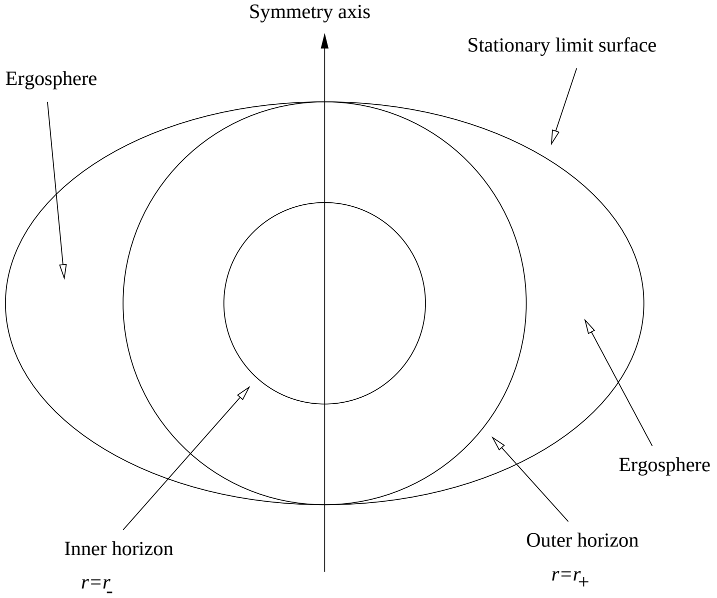

Figure 12: The Ergosphere.

The stationary limit surface represents the inner boundary of the region where particles travelling on timelike curves can travel on an orbit of the Killing vector $K = \partial/\partial t$. In other words, for all points outside the stationary limit surface, it is possible for a particle to follow a timelike path with r, θ and φ held fixed; that is to say, it is at rest as seen from infinity.

The stationary limit surface is a timlike surface at all points except these two points of contact with the outer horizon. (Of course the outer horizon itself is a null surface everywhere.) This can be seen by calculating the magnitude squared of the differential of the function $f = r^2 - 2m\,r + a^2\,\cos^2\theta$ whose vanishing defines the stationary limit surface:

$$\begin{aligned} df &= 2(r-m)\,dr - 2a^2\,\sin\theta\,\cos\theta\,d\theta = f_r\,dr + f_\theta\,d\theta\ ; \\ |df|^2 &= g^{rr}\,(f_r)^2 + g^{\theta\theta}\,f_\theta^2 = \frac{4\Delta\,(r-m)^2 + a^4\,\sin^2 2\theta}{\rho^2}\ . \end{aligned} \tag{5.28}$$

(Since $g^{r\theta} = 0$.) Hence, on the stationary limit surface defined by $f = 0$, we have

$$|df|^2 = \frac{4a^2\,m^2\,\sin^2\theta}{\rho^2}\ . \tag{5.29}$$

This shows that the surface is timelike for all θ except for $\theta = 0$ and $\theta = \pi$, where it becomes null.

Since the stationary limit surface is generically timelike, this means that it can be crossed in both directions by particles following timelike paths. The only exception is at the north and south poles, where it coincides with the outer event horizon. The situation can be clarified with the aid of a diagram. Owing to the limitations of the 2-dimensional page, we shall just take a section through the equatorial plane, $\theta = \frac{1}{2}\pi$. Figure 13 depicts this, with r plotted radially, and φ as the angular coordinate. Black dots represent events where flashes of light were emitted, and the circles show the light-fronts a short while later.

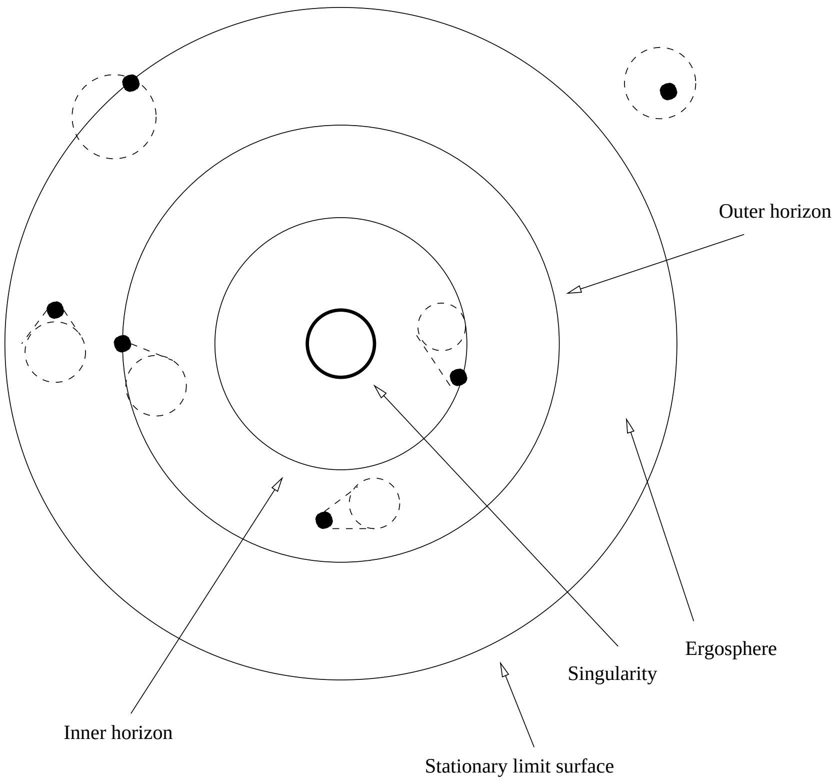

Figure 13: Light-cones in equatorial plane.

On the stationary limit surface, the point of emission lies on the light-front, since K is null there. The light-front itself lies partially inside and partially outside the stationary limit surface, since it is a timelike surface. Inside the ergosphere, the emission point lies outside the lightfront, since K is spacelike. At $r = r_+$ K is still spacelike, and so the emission point still lies outside the wave-front. But now, for emssion points on $r = r_+$, the wave-fronts lie entirely within the horizon, since $r = r_+$ is a null surface.

The third case to consider is the extremal situation when $a^2 = m^2$, implying that the two roots $r_\pm = m \pm \sqrt{m^2 - a^2}$ become coincident, at $r = m$. The story is similar to that of the extremal Reissner-Nordström solution with $e^2 = m^2$, and looking at Figure 11 we can see that the effect of setting $a^2 = m^2$ is to squeeze down the Penrose diagram so that all the regions II are removed. The result is presented in Figure 14.

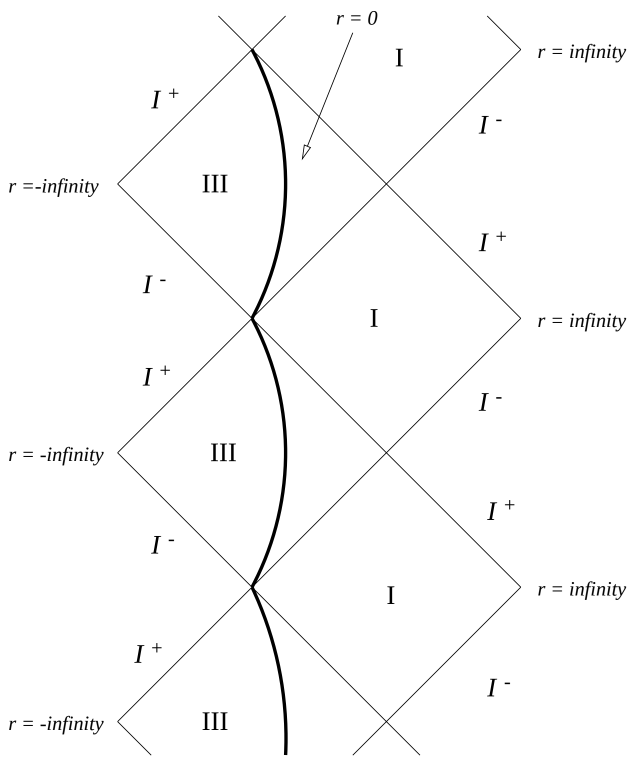

Figure 14: The maximal analytic extension of $a^2 = m^2$ Kerr spacetime.

5.3 Killing horizons and surface gravity

5.3.1 Killing horizons

There are several useful concepts that it now becomes appropriate to introduce. First, is the notion of a *Killing Horizon*,[1] which is defined as follows. A null hypersurface $\mathcal{N}$ is a Killing Horizon of a Killing vector field ξ if, on $\mathcal{N}$, ξ is normal to $\mathcal{N}$.

Now, if a hypersurface is defined by $f = 0$, then vector fields normal to this surface are of the form

$$\ell^\mu = h\, g^{\mu\nu}\, \partial_\nu f \ , \tag{5.30}$$

where h is any non-vanishing function. Consequently, the hypersurface is a Killing horizon of a Killing vector ξ if, firstly, $\ell^\mu\, \ell_\mu = 0$ (i.e. it is null), and secondly $\xi^\mu = \psi\, \ell^\mu$ for some non-vanishing function $\psi(x)$.

A further observation is that the ℓ^μ is not only normal to the null surface $\mathcal{N}$, but it is also *tangent* to $\mathcal{N}$. This follows from the fact that, by definition, any vector t^μ tangent to a surface is orthogonal to the normal vector ℓ^μ, i.e. $t^\mu\, \ell_\mu = 0$. But since ℓ^μ is null here, it follows that it itself satisfies the condition for being a tangent vector. This means that there must exist some curve $x^\mu = x^\mu(\lambda)$ in $\mathcal{N}$ such that

$$\ell^\mu = \frac{dx^\mu}{d\lambda} \ , \tag{5.31}$$

where λ parameterises the curve.

The curves $x^\mu(\lambda)$ are in fact geodesics. To see this, recall that $\ell^\mu = dx^\mu/d\lambda$ is given by (5.30), and now calculate $\ell^\rho\, \nabla_\rho\, \ell^\mu$:

$$\begin{aligned}
\ell^\rho\, \nabla_\rho\, \ell^\mu &= (\ell^\rho\, \partial_\rho h)\, g^{\mu\nu}\, \partial_\mu f + h\, g^{\mu\nu}\, \ell^\rho\, \nabla_\rho\, \partial_\nu\, f \ , \\
&= (\ell^\rho\, \partial_\rho\, \log h)\, \ell^\mu + h\, g^{\mu\nu}\, \ell^\rho\, (\nabla_\nu\, \partial_\rho\, f) \ , \\
&= \ell^\mu\, \frac{d\log h}{d\lambda} + h\, \ell^\rho\, \nabla^\mu (h^{-1}\, \ell_\rho) \ , \\
&= \ell^\mu\, \frac{d\log h}{d\lambda} + \ell^\rho\, \nabla^\mu\, \ell_\rho - \ell^2\, (\partial^\mu\, \log h) \ , \\
&= \ell^\mu\, \frac{d\log h}{d\lambda} + \tfrac{1}{2}\partial^\mu\, (\ell^2) - \ell^2\, (\partial^\mu\, \log h) \ .
\end{aligned} \tag{5.32}$$

(The indices ρ and ν in the second term of the second line could be interchanged on account of the fact that second covariant derivatives commute on scalar fields.) Now, we know that

[1] This does not, of course, allude to any lethal properties of the event horizon! Indeed, as we have seen, one can pass through the event horizon of a black hole quite unscathed, and, while one might view the unfolding developments with a measure of concern, it is only later that one's situation becomes precarious.

ℓ^μ is null on $\mathcal{N}$, so $\ell^2 = 0$ there. This does not mean that $\partial^\mu(\ell^2)$ vanishes on $\mathcal{N}$, but the fact that $\ell^2 = 0$, which is constant, on $\mathcal{N}$ does mean that $t^\mu\, \partial_\mu(\ell^2) = 0$ for any vector t^μ tangent to $\mathcal{N}$. In view of the previous discussion, this means that $\partial_\mu(\ell^2)$ must be proportional to ℓ_μ on $\mathcal{N}$, and hence we have that

$$\ell^\rho\, \nabla_\rho\, \ell^\mu\Big|_{\mathcal{N}} = \alpha\, \ell^\mu \tag{5.33}$$

for some function α. Recalling that the function h in (5.30) is still at our disposal, we see that by choosing it appropriately, we can make α vanish. This would imply that $x^\mu(\lambda)$ on $\mathcal{N}$ satisfies the geodesic equation

$$\ell^\rho\, \nabla_\rho\, \ell^\mu = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\,\rho}\, \frac{dx^\nu}{d\lambda}\, \frac{dx^\rho}{d\lambda} = 0 \tag{5.34}$$

on $\mathcal{N}$, with λ being an affine parameter. (The more general equation (5.33) is still the geodesic equation, but with the parameter λ not an *affine* parameter.) One can define the null geodesics $x^\mu(\lambda)$ with affine parameter λ, for which the tangent vectors $\ell^\mu = dx^\mu/d\lambda$ are normal to the null surface $\mathcal{N}$, to be the *generators* of $\mathcal{N}$.

Let us now look at the event horizons in the Kerr spacetime. It is convenient to use the metric given in (5.24), since this is valid on the two horizons. It is not difficult to calculate the components of the inverse metric in this coordinate system, and in particular, that

$$g^{r\,u_+} = \frac{r^2 + a^2}{\rho^2}\,, \qquad g^{rr} = \frac{\Delta}{\rho^2}\,, \qquad g^{r\theta} = 0\,, \qquad g^{r\,\varphi_+} = \frac{a}{\rho^2}\,. \tag{5.35}$$

Thus on the horizons, where $\Delta = 0$, we have that the normal vectors are proprotional to

$$\frac{r_\pm^2 + a^2}{\rho^2}\, \frac{\partial}{\partial u_+} + \frac{a}{\rho^2}\, \frac{\partial}{\partial \varphi_+}\,. \tag{5.36}$$

Now it is easily seen from (5.25) that $\partial/\partial v = \partial/\partial t$ and $\partial/\partial\varphi_+ = \partial/\partial\varphi$, and hence we deduce that the event horizons $r = r_\pm$ are Killing horizons of

$$\xi = K + \frac{a}{r_\pm^2 + a^2}\, L = \frac{\partial}{\partial t} + \frac{a}{r_\pm^2 + a^2}\, \frac{\partial}{\partial\varphi}\,. \tag{5.37}$$

In particular, we have that the outer horizon is a Killing horizon of the Killing vector

$$\xi = \frac{\partial}{\partial t} + \Omega_H\, \frac{\partial}{\partial\varphi} \tag{5.38}$$

where

$$\Omega_H \equiv \frac{a}{r_+^2 + a^2} = \frac{a}{2m\, r_+}\,. \tag{5.39}$$

In fact Ω_H can be interpreted as the angular velocity of the horizon. To see this, we first note that the angular velocity of a particle, as measured with respect to infinity, is given by

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{U^\varphi}{U^t} \,, \tag{5.40}$$

where $U^\mu = dx^\mu/d\tau$ is its 4-velocity. Thus we have

$$U = U^\mu\, \partial_\mu = U^t\, \frac{\partial}{\partial t} + U^\varphi\, \frac{\partial}{\partial \varphi} = U^t\,\Big(\frac{\partial}{\partial t} + \Omega\, \frac{\partial}{\partial \varphi}\Big) \,. \tag{5.41}$$

This means that particles moving on the orbits of ξ rotate with angular velocity Ω_H with respect to static particles that move on the orbits of K. In other words, particles moving on the orbits of ξ have angular velocity Ω_H relative to the stationary frame at infinity. Finally, since we showed that the event horizon is a Killing horizon of ξ, it follows that the null geodesic generators of the horizon themselves follow orbits of ξ. Thus the outer horizon of the Kerr black hole is rotating with angular velocity Ω_H with respect to the stationary frame at infinity.

5.3.2 Surface gravity

We saw in the previous discussion that if $\mathcal{N}$ is a Killing horizon of the vector field ξ, then if ℓ^μ is a normal vector to $\mathcal{N}$ in the affine parametrisation, implying $\ell^\nu\, \nabla_\nu\, \ell^\mu = 0$, then there exists a function ψ such that $\xi^\mu = \psi\, \ell^\mu$. It then follows that on $\mathcal{N}$ we shall have

$$\xi^\nu\, \nabla_\nu\, \xi^\mu = \kappa\, \xi^\mu \,, \tag{5.42}$$

where

$$\kappa = \xi^\nu\, \partial_\nu\, \log|\psi| \,. \tag{5.43}$$

The surface gravity κ may be expressed in a variety of different ways, which can be derived from (5.42). First, observe that if we view ξ as the 1-form $\xi = \xi_\mu\, dx^\mu$, then the fact that ξ is normal to $\mathcal{N}$ means that $\xi\wedge d\xi|_{\mathcal{N}} = 0$. This is known as Frobenius' theorem. This can be rewritten in terms of indices as

$$\xi_{[\mu}\, \partial_\nu\, \xi_{\rho]}\Big|_{\mathcal{N}} = 0 \,. \tag{5.44}$$

Now since ξ is a Killing vector, it follows from (5.7) that

$$\nabla_\mu\, \xi_\nu = \nabla_{[\mu}\, \xi_{\nu]} = \partial_{[\mu}\, \xi_{\nu]} \,, \tag{5.45}$$

and hence (5.44) can be rewritten as

$$\xi_\rho\, \nabla_\mu\, \xi_\nu = \xi_\nu\, \nabla_\mu\, \xi_\rho - \xi_\mu\, \nabla_\nu\, \xi_\rho \,. \tag{5.46}$$

Multiplying by $\nabla^\mu \xi^\nu$, we obtain

$$\begin{aligned}
\xi_\rho\,(\nabla^\mu \xi^\nu)\,(\nabla_\mu \xi_\nu)\Big|_{\mathcal{N}} &= -2(\xi_\mu\,\nabla^\mu \xi^\nu)\,(\nabla_\nu\,\xi\rho)\Big|_{\mathcal{N}}\,,\\
&= -2\,\kappa\,(\xi^\nu\,\nabla_\nu\,\xi\rho)\Big|_{\mathcal{N}}\,,\\
&= -2\kappa^2\,\xi_\rho\Big|_{\mathcal{N}}\,, \qquad (5.47)
\end{aligned}$$

where we have twice made use of the equation (5.42). Thus aside from singular points on $\mathcal{N}$ where ξ_ρ vanishes, we have

$$\kappa^2 = -\tfrac{1}{2}(\nabla^\mu \xi^\nu)\,(\nabla_\mu \xi_\nu)\Big|_{\mathcal{N}}\,. \qquad (5.48)$$

In fact points where ξ_ρ vanishes are arbitrarily close to points where it is non-zero, so by continuity the expression (5.48) for κ is valid everywhere on $\mathcal{N}$.

A simpler expression for κ can be given, which does not require the use of the covariant derivative, using (5.45). Thus we obtain

$$\kappa^2 = -\tfrac{1}{2}g^{\mu\rho}\,g^{\nu\sigma}\,(\partial_{[\mu}\,\xi_{\nu]})\,(\partial_{[\rho}\,\xi_{\sigma]})\Big|_{\mathcal{N}}\,. \qquad (5.49)$$

Another way of expressing κ, which can sometimes be useful for calculational purposes, is obtained by first noting that any Killing vector satisfies the identity

$$\nabla_\mu\,\nabla_\nu\,\xi_\rho = R^\sigma{}_{\mu\nu\rho}\,\xi_\sigma\;. \qquad (5.50)$$

This can be proven using the general result, following from the definitions in section 2, that

$$[\nabla_\mu, \nabla_\nu]\,V^\rho = R^\rho{}_{\sigma\mu\nu}\,V^\sigma\;. \qquad (5.51)$$

By taking a covariant derivative of (5.7), and using this result and the cyclic identity for the Riemann tensor, (5.50) follows. By taking a contraction of (5.50), we obtain the identity

$$\Box\,\xi_\mu = -R_{\mu\nu}\,\xi^\nu\;, \qquad (5.52)$$

valid for any Killing vector, where $\Box = \nabla^\mu\,\nabla_\mu$ is the covariant Laplacian. Now, noting that $\Box\xi^2 = 2(\nabla^\mu\,\xi^\nu)\,(\nabla_\mu\,\xi_\nu) + 2\xi^\mu\,\Box\,\xi_\mu$, where ξ^2 means $\xi^\mu\,\xi_\mu$, we obtain the result that

$$\kappa^2 = -\tfrac{1}{4}(\Box\,\xi^2 + 2R_{\mu\nu}\,\xi^\mu\,\xi^\nu)\Big|_{\mathcal{N}}\,. \qquad (5.53)$$

This formula is especially easy to use in cases where the spacetime is Ricci flat, such as in the Kerr metric. Note also that the Laplacian of a scalar field ϕ is expressible in the simple form

$$\Box\,\phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\Big(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\,\phi\Big)\;, \qquad (5.54)$$

allowing the first term in (5.53) to be evaluated without the need for explicit computation of the Christoffel connection.

Finally, we can obtain the following even simpler expression for κ:

$$\kappa^2 = (\partial^\mu \lambda)\,(\partial_\mu \lambda)\Big|_{\mathcal{N}}\,, \tag{5.55}$$

where $\lambda^2 \equiv -|\xi|^2 = -\xi^\mu\,\xi_\mu$. The proof is surprisingly tricky. We can rewrite the Frobenius condition (5.44) that holds on the horizon as $\xi_{[\mu}\,\nabla_\nu\,\xi_{\rho]} = 0$. By simply writing out the three terms in this, it is then easy to see that we can write the square of the Frobenius condition as

$$3(\xi^{[\mu}\,\nabla^\nu\,\xi^{\rho]})(\xi_{[\mu}\,\nabla_\nu\,\xi_{\rho]}) = \xi^\mu\,\xi_\mu\,(\nabla^\nu\,\xi^\rho)(\nabla_\nu\,\xi_\rho) - 2(\xi^\mu\,\nabla^\nu\,\xi^\rho)(\xi_\nu\,\nabla_\mu\,\xi_\rho)\,, \tag{5.56}$$

which is valid everywhere. Now since $\xi_{[\mu}\,\nabla_\nu\,\xi_{\rho]}$ vanishes on the horizon, it follows that the gradient of the left-hand side of (5.56) vanishes on the horizon. On the other hand, we know from (5.42) that the gradient of $|\xi|^2$ does not vanish on the horizon, provided that κ is non-zero. This means that by l'Hospital's rule, it must be that we can divide (5.56) by $|\xi|^2$ and then take the limit as we approach the horizon, and the left-hand side will still vanish. Thus we are able to deduce that in the limit of approaching the horizon, we have

$$(\nabla^\nu\,\xi^\rho)(\nabla_\nu\,\xi_\rho) = \frac{2(\xi^\mu\,\nabla^\nu\,\xi^\rho)(\xi_\nu\,\nabla_\mu\,\xi_\rho)}{|\xi|^2}\,. \tag{5.57}$$

Having successfully negotiated this tricky step, the rest is plain sailing. The right-hand side in (5.57) can be immediately rewritten as

$$\frac{\partial^\rho(\xi^\nu\,\xi_\nu)\,\partial_\rho(\xi^\mu\,\xi_\mu)}{2|\xi|^2}\,, \tag{5.58}$$

which is nothing but $-\frac{1}{2}\partial^\rho\lambda\,\partial_\rho\lambda$. From (5.48), the result (5.55) now immediately follows.

Note that from its definition so far, the normalisation for κ is undetermined, since it scales under constant scalings of the Killing vector ξ. Once cannot normalise ξ at the horizon, since $\xi^2 = 0$ there, but its normalisation can be specified in terms of the behaviour of ξ at infinity. It is conventional to define κ in terms of the Killing vector combination of the form $K + \Omega_H\,L$, such as arises in (5.38) in the Kerr solution. More generally, this can be stated as follows. There is a unique Killing vector (up to scale) that is timelike at arbitrarily large distances in the asymptotically flat regions. (In our coordinates, this is precisely $K = \partial/\partial t$.) This vector, which we shall denote generically by K, may be normalised canonically by requiring that it have magnitude-squared equal to -1 at infinity. Then the Killing vector ξ of the Killing horizon is defined to be $\xi = K + \cdots$, where the ellipses denote whatever additional spacelike Killing vectors appear in the calculated expression for ξ.

Let us now examine why the quantity κ is called the *surface gravity.* It has the interpretation of being the acceleration of a static particle near the horizon, as measured at spatial infinity. One can see this as follows. Let us consider a particle near the horizon, moving on an orbit of ξ^μ; this means that its 4-velocity $u^\mu = dx^\mu/d\tau$ is proportional to ξ^μ. Since the 4-velocity must satisfy $u^\mu\, u_\mu = -1$, this means that we must have

$$u^\mu = \lambda^{-1}\, \xi^\mu \ , \tag{5.59}$$

where, as above, we have defined the function λ by $\lambda^2 = -\xi^\mu\, \xi_\mu$. Now, the 4-acceleration of the particle is given by

$$a^\mu = \frac{Du^\mu}{D\tau} \equiv \frac{dx^\nu}{d\tau}\, \nabla_\nu\, u^\mu = u^\nu\, \nabla_\nu\, u^\mu \ . \tag{5.60}$$

Using (5.59), we see that this gives

$$\begin{aligned} a^\mu &= \lambda^{-2}\, \xi^\nu\, \nabla_\nu\, \xi^\mu - \lambda^{-3}\, \xi^\mu\, \xi^\nu\, \nabla_\nu\, \lambda \\ &= -\lambda^{-2}\, \xi^\nu\, \nabla^\mu\, \xi_\nu - \tfrac{1}{2}\lambda^{-4}\, \xi^\mu\, \xi^\nu\, \nabla_\nu(\xi^\rho\, \xi_\rho) \\ &= -\tfrac{1}{2}\lambda^{-2}\, \partial^\mu\, (\xi^\nu\, \xi_\nu) - \lambda^{-4}\, \xi^\mu\, \xi^\nu\, \xi^\rho\, \nabla_\nu\, \xi_\rho \\ &= \lambda^{-1}\, \partial^\mu\, \lambda \ . \end{aligned} \tag{5.61}$$

In the steps above, we have used the fact that $\nabla_\mu\, \xi_\nu$ is antisymmetric in μ and ν, since ξ is a Killing vector. The upshot from this is that the magnitude of the 4-acceleration is given by

$$|a| = \sqrt{g^{\mu\nu}\, a_\mu\, a_\nu} = \lambda^{-1}\, \sqrt{g^{\mu\nu}\, \partial_\mu\lambda\, \partial_\nu\lambda} \ . \tag{5.62}$$

As the particle approaches the horizon, the factor $\sqrt{g^{\mu\nu}\, \partial_\mu\lambda\, \partial_\nu\lambda}$ becomes equal to the surface gravity (see (5.55)), but the prefactor λ^{-1} diverges, owing to the fact that ξ becomes null on the horizon. Thus the *proper acceleration* of a particle on an orbit of ξ diverges on the horizon (which is why the particle is inevitably drawn through the horizon). However, suppose we measure the acceleration as seen by a static observer at infinity. For such an observer, there will be a scaling factor relating the proper time τ of the particle to the time t at infinity, since $d\tau^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu$. If the black hole were non-rotating, so that ξ were simply $\partial/\partial t$, we would have $d\tau^2 = -g_{00}\, dt^2$, which could be written nicely as $d\tau^2 = -\xi^\mu\, \xi^\nu\, g_{\mu\nu}\, dt^2$. Clearly this latter expression, being generally covariant, will always be valid, even in the rotating case, and so we will have $d\tau = \lambda\, dt$. Thus the acceleration of the particle, as measured by the static observer at infinity, will be $|a|\, d\tau/dt$, which is $\lambda\, |a|$. Consequently, the acceleration of a particle near to the horizon that is on an orbit of ξ, as measured by a static observer at infinity, will be equal to κ. This explains why κ is called the surface gravity.

It is straightforward, using (5.55), to calculate the surface gravity for the Kerr metric. After some algebra, one finds that the result for the surface gravity on the outer horizon $r = r_+$ is expressible as

$$\kappa = \frac{\sqrt{m^2 - a^2}}{2m\, r_+}\,. \tag{5.63}$$

Note that if we specialise to the case of the Schwarzschild solution, by setting $a = 0$, we get $\kappa = 1/(4m)$. This can indeed be seen, in a Newtonian limit, to be in accordance with one's expectations for the strength of the gravitational field on the horizon; one would expect a Newtonian gravitational acceleration $a = m/r^2$ at a distance r from a point mass m, and so on the horizon $r = 2m$, this gives $a = 1/(4m)$.

5.3.3 Non-degenerate Killing Horizons

It is instructive to look at the surface gravity in a little more detail. Firstly, we may observe that κ is constant on the orbits of ξ. In other words, $\xi^\mu\, \partial_\mu \kappa^2 =$. To see this, consider first the derivative of κ with respect to any direction t^μ tangential to $\mathcal{N}$. Using (5.48 and (5.50), we have

$$t^\rho\, \partial_\rho\, \kappa^2 = -(\nabla^\mu\, \xi^\nu)\, t^\rho\, \nabla_\rho\, \nabla_\mu\, \xi_\nu\Big|_{\mathcal{N}} = -(\nabla^\mu\, \xi^\nu)\, t^\rho\, R_{\sigma\rho\mu\nu}\, \xi^\sigma\Big|_{\mathcal{N}}\,. \tag{5.64}$$

Note that we are allowed to take the derivative despite the restriction that (5.48) is defined on the surface $\mathcal{N}$ precisely because the direction of the derivative $t^\rho\, \partial_\rho$ is tangential to $\mathcal{N}$. Now, as we have observed, ξ^μ itself is tangential to $\mathcal{N}$, despite also being normal to $\mathcal{N}$, owing to the fact that it is null on $\mathcal{N}$. Thus we may apply (5.64) to the case $t^\mu = \xi^\mu$, whereupon we immediately obtain $\xi^\mu\, \partial_\mu\, \kappa^2 = 0$, by virtue of the antisymmetry of $R_{\sigma\rho\mu\nu}$ in its first two indices.

A non-degenerate Killing horizon is defined to be one on which the surface gravity κ is non-zero. In such circumstances a particular orbit of ξ, on which, as we have seen, κ is equal to some (non-zero) constant, will coincide with only part of the null generator of $\mathcal{N}$. We can see this, and what it implies, in the following way. We can choose a parameter y along the orbit of ξ in such a way that $\xi = \partial/\partial y$. Since the orbit of ξ lies in $\mathcal{N}$, this means that we can think of y as one of the coordinates lying in the surface $\mathcal{N}$. Now, we introduced previously the normal $\ell^\mu = dx^\mu/d\lambda$ to $\mathcal{N}$, where λ is chosen to be an affine parameter so that (5.34) holds. It follows that on the orbit of ξ we have

$$\xi = \frac{\partial}{\partial y} = \frac{d\lambda}{dy}\frac{\partial}{\partial\lambda} = \psi\,\ell\;, \tag{5.65}$$

where $\psi = \frac{d\lambda}{dy}$ satisfies

$$\frac{d\log|\psi|}{dy} = \kappa\;. \tag{5.66}$$

(See equation (5.42), and the equation in the line below it.) Now, we have already established that κ is constant on an orbit of ξ, and so this means that (5.66) can be integrated to give $\psi = \pm\kappa\, e^{\kappa\, y}$. (We have, without losing generality, made a convenient choice of integration constant.) One further integration then gives

$$\lambda = \pm e^{\kappa\, y}\;, \tag{5.67}$$

where again we have, w.o.l.o.g., made a convenient choice of integration constant.

We see from (5.67) that for a specific choice of integration constant, the full range of the coordinate y on the orbit of ξ, namely $-\infty < y < \infty$, covers only half of the full range of the affine parameter λ, namely either $0 < \lambda < \infty$ or $-\infty < \lambda < 0$, depending on the sign choice made in (5.67). There is a bifurcation point at $\lambda = 0$, which is a fixed point of the orbit of ξ (meaning that ξ vanishes there). It can be shown to correspond to a 2-sphere in the spacetime. The full Killing horizon, generated by λ, comprises two parts, which bifurcate at $\lambda = 0$. The full Killing horizon is known as a *Bifurcate Killing Horizon*, and $\lambda = 0$ is known as the *Bifurcation 2-sphere*.

One easily sees now that if $\mathcal{N}$ is a bifurcate Killing horizon of ξ, then κ^2 is constant on $\mathcal{N}$. We have already seen that κ^2 is constant on the orbits of ξ, and that its value is equal to its limiting value on the bifurcation 2-sphere B. We need only show that κ^2 is constant on this 2-sphere in order to complete the proof. We can do this by using equation (5.64) again. Taking t^μ to be any vector tangent to B, we immediately see that $t^\mu\,\partial_\mu\kappa^2 = 0$ on B, and hence that κ^2 is constant on B. Thus we have shown that κ^2 is constant on the entire bifurcate Killing horizon $\mathcal{N}$.

An example is probably useful at this point. Let us consider in detail the horizon at $r = 2m$ in the Schwarzschild spacetime. As we saw in chapter 3, it is best to use the Kruskal coordinate system with $\tilde v = e^{v/(4m)}$ and $\tilde u = -e^{-u/(4m)}$, where $v = t + r^*$ and $u = t - r^*$. The original radial coordinate r is given in terms of $\tilde v$ and $\tilde u$ by $\tilde v\,\tilde u = -(r-2m)\,e^{r/(2m)}$. Thus the event horizon, at $r = 2m$, corresponds to $\tilde v\,\tilde u = 0$, implying that at least one of $\tilde v$ or $\tilde u$ vanishes. In fact, as we see in Figure 4, the event horizon corresponds to the entire pair of 45-degree $\tilde v$ and $\tilde u$ axes. See Figure 15. From the definitions of the Kruskal coordinates, and by applying the chain rule, we see that the timelike Killing vector $K = \partial/\partial t$ in Kruskal coordinates is

$$K = \frac{\partial}{\partial t} = \frac{\tilde v}{4m}\,\frac{\partial}{\partial \tilde v} - \frac{\tilde u}{4m}\,\frac{\partial}{\partial \tilde u}\ . \tag{5.68}$$

Now, let us consider the vector ℓ normal to the horizon. Since the horizon is defined by $f(r) = 0$, where f could be, for example, $r - 2m$, it follows that the differential of f is proportional to dr. Since any function times a normal vector is, *a priori*, an equally good normal vector we may as well think of the differential of f as being proportional to dr^*, or in other words, proportional to $dv - du$. But this translates, in Kruskal coordinates, into a differential of the form $df = \tilde v^{-1}\, d\tilde v - \tilde u^{-1}\, d\tilde u$. Our discussion of the horizon will concentrate on the two arms of the cross, one specified by $\tilde u = 0$, and the other specified by $\tilde v = 0$. Clearly in either case, our differential diverges. So again we use the fact that we can make functional rescalings of df, in order to make a choice that is non-singular on the horizon. Thus for $\tilde u = 0$ we can simply rescale df by $\tilde u$ before setting $\tilde u = 0$. This means that on the $\tilde u = 0$ branch of the horizon, we have a differential of the form $df = d\tilde u$ (we use the same symbol f, although of course this is different from the previous one; this is just to avoid an unecessary proliferation of symbols). The normal vector is then obtained by raising the index on the components of this differential. The only non-vanishing comonent is $f_{\tilde u} = 1$. Bearing in mind that the metric in Kruskal coordinates has the off-diagonal structure given in (3.48), we see that the only relevant non-zero component of the inverse metric will be $g^{\tilde u\tilde v}$, and thus the only non-vanishing component of the normal vector will be $\ell^{\tilde v}$. A similar anaysis for the $\tilde v = 0$ branch of the horizon shows that there we will have $\ell^{\tilde u}$ as the only non-vanishing component of the normal vector.

In summary, we have the following normal vectors on the horizon:

$$\begin{aligned} \tilde u = 0: &\qquad \ell = \frac{\partial}{\partial \tilde v}\ , \\ \tilde v = 0: &\qquad \ell = \frac{\partial}{\partial \tilde u}\ . \end{aligned} \tag{5.69}$$

This might seem counter-intuitive, in that one might have imagined, for example, that the vector normal to $\tilde u$ =constant would be $\partial/\partial\tilde u$, rather than $\partial/\partial \tilde v$. There is nothig mysterious about this point, however; it is simply a consequence of the Minkowskian signature of the spacetime metric, and the fact that the normal vector is null. Thus what is parallel is perpendicular.

Of course we are free to multiply the normal vectors by any functions, and they will still be normal vectors to the horizon. But by good luck the ones that we have written here are actually the ones that we are looking for. Recall from our previous discussion that $\ell = \partial/\partial\lambda$

defines geodesic curves $x^\mu(\lambda)$, and that it is convenient to choose the parameterisation so that λ is an affine parameter, implying that $\ell^\nu \nabla_\nu \ell^\mu = 0$. It is a straightforward matter to check that our normal vectors (5.69) do indeed satisfy this condition. The verification involves calculating the necessary Christoffel connection components using (2.22), and using the definition of the covariant derivative given in (2.23).

Finally, we are ready to look at the surface gravity everywhere on the horizon. Consider the branch $\tilde{u} = 0$ first. From (5.68), we see that the timelike Killing vector K becomes $K = \frac{\tilde{v}}{4m}\, \partial/\partial\tilde{v}$ when $\tilde{u} = 0$. Comparing with (5.69), we see that this is proportional to the normal vector, and thus $\tilde{u} = 0$ is a Killing horizon with respect to K. The function of proportionality in the relation $K = \psi\, \ell$ is $\psi = \tilde{v}/(4m)$. From (5.43) we therefore have that the surface gravity on $\tilde{u} = 0$ is given by

$$\kappa = K^\nu\, \partial_\nu\, \log\Big|\frac{\tilde{v}}{4m}\Big| = \frac{\tilde{v}}{4m}\, \frac{d}{d\tilde{v}}\, \log\Big|\frac{\tilde{v}}{4m}\Big| = \frac{1}{4m}\,. \tag{5.70}$$

Note that this agrees with what we found earlier, when we set $a = 0$ in the result (5.63) for the surface gravity in the Kerr spacetime.

Now let us repeat the computation for the $\tilde{v} = 0$ branch of the event horizon. Now, we have from (5.68) that the timelike Killing vector is $K = -\frac{\tilde{u}}{4m}\, \partial/\partial\tilde{u}$. Again, this is proportional to the normal vector ℓ given in (5.69), this time with $K = \psi\, \ell$ for $\psi = -\tilde{u}/(4m)$. This time the computation of the surface gravity using (5.43) gives

$$\kappa = K^\nu\, \partial_\nu\, \log\Big| - \frac{\tilde{u}}{4m}\Big| = -\frac{\tilde{u}}{4m}\, \frac{d}{d\tilde{v}}\, \log\Big| - \frac{\tilde{u}}{4m}\Big| = -\frac{1}{4m}\,. \tag{5.71}$$

Thus the surface gravity is actually negative on the $\tilde{v} = 0$ branch of the horizon. This is intuitively reasonable (if anything involving white holes could be said to be either intuitive or reasonable!), since we saw that actually this branch of the horizon represented a one-way membrane for which future-directed timelike or null rays can cross only outwards, but not inwards. The negative surface gravity implies, for example, that there is a tendency for objects to be propelled outwards from the $\tilde{v} = 0$ horizon into the region I in the Penrose diagram 5. The situation is summarised in Figure 15.

The $\tilde{u} = 0$ and $\tilde{v} = 0$ branches of the event horizon in this Schwarzschild example together make up a bifurcate Killing horizon, of the kind that we discussed abstractly previously. The intersection, or bifuraction point, at $\tilde{u} = \tilde{v} = 0$, is the bifurcation 2-sphere (remember the two suppressed coordinates θ and φ). We can indeed see from (5.68) that it is a fixed-point of the orbits of the Killing vector K. We note also that, as we showed in the previous general discussion, the value of κ^2 is indeed constant over the entire bifurcate

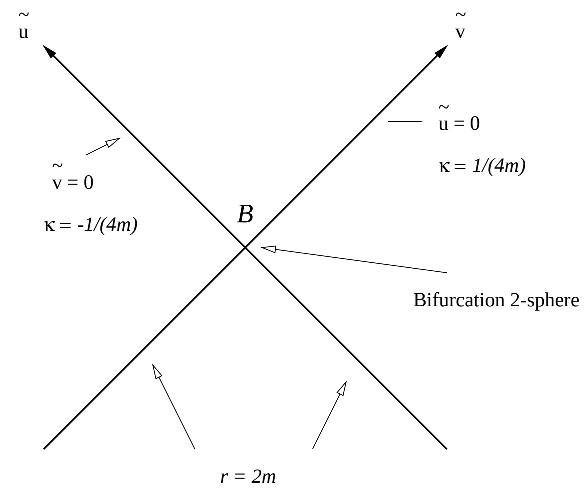

Figure 15: The bifurcate horizon in Schwarzschild spacetime.

Killing horizon. The only point that perhaps we would not have foreseen without the aid of an example is that although κ^2 is constant, κ itself can take opposite signs on different pieces of the horizon.

6 Black-Hole Thermodynamics

It was long believed that black holes are truely black, since nothing could escape from the event horizon. In the words of a bumper sticker with a distinctly 60's flavour, which used to be seen occasionally, "Black holes are out of sight." In fact, appropriately enough, it was in the 1970's that this picture of black holes was revolutionised by the work of Stephen Hawking, who showed that once quantum effects are taken into account black holes actually radiate particles, gradually losing their mass in the process. In fact Hawking showed that they radiate with a black-body spectrum, at a temperature

$$T = \frac{\kappa}{2\pi} \,, \tag{6.1}$$

where κ is the surface gravity.

The discovery that black holes emit black-body radiation was in a certain sense not entirely unforeshadowed by previous developments. It had been known for some time that classical black holes obey a set of laws that exhibit a remarkable parallelism to the basic laws of thermodynamics. We shall not attempt to give a complete discussion of the laws of black-hole dynamics here, nor shall we give a complete or rigorous treatment of the Hawking results. One could base an entire lecture course on these topics in themselves. Instead, we shall focus on one particular aspect of the classical dynamical laws for black holes, namely the analogue of the First Law of Thermodynamics. Armed with this, and with a relatively simple alternative derivation of the black-body radiation result, we shall be able to discuss some of the essential facts about the Hawking temperature and the entropy of black holes.

6.1 First Law of Black-Hole Dynamics

To begin, let us collect together a few results for the Kerr black hole. Several have already been discussed, and the only new result we shall need is an expression for the area of the outer horizon. Since this is defined by $r = r_+ = m + \sqrt{m^2 - a^2}$, we can determine this by looking at the metric on the surface $r = r_+$ at constant time. In other words, we first set $dr = 0$ and $dt = 0$ in (5.11), giving the two-dimensional metric

$$ds^2 = \rho^2\, d\theta^2 + \frac{\Big((r^2 + a^2)^2 - \Delta\, a^2\, \sin^2\theta\Big)\, \sin^2\theta}{\rho^2}\, d\varphi^2 \ . \tag{6.2}$$

We now set $r = r_+$, obtaining the metric

$$ds^2 = \rho_+^2\, d\theta^2 + \Big(\frac{2m\, r_+}{\rho_+}\Big)^2\, \sin^2\theta\, d\varphi^2 \tag{6.3}$$

on the outer horizon, where $\rho_+^2 = r_+^2 + a^2\, \cos^2\theta$. The area is therefore given by

$$A = 2m\, r_+ \int \sin\theta\, d\theta\, d\varphi = 8\pi\, m\, r_+ \ . \tag{6.4}$$

The other results that we need are for the surface gravity κ (5.63), the angular velocity of the horizon Ω_H (5.39), and the angular momentum $J = a\, m$. It is straightfoward now to show that the following identity holds:

$$m = \frac{\kappa\, A}{4\pi} + 2\Omega_H\, J \ . \tag{6.5}$$

Of course just presented in isolation, this result appears to be somewhat ill-motivated, since there are obviously many relations that can be written down involving the mutually-dependent quantities appearing here. The point is that this particular relation arises if one calculates the total energy E for a stationary black hole, using the ADM construction. This can be re-expressed in the form of a so-called Komar integral, as $E = -\int dS_{\mu\nu}\, \nabla^\mu\, K^\nu$, where the integral is taken over the two-dimensional boundary of a three-dimensional spacelike hypersurface. (Which will be taken to be the future event horizon plus the boundary sphere at infinity (i.e. at i^0).) After some manipulations, it can be shown that the energy (equals mass) is given by (6.5), which is known as the Smarr formula. The substitution of the explicit expressions for κ, A, Ω_H and J is then just giving a confirmation of the more generally-derived result.

Now, from (6.5), we can deduce the following *First Law of Black Hole Dynamics.* This is concerned with what happens if we perturb a black hole with given mass m and angular momentum J, having surface gravity κ and angular velocity Ω_H, so that it adjusts to a configuration with mass $m+\delta m$ and angular momentum $J+\delta J$. A crucial result, known as the "*No-hair theorem*, asserts that if (M,g) is an asymptotically-flat, stationary, vacuum spacetime that is non-singular on and outside the event horizon, then (M,g) must be a Kerr solution. Consequently, there is a unique vacuum solution for a black hole with mass m and angular momentum J. The proof of this result is a bit like a glorified version of the proof in electrodynamics that a given distribution of charges and boundary condition results in a unique electrostatic potential.

Now, using the uniqueness theorem, we know that m must be expressible as some function of the area A of the horizon, and the angular momentum J; thus $m=m(A,J)$. Since A and J both have dimensions of m^2, it must be that $m(A,J)$ is an homogeneous function of degree $\frac{1}{2}$, and so by Euler's theorem

$$\begin{aligned} A\,\frac{\partial m}{\partial A}+J\,\frac{\partial m}{\partial J} &= \tfrac{1}{2}m \\ &= \frac{\kappa}{8\pi}\,A+\Omega_H\,J\ , \end{aligned} \tag{6.6}$$

where the second line follows using the Smarr formula (6.5). Thus we have

$$A\Big(\frac{\partial m}{\partial A}-\frac{\kappa}{8\pi}\Big)+J\Big(\frac{\partial m}{\partial J}-\Omega_H\Big)=0\ . \tag{6.7}$$

Since A and J are treated as the independent variables here, it follows that the coefficients of A and J must independently vanish in this equation, and hence we deduce that

$$dm=\frac{\kappa}{8\pi}\,dA+\Omega_H\,dJ\ . \tag{6.8}$$

This is known as the first law of black-hole dynamics. Delving deep into one's memory, one can recognise this as being closely parallel to the first law of thermodynamics,

$$dE=T\,dS+\sum_i \Phi_i\,dX_i\ , \tag{6.9}$$

where T is the temperature, S is the entropy, and Φ_i represents the curiously-named chemical potentials, with their conjugate thermodynamic variables X_i.

Let us remark at this point that there is relatively straightforward generalisation of the Kerr solution to the so-called Kerr-Newman solution for a charged rotating black hole. The metric is of the same form as (5.18), except that now the function Δ is given by

$$\Delta=r^2-2m\,r+a^2+e^2\ , \tag{6.10}$$

where e is the electric charge. This is a solution of the Einstein-Maxwell equations, with the gauge potential A given by

$$A = \phi\, dt = \frac{2e\, r}{\rho^2}\, (dt - a\, \sin^2\theta\, \partial\varphi)\ . \tag{6.11}$$

It reduces to the Reissner-Nordström solution if the rotation parameter a is set to zero. If one computes the expression analogous to (6.8) in this case, allowing now for the perturbed black hole to settle down to a new value of e as well as m and J, one finds

$$dm = \frac{\kappa}{8\pi}\, dA + \Omega_H\, dJ + \phi_H\, de\ , \tag{6.12}$$

where ϕ_H is the value of the electrostatic potential on the horizon.

6.2 Hawking Radiation in the Euclidean Approach

Before the semi-classical quantum calculations by Hawking, the similarities between the first laws of black-hole dynamics and thermodynamics were seen as curious parallels, but little more than that. Hawking's surprising result, that a black hole radiates like a thermodynamic system at temperature $T = \kappa/(2\pi)$, changed all that. It now became apparent that the first law of black-hole dynamics *is* the first law of thermodynamics. It is the work of but a moment to see that the entropy of the black hole is thus given by $S = \frac{1}{4}A$, where A is the area of the event horizon.

Hawking first derived the black hole radiation by means of a careful analysis of what is meant by the vacuum in quantum field theory in curved spacetime, and in particular, how the vacuum for an observer at $\mathcal{I}^+$ is related to the vacuum for an observer at $\mathcal{I}^-$. The outcome from this analysis is that in the balck-hole background, a zero-particle initial state becomes a state populated by a thermal distribution of particles with respect to the observer at $\mathcal{I}^+$. Rather than going into the details of this derivation, which is quite involved, let us instead follow a route that was developed later, once the thermodynamic implications had been digested. It is a little heuristic, in the sense that it perhaps lacks a fully rigorous justification, but it is elegant from the point of view of geometry and topology, which is, it may be recalled, the theme of these lectures.

Let us begin by considering the Schwarzschild solution. We then perform a Wick rotation of the time coordinate, by writing $t = -\mathrm{i}\,\tau$. The original metric (3.2) then becomes

$$ds^2 = \Big(1 - \frac{2m}{r}\Big)\, d\tau^2 + \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 + r^2\, d\Omega^2\ . \tag{6.13}$$

Now, consider the following transformation of the radial coordinate:

$$R = 4m\,\Big(1 - \frac{2m}{r}\Big)^{1/2}\ , \tag{6.14}$$

in terms of which the metric (6.13) becomes

$$ds^2 = \Big(\frac{r}{2m}\Big)^4\, dR^2 + R^2\,\Big(\frac{d\tau}{4m}\Big)^2 + r^2\, d\Omega^2\ . \tag{6.15}$$

Now the coordinate R vanishes as r approaches the "horizon" at $r = 2m$. If we look at the form of the metric (6.15) near $r = 2m$, we see that it approaches

$$ds^2 = dR^2 + R^2\,\Big(\frac{d\tau}{4m}\Big)^2 + 4m^2\, d\Omega^2\ . \tag{6.16}$$

This has a singularity at $R = 0$, but under appropriate conditions, namely if $\tau/(4m)$ has period 2π, this is nothing but the familar coordinate singularity at the origin of two-dimensional polar coordinates. (Compare with $ds^2 = dr^2 + r^2\, d\theta^2$.) Of course, if τ is assigned any other period there will be a genuine curvature singularity at $R = 0$, since then the metric is like the metric on a cone, which has a delta-function singularity in its curvatuure at the apex. However, if we proceed by making the assumption that this calculation is trying to tell us something, then we would naturally choose to take τ to have the special periodicity for which the nice singularity-free interpretation can be given. The upshot is that we arrive at the interpretation of the Euclideanised Schwarzschild metric as the metric on a smooth manifold defined by

$$0 \le \tau \le 8\pi\, m\ , \qquad 2m \le r \le \infty\ , \tag{6.17}$$

with the angular coordinates θ and φ on the 2-sphere precisely as usual.

This Euclideanised Schwarzschild manifold is completely free of curvature singularities; it makes no more sense to ask what happens for r less than $2m$ here than it does to ask what happens for r less than zero in plane-polar coordinates. The manifold with $r \ge 2m$ is complete. The interesting point is that in terms of the original Schwarzschild spacetime, we have been led to perform a periodic identification in imaginary time, with period $8\pi\, m$. Now, those familiar with statistical mechanics may have encountered such a kind of periodicity before. A statistical system in thermal equilibrium at temperature T can be described in terms of Green functions and partition functions evaluated in states that are periodic in imaginary time, with period $\beta = 1/T$. A very rough sketch of why this is the case, which can easily be fleshed out with a little more effort, is the following. The time-evolution operator in quantum mechanics is $U = e^{\mathrm{i} H\, t}$, where H is the Hamiltonian. Thus an evaluation of operators between initial and final energy eigenstates at times differing by an amount $\mathrm{i}\,\beta$ will acquire a factor $e^{-E\,\beta}$, which can be recognised as the Boltzmann factor $e^{-E/T}$ for a system at temperature $T = 1/\beta$. Thus we arrive at the tentative conclusion that the Euclideanised

Schwarzschild manifold is describing a system in thermal equilibrium at temperature

$$T = \frac{1}{8\pi\, m} \ . \tag{6.18}$$

This is precisely the temperature already found by Hawking for the black-body radiation emitted by the Schwarzschild black hole. Recall that in the Schwarzschild spacetime, we saw previously that the surface gravity on the future horizon is given by $\kappa = 1/(4m)$, and so indeed the temperature is $T = \kappa/(2\pi)$.

A similar calculation can easily be performed for the Reissner-Nordström solution. In fact, it is quite instructive to do the calculation for a more general class of static metrics, in order to bring out the relation between the surface gravity and the periodicity of τ more transparently. Consider, therefore, a metric of the form

$$\text{Minkowskian}: \qquad ds^2 = -f\, dt^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2 \ , \tag{6.19}$$

$$\text{Euclidean}: \qquad ds^2 = f\, d\tau^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2 \ , \tag{6.20}$$

where we give both its original Minkowskian-signature form, and its form after Euclideanisation. Let us suppose that f, which is taken to be a function only of r, has a zero at some point $r = r_0$. This would correspond to an event horizon. Let us then define a new radial coordinate $R = f^{1/2}$. Thus we have $dR = \frac{1}{2} f^{-1/2}\, f'\, dr$, and hence, in the vicinity of $r = r_0$, the metric (6.20) approaches

$$ds^2 = \frac{4}{f'(r_0)^2}\Big(dR^2 + \tfrac{1}{4} f'(r_0)^2\, R^2\, d\tau^2\Big) + r_0^2\, d\Omega^2 \ . \tag{6.21}$$

Thus we see that $R = 0$ is like the origin of polar coordinates provided that we identify τ with period $\Delta\tau$ given by

$$\Delta\tau = \Big|\frac{4\pi}{f'(r_0)}\Big| \ . \tag{6.22}$$

On the other hand, we can perform a calculation of the surface gravity on the horizon at $r = r_0$ in the metric (6.19). This is a Killing horizon with respect to the timelike Killing vector $K = \partial/\partial t$. Using the expression (5.55) we have $\lambda^2 = -g_{\mu\nu}\, K^\mu\, K^\nu = -g_{tt} = f$, and hence

$$\kappa^2 = g^{\mu\nu}\, \partial_\mu\, \lambda\, \partial_\nu\, \lambda = g^{rr}\, (f^{1/2})'^2\Big|_{r=r_0} = \tfrac{1}{4} f'(r_0)^2 \ . \tag{6.23}$$

Thus we see that $\kappa = \pm\frac{1}{2} f'(r_0)$, and thus comparing with (6.22) we have the relation

$$\Delta\tau = \Big|\frac{2\pi}{\kappa}\Big| \ . \tag{6.24}$$

For a metric such as Kerr, which is stationary but not static, the calculation is a little more tricky. The "Euclidean philosophy" now would be that we should consider operators that are sandwiched between in and out states that have coordinate values related by $(t, r, \theta, \varphi) \sim (t + \mathrm{i}\,\beta, r, \theta, \varphi + \mathrm{i}\,\Omega_H\,\beta)$. Thus in the Euclideanised metric we should make everything real by taking $t = -\mathrm{i}\,\tau$ and $\Omega_H = \mathrm{i}\,\widetilde\Omega_H$, where $\widetilde\Omega_H$ is real. This means that we should take the rotation parameter a to be imaginary, $a = \mathrm{i}\,\alpha$. Thus the Kerr metric (5.11) Euclideanises to become

$$\begin{aligned} ds^2 &= \frac{(\Delta + \alpha^2\,\sin^2\theta)}{\rho^2}\,d\tau^2 - \frac{4m\,\alpha\,r\,\sin^2\theta}{\rho^2}\,d\tau\,d\varphi \\ &\quad + \frac{\Big((r^2-\alpha^2)^2 + \Delta\,\alpha^2\,\sin^2\theta\Big)\,\sin^2\theta}{\rho^2}\,d\varphi^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2\ , \end{aligned} \tag{6.25}$$

where

$$\rho^2 = r^2 - \alpha^2\,\cos^2\theta\ , \qquad \Delta = r^2 - 2m\,r - \alpha^2\ . \tag{6.26}$$

We shall want to examine the behaviour of this metric in the vicinity of $r_+ = m + \sqrt{m^2+\alpha^2}$, where Δ first vanishes as one approaches from large r. We shall introduce a new radial coordinate R, defined by $R = \Delta^{1/2}$, and then take the limit when R is very small. We can in fact judiciously set $r = r_+$ at the outset in certain places in the metric (6.25), namely in those places where no singularity will result from doing so. Thus near to $r = r_+$, the metric approaches

$$ds^2 = \frac{(\Delta + \alpha^2\,\sin^2\theta)}{\rho_+^2}\,d\tau^2 - \frac{4m\,\alpha\,r_+\,\sin^2\theta}{\rho_+^2}\,d\tau\,d\varphi + \frac{4m^2\,r_+^2\,\sin^2\theta}{\rho_+^2}\,d\varphi^2 + \frac{\rho_+^2}{\Delta}\,dr^2 + \rho_+^2\,d\theta^2\ , \tag{6.27}$$

where $\rho_+^2 = r_+^2 - \alpha^2\,\cos^2\theta$, and we have used the fact that $r_+^2 - \alpha^2 = 2m\,r_+$. Note that ρ_+^2 is non-vanishing for all θ. The metric (6.27) can be reorganised, by completing the square, so that it becomes

$$ds^2 = \frac{\rho_+^2}{\Delta}\,dr^2 + \frac{\Delta}{\rho_+^2}\,d\tau^2 + \frac{4m^2\,r_+^2\,\sin^2\theta}{\rho_+^2}\,(d\varphi - \widetilde\Omega_H\,d\tau)^2 + \rho_+^2\,d\theta^2\ , \tag{6.28}$$

where $\widetilde\Omega_H = \alpha/(2m\,r_+)$ is the "angular momentum" on the horizon in his Euclideanised metric (see (5.39)). Now, making our substitution $R = \Delta^{1/2}$, and noting that near to $r = r_+$ we can consequently write $2R\,dR = d[(r-r_+)(r-r_-)] \sim dr\,(r_+ - r_-) = 2\sqrt{m^2+\alpha^2}\,dr$, we see that near $r = r_+$ the Euclideanised Kerr metric approaches

$$ds^2 = \frac{\rho_+^2}{m^2+\alpha^2}\,dR^2 + \frac{R^2}{\rho_+^2}\,d\tau^2 + \frac{4m^2\,r_+^2\,\sin^2\theta}{\rho_+^2}\,(d\varphi - \widetilde\Omega_H\,d\tau)^2 + \rho_+^2\,d\theta^2\ . \tag{6.29}$$

We now have to examine in detail what happens as R approaches zero. If θ is equal to 0 or π, the prefactor of $(d\varphi - \widetilde\Omega_H\,d\tau)^2$ vanishes, and consequently we shall have a conical singularity at $R = 0$ in the (R, τ) plane unless τ has the appropriate periodicity. Noting that at $\theta = 0$ or $\theta = \pi$ we have $\rho_+^2 = r_+^2 - \alpha^2 = 2m\, r_+$, we see that the relevant two-dimensional part of the metric is

$$ds^2 = \frac{2m\, r_+}{m^2 + \alpha^2} \left[dR^2 + R^2 \left(\frac{m^2 + \alpha^2}{4m^2\, r_+^2} \right) d\tau^2 \right] , \tag{6.30}$$

and thus the conical singularity is avoided if τ is identified periodically with period

$$\Delta\tau = \frac{4\pi\, m\, r_+}{\sqrt{m^2 + \alpha^2}} . \tag{6.31}$$

If θ takes any other generic value $0 < \theta < \pi$, the prefactor of $(d\varphi - \widetilde{\Omega}_H\, d\tau)^2$ in (6.29) is non-zero, and no further conditions arise.

Comparing (6.31) with the expression for the surface gravity for the Kerr metric that we obtained in (5.63), we see that the periodicity of τ is again given by

$$\Delta\tau = \frac{2\pi}{\kappa} , \tag{6.32}$$

where κ is given by (5.63) with $a = \mathrm{i}\, \alpha$.

6.3 Hawking Radiation in the Lorentzian Approach

We have already seen in the framework of the Euclideanisation approach that a black hole seems to behave like a black body radiating with temperature $T = \kappa/(2\pi)$, where κ is its surface gravity. In this chapter, we shall look at this problem from a more traditional point of view, by considering the behaviour of quantised matter fields in a black-hole background geometry. This is the way Hawking first derived the result.

It is not the main intention in this course to study quantum field theory, and so we shall not dwell too much on the fine points of the subject. We can, however, easily give a brief review of the salient points. We shall, for simplicity, restrict attention to the case of a scalar field theory; the extension to fermions, and to fields of higher spin, is straightforward, and the eventual conclusions are similar.

6.3.1 Scalar quantum field theory in curved spacetime

Consider a real scalar field ϕ satisfying the massive Klein-Gordan equation

$$\square\, \phi - m^2\, \phi = 0 . \tag{6.33}$$

One can define a natural inner product $\{\ ,\ \}$ on such fields, as

$$\{\phi_1, \phi_2\} \equiv \int_\Sigma d\Sigma_\mu\, \phi_1 \overset{\leftrightarrow}{\partial}{}^\mu\, \phi_2 , \tag{6.34}$$

where Σ is some Cauchy 3-surface on which initial data, once specified for ϕ_i, evolves deterministically according to the Klein-Gordan equation. The symbol $\overset{\leftrightarrow}{\partial}{}^{\mu}$ denotes the "two-edged" derivative, $\phi_1 \overset{\leftrightarrow}{\partial}{}^{\mu} \phi_2 \equiv \phi_1 \partial^\mu \phi_2 - (\partial^\mu \phi_1) \phi_2$. The inner product is independent of the choice of Cauchy surface Σ, as can be seen by considering two such surfaces Σ and Σ'. In the usual way, one imagines that these two surfaces together form the boundary of a closed 4-volume V in spacetime. Either they already do this as they stand, or else the "deficit" that completes the bounding 3-surface consists of a hypersurface on which the fields are all assumed to vanish. By Gauss' law, the difference between the inner products on the two Cauchy surfaces can then be turned into a 4-volume integral that vanishes:

$$\{\phi_1, \phi_2\}_\Sigma - \{\phi_1, \phi_2\}_{\Sigma'} = \int_\Sigma d\Sigma_\mu \phi_1 \overset{\leftrightarrow}{\partial}{}^{\mu} \phi_2 - \int_{\Sigma'} d\Sigma_\mu \phi_1 \overset{\leftrightarrow}{\partial}{}^{\mu} \phi_2 = \int_V \nabla_\mu (\phi_1 \overset{\leftrightarrow}{\partial}{}^{\mu} \phi_2) = 0 \ , \tag{6.35}$$

where the last step follows upon use of the Klein-Gordan equation (6.33).

The inner product $\{\ ,\ \}$ is antisymmetric: $\{\phi_1, \phi_2\} = -\{\phi_2, \phi_1\}$. This is the well-known feature of the scalar field, that it has an indefinite-signature inner product, leading to negative-norm states, and so on. A basis ϕ_i, ϕ_i' for the solutions ϕ can be chosen so that $\{\phi_i, \phi_j'\} = \delta_{ij}$. If we group these pairs of real solutions into complex solutions,

$$\psi_i = \tfrac{1}{\sqrt{2}}(\phi_i - \mathrm{i}\,\phi_i') \ , \qquad \bar\psi_i = \tfrac{1}{\sqrt{2}}(\phi_i + \mathrm{i}\,\phi_i') \ , \tag{6.36}$$

then we can define an inner product (ψ_i, ψ_j) by

$$(\psi_i, \psi_j) = \mathrm{i} \int_\Sigma d\Sigma_\mu \,\bar\psi_i \overset{\leftrightarrow}{\partial}{}^{\mu} \psi_j = \delta_{ij} \ , \tag{6.37}$$

Of course the inner product here is still of indefinite signature, since we have

$$(\psi_i, \psi_j) = -(\bar\psi_i, \bar\psi_j) = \delta_{ij} \ , \qquad (\psi_i, \bar\psi_j) = (\bar\psi_i, \psi_j) = 0 \ . \tag{6.38}$$

Note that this is sesquilinear, in the sence that for complex constants a and b, we have $(a\,\psi, b\,\phi) = \bar a\, b\,(\psi, \phi)$.

Classical solutions can be expanded in a complete set of basis functions ψ_i and $\bar\psi_i$. Upon second quantisation, the expansion becomes

$$\Phi(x) = \sum_i (a_i\, \psi_i(x) + \alpha_i^\dagger\, \bar\psi_i(x)) \ , \tag{6.39}$$

where the "Fourier coefficients" a_i and their Hermitean conjugates $a_i^\dagger$ are operators in a Hilbert space, satisfying the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \ , \qquad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger] \ . \tag{6.40}$$

One then builds up the Fock space by defining a vacuum state $|\text{vac}\rangle$, such that

$$a_i\,|\text{vac}\rangle = 0\ , \qquad \langle\text{vac}|\text{vac}\rangle = 1\ , \tag{6.41}$$

and then acting with the creation operators $a_i^\dagger$ to get the basis

$$|\text{vac}\rangle\ , \qquad a_i^\dagger\,|\text{vac}\rangle\ , \qquad a_i^\dagger\,a_j^\dagger\,|\text{vac}\rangle\ , \qquad \textit{etc.}\ . \tag{6.42}$$

Note that the inner product $\langle\ |\ \rangle$ is positive-definite.

This choice of basis for the Hilbert space is determined by the choice of vacuum state $|\text{vac}\rangle$, but this in turn depends on the choice of the basis ψ_i. This is not unique, and it is clear that a new basis ψ_i', related to ψ_i by

$$\psi_i' = A_{ij}\,\psi_j + B_{ij}\,\bar\psi_j \tag{6.43}$$

will satisfy the same basis relations (6.38), provided that the constants A_{ij} and B_{ij} obey, in the obvious matrix notation,

$$A\,A^\dagger - B\,B^\dagger = 1\ , \qquad A\,B^T - B\,A^T = 0\ . \tag{6.44}$$

Note that (6.43) can be inverted, to give

$$\psi_i = A'_{ij}\,\psi_j + B'_{ij}\,\bar\psi_j\ , \tag{6.45}$$

where

$$A' = A^\dagger\ , \qquad B' = -B^\dagger\ . \tag{6.46}$$

(It is easy to check by substitution that this indeed gives the required inverse relation; it is not so easy to *derive* (6.46).) Since A' and B' must themselves satisfy the same relations as A and B in (6.44), it follows from (6.46) that

$$A^\dagger\,A - B^T\,\bar B = 1\ , \qquad A^\dagger\,B - B^T\,\bar A = 0\ . \tag{6.47}$$

If a spacetime is stationary, meaning that it admits a timelike Killing vector K, then one can give a meaning to the notion of positive and negative freqency. Specifically, one can choose a basis $\{u_i\}$ of positive-frequency solutions of the Klein-Gordan equation, which satisfy the condition

$$K(u_i) \equiv K^\mu\,\partial_\mu\,u_i = -\mathrm{i}\,\omega_i\,u_i\ , \qquad \omega_i > 0 \tag{6.48}$$

for each i. This follows from the fact that if u_i is a solution of the Klein-Gordan equation, then so is $K^\mu\,\partial_\mu\,u_i$ (prove by using the properties of Killing vectors to show that

$\Box(K^\mu\,\partial_\mu\,u_i) = K^\mu\,\partial_\mu\Box\,u_i)$. Furthermore, K is an anti-Hermitean operator, and so solutions of the Klein-Gordan equation can be simultaneously eigenfunctions of K with imaginary eigenvalues. One can choose a basis for u_i such that $(u_i, u_j) = \delta_{ij}$. Of course we will also have $(u_i, \bar{u}_j) = 0$, since $\bar{u}_j$ has a negative frequency, and eigenfunctions with different frequencies are always orthogonal.

In this basis, the vacuum $|\text{vac}\rangle$ is the state of lowest energy, and $a_i^\dagger\,|\text{vac}\rangle$, $a_i^\dagger\,a_j^\dagger\,|\text{vac}\rangle$ describe 1-particle, 2-particle, *etc.* states. The particle number is measured by the number operator

$$N = \sum_i a_i^\dagger\,a_i\ . \tag{6.49}$$

6.3.2 Particle production in non-stationary spacetimes

Consider a spacetime that is stationary for $t < t_-$, stationary for $t > t_+$, and non-stationary during the intervening period $t_- < t < t_+$. We may denote the associated components of the total spacetime by M_-, M_+ and M_0 respectively. We assume that the metrics in M_- and M_+ are isometric.

In the region M_-, we may expand a Klein-Gordan field Φ in terms of a complete set of positive-frequency modes u_i as defined previously, so that

$$\Phi(x) = \sum_i (a_i\,u_i(x) + \alpha_i^\dagger\,\bar{u}_i(x))\ . \tag{6.50}$$

Since the functions $u_i(x)$ are solutions of the Klein-Gordan equation in M_- and in M_+ (the metrics are assumed to be the same in the two stationary regions), it follows that we can also expand the Klein-Gordan field in M_+ in terms of the same eigenfunctions. However, since the $u_i(x)$ do not solve the Klein-Gordan equation in the intervening region M_0, it means that there will be some general rotation of the basis, of the form (6.43). In other words, the Klein-Gordan field in the region M_+ can be expanded as

$$\Phi(x) = \sum_i (a_i\,v_i + a_i^\dagger\,\bar{v}_i)\ , \tag{6.51}$$

where

$$v_i = A_{ij}\,u_j + B_{ij}\,\bar{u}_j\ . \tag{6.52}$$

Since the inner product (,) is independent of the choice of hypersurface, the coefficients A_{ij} and B_{ij} must satisfy the conditions (6.44).

If we nonetheless choose to expand $\Phi(x)$ in the region M_+ in terms of the original basis u_i, then we will have new expansion coefficients a'_i and ${a'_i}^\dagger$,

$$\Phi(x) = \sum_i (a'_i\,u_i + {a'_i}^\dagger\,\bar{u}_i)\ . \tag{6.53}$$

Substitution of (6.51) and (6.52) immediately leads to the so-called *Bogoliubov transformation*

$$a_i' = a_j\, A_{ji} + a_j^\dagger\, \bar{B}_{ji}\ . \tag{6.54}$$

It is a simple exercise to verify from this that the relations (6.44) imply and are implied by the statement that both the unprimed and the primed operators $(a_i, a_i^\dagger)$ satisfy (6.40).

Not that in the special case where the Bogoliubov coefficients B_{ij} vanish, the rotation of basis (6.52) does not mix positive-frequency with negative-frequency modes, and the transformation is a purely unitary one; as can be seen from (6.44) and (6.47), we have $A\,A^\dagger = A^\dagger\,A = 1$. This just rotates the annihilation operators a_i around amongst themselves, and therefore it does not change the vacuum state $|\mathrm{vac}\rangle$, which is defined by $a_i\,|\mathrm{vac}\rangle = 0$.

Now let us look at the general case, when B_{ij} is non-zero. The number operators for particles in the i'th mode in the regions M_- and M_+ are given by

$$N_i = a_i^\dagger\, a_i\ , \qquad N_i' = a_i'^\dagger\, a_i'\ , \qquad \text{no sum}\ , \tag{6.55}$$

respectively. Suppose we consider the zero-particle state in M_-, namely the vacuum $|\mathrm{vac}\rangle$ defined by $a_i\,|\mathrm{vac}\rangle = 0$. If we calculate the expectation value of N_i' in the region M_+, with respect to this vacuum, is therefore given by

$$\begin{aligned}
\langle N_i'\rangle &= \langle\mathrm{vac}|N_i'|\mathrm{vac}\rangle = \langle\mathrm{vac}|a_i'^\dagger\, a_i'|\mathrm{vac}\rangle\ ,\\
&= \langle\mathrm{vac}|(a_j\, B_{ji})\,(a_k^\dagger\, \bar{B}_{ki})|\mathrm{vac}\rangle\ ,\\
&= \langle\mathrm{vac}|a_j\, a_k^\dagger|\mathrm{vac}\rangle\, B_{ji}\, \bar{B}_{ki}\ ,\\
&= \langle\mathrm{vac}|[a_j, a_k^\dagger]|\mathrm{vac}\rangle\, B_{ji}\, \bar{B}_{ki}\ ,\\
&= (B^\dagger\, B)_{ii}\ , \qquad \text{no sum}\ .
\end{aligned} \tag{6.56}$$

This is the expected number of particles in the i'th mode in region M_+, measured with respect to the vacuum state $|\mathrm{vac}\rangle$ defined in M_-. In other words, particles have been created as a result of the non-unitary change of basis. The total number of particles, summed over all modes, is $\sum_i (B^\dagger\, B)_{ii}$, or in other words $\mathrm{tr}(B^\dagger\, B)$. This vanishes if and only if $B = 0$, since $(B^\dagger\, B)$ is positive semi-definite.

6.3.3 Hawking radiation from Schwarzschild black hole

Consider a massless Klein-Gordan field $\Phi(x)$ in the Schwarzchild background. This satisfies

$$0 = \Box\,\Phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\,(\sqrt{-g}\; g^{\mu\nu}\,\partial_\nu\,\Phi)\ , \tag{6.57}$$

and hence, from the form of the Schwarzschild solution (3.15) obtained in section 3, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big[r^2\Big(1-\frac{2m}{r}\Big)\frac{\partial\Phi}{\partial r}\Big]-\Big(1-\frac{2m}{r}\Big)^{-1}\frac{\partial^2\Phi}{\partial t^2}+\frac{1}{r^2}\nabla^2_{(\theta,\varphi)}\Phi=0\;, \tag{6.58}$$

where $\nabla^2_{(\theta,\varphi)}$ denotes the usual Laplacian on the unit 2-sphere. Thus we can separate variables by writing

$$\Phi(x)=f(r)\,e^{-\mathrm{i}\,\omega\,t}\,Y_{\ell m}(\theta,\varphi)\;, \tag{6.59}$$

where the radial function $f(r)$ satisfies

$$\frac{1}{r^2}\frac{d}{dr}\Big[r^2\Big(1-\frac{2m}{r}\Big)\frac{df}{dr}\Big]+\Big\{\omega^2\Big(1-\frac{2m}{r}\Big)^{-1}-\frac{\ell(\ell+1)}{r^2}\Big\}f=0\;. \tag{6.60}$$

This can be rewritten in terms of the coordinate r^* defined in (3.38) as

$$\frac{1}{r^2}\Big(1-\frac{2m}{r}\Big)^{-1}\frac{d}{dr^*}\Big(r^2\frac{df}{dr^*}\Big)+\Big\{\omega^2\Big(1-\frac{2m}{r}\Big)^{-1}-\frac{\ell(\ell+1)}{r^2}\Big\}f=0\;. \tag{6.61}$$

Near the horizon at $r=2m$, we see that the radial wave equation (6.61) becomes approximately

$$\frac{d^2f}{dr^{*2}}+\omega^2\,f=0\;, \tag{6.62}$$

with solution $f\sim e^{\pm\mathrm{i}\,\omega\,r^*}$. In particular, this means that an *outgoing* wavefunction near the horizon will have form

$$\Phi_\omega\sim e^{-\mathrm{i}\,\omega\,u}\;, \tag{6.63}$$

where $u=t-r^*$ is the retarded Eddington-Finkelstein coordinate. (We suppress the $Y_{\ell m}(\theta,\varphi)$ factor, since it is inessential for this discussion.)

Consider an outgoing (massless) particle in the geometric optics approximation, which means that it is viewed as a null geodesic, travelling on a path of constant u. (Recall the discussion of radial null geodesics in Chapter 3.) Now we have the relation (3.47) between the Eddington-Finkelstein and Kruskal coordinates, which becomes, using the expression $\kappa=1/(4m)$ for the surface gravity,

$$\tilde v=e^{\kappa v}\;,\qquad \tilde u=-e^{-\kappa u}\;. \tag{6.64}$$

Thus the outgoing null geodesic $\tilde u=0$ just grazes the future horizon $\mathcal{H}^+$, which corresponds to $\tilde u=0$ in the Kruskal representation of the horizon (see Figure 15). A null ray γ, which reaches $\mathcal{I}^+$, must therefore have $\tilde u<0$. Let us consider one that is close to $\mathcal{H}^+$, and thus has $\tilde u=-\varepsilon$ for some small positive constant ε, and trace it back from $\mathcal{I}^+$ into the star. Thus on this ray we have $u=-\kappa^{-1}\log\varepsilon$ near to $\mathcal{H}^+$, and so the wavefunction near $\mathcal{H}^+$ is of the form

$$\Phi_\omega\sim e^{\frac{\mathrm{i}\,\omega}{\kappa}\,\log\varepsilon}\;. \tag{6.65}$$

Note that since $\tilde{u}$ is the affine parameter on the ingoing null geodesics, ε is the affine distance from the ray γ to the ray γ_H at $\tilde{u}=0$ that grazes the future horizon $\mathcal{H}^+$. Note that the ray γ_H is the null geodesic generator of $\mathcal{H}^+$.

If we could solve for the wavefunctions exactly, in terms of special functions with known analytic and asymptotic properties, it would be completely straightforward to study how this outgoing late-time solution Φ_ω matched on to a solution in the distant past at $\mathcal{I}^-$. Since this cannot be done, we must resort to slightly indirect arguments to establish how the matching works. In the geometric optics approximation, one just has to continue the rays γ and γ_H back to $\mathcal{I}^-$. The ray γ_H will intercept $\mathcal{I}^-$ at some fixed value of v; we may as well let this value be 0, since no loss of generality is involved. Now, the affine distance between the continuations of γ_H and γ to $\mathcal{I}^-$ will ε. The affine parameter for outging null geodesics at $\mathcal{I}^-$ is v itself, since the metric there is just $ds^2=-dv\,du+r^2\,d\Omega^2$. Thus the null ray γ has $v=-\varepsilon$. Consequently, (6.65) can be expressed as

$$\Phi_\omega(v)\sim e^{\frac{\mathrm{i}\,\omega}{\kappa}\,\log(-v)}\ , \tag{6.66}$$

when $v<0$. If $v>0$, an ingoing null ray from $\mathcal{I}^-$ goes through $\mathcal{H}^+$, and never makes it out to $\mathcal{I}^+$. Thus we have $\Phi_\omega(v)=0$ for $v>0$.

The Fourier transform of $\Phi_\omega(v)$ is therefore given by

$$\begin{aligned}\widetilde{\Phi}_\omega(\omega') &= \int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega'\,v}\,\Phi_\omega(v)\,dv\ ,\\ &= \int_{-\infty}^{0} e^{\mathrm{i}\,\omega'\,v+\frac{\mathrm{i}\,\omega}{\kappa}\,\log(-v)}\,dv\ .\end{aligned} \tag{6.67}$$

The integrand has a branch cut, which may be taken to lie along the positive v axis. When ω' is positive, the integration contour may be swung round in the complex v plane from the negative v axis to the positive imaginary v axis, by defining $v=\mathrm{i}\,x$ with x real, and running from ∞ to 0. Since we then have $\log(-v)=\log(x\,e^{-\mathrm{i}\,\pi/2})=\log x-\mathrm{i}\,\pi/2$, it follows that

$$\widetilde{\Phi}_\omega(\omega')=-\mathrm{i}\,e^{\frac{\pi\,\omega}{2\kappa}}\int_0^\infty e^{-\omega'\,x+\frac{\mathrm{i}\,\omega}{\kappa}\,\log x}\,dx \tag{6.68}$$

when $\omega'>0$. On the other hand, when ω' is negative we can instead swing the contour round to the negative imaginary axis, by taking $v=-\mathrm{i}\,x$, with x real and running from ∞ to 0. Thus we have

$$\widetilde{\Phi}_\omega(\omega')=\mathrm{i}\,e^{-\frac{\pi\,\omega}{2\kappa}}\int_0^\infty e^{\omega'\,x+\frac{\mathrm{i}\,\omega}{\kappa}\,\log x}\,dx \tag{6.69}$$

when $\omega'<0$. Comparing these two expressions, we see that if ω' is positive, we have

$$\widetilde{\Phi}_\omega(-\omega')=-e^{-\frac{\pi\,\omega}{\kappa}}\,\widetilde{\Phi}_\omega(\omega')\ . \tag{6.70}$$

The upshot from the above calculation is that we have established that a pure single-frequency mode of positive frequency ω on $\mathcal{I}^+$ at late times matches onto an entire chorus of modes on $\mathcal{I}^-$ at early times, having both positive and negative frequencies. The Fourier transform $\widetilde{\Phi}_\omega(\omega')$ gives precisely the Bogoliubov coefficients A and B that we discussed previously, with

$$\begin{aligned} A_{\omega\,\omega'} &= \widetilde{\Phi}_\omega(\omega')\ , \\ B_{\omega\,\omega'} &= \widetilde{\Phi}_\omega(-\omega') = -e^{-\frac{\pi\,\omega}{\kappa}}\,\widetilde{\Phi}_\omega(\omega')\ , \end{aligned} \tag{6.71}$$

where ω' here is taken to be positive. In particular, we deduce that

$$B_{ij} = -e^{-\frac{\pi\,\omega_i}{\kappa}}\,A_{ij}\ . \tag{6.72}$$

Since the A and B matrices satisfy the relations (6.44), it follows, in particular, that

$$\begin{aligned} \delta_{ij} &= A_{ik}\,\bar{A}_{jk} - B_{ik}\,\bar{B}_{jk}\ , \\ &= \Big(e^{\frac{\pi(\omega_i+\omega_j)}{\kappa}} - 1\Big)B_{ik}\,\bar{B}_{jk}\ . \end{aligned} \tag{6.73}$$

Consequently, we have

$$(B\,B^\dagger)_{ii} = \frac{1}{e^{\frac{2\pi\,\omega_i}{\kappa}} - 1}\ . \tag{6.74}$$

This is not yet quite what we want, since we have effectively worked out the "inverse" of the desired result. Our calculation has related a single positive-frequency mode at late times to a chorus of positive and negative frequency modes at early times. We should really be looking at the problem the other way around, since we want to see what the distribution of particle numbers is at late times, given a zero-particle vacuum state at the initial time. This is achieved by using the inversion discussed in the previous subsection, and given in (6.46). Thus from (6.56), we have that the occupancy number in the mode i at $\mathcal{I}^+$, given an initial vacuum $\mathcal{I}^-$, is

$$\langle N_i\rangle = (B'^\dagger\,B')_{ii} = (\bar{B}\,B^T)_{ii} = (B\,B^\dagger)_{ii}\ , \tag{6.75}$$

where, in the last step, we have used the fact that $B\,B^\dagger$ is real. Thus we arrive at the final conclusion that the occupancy number in the i'th mode at $\mathcal{I}^+$ is

$$\langle N_i\rangle = \frac{1}{e^{\frac{2\pi\,\omega_i}{\kappa}} - 1}\ . \tag{6.76}$$

This shows that the states are occupied in a Planckian distribution, corresponding to black-body radiation at temperature

$$T = \frac{\kappa}{2\pi}\ . \tag{6.77}$$

7 Higher-Dimensional Theories, and Kaluza-Klein Reduction

So far, our discussion has been concerned with a relatively restricted class of theories in four dimensions, namely pure Einstein gravity, and also gravity coupled to Maxwell theory. There are, of course, many more complicated theories that one could consider already in four dimensions, for example by considering Einstein gravity coupled to non-abelian Yang-Mills theory rather than the abelian Maxwell theory, or by adding additional "matter fields," which might include scalars with a variety of different possible couplings and self-interactions. *A priori*, the possibilities for such theories are unlimited. There are, however, certain classes of theory that are singled out as being rather distinguished, if one imposes further requirements based on symmetry principles. The most important of these is supersymmetry, which is a symmetry that rotates bosons and fermions into each other. These theories are very tightly constrained, especially when the number of supersymmetry transformation parameters is large. Supersymmetric theories including gravity are known as supergravities, and in four dimensions they exist with up to $N = 8$ supersymmetries.

One can also consider theories in more than four dimensions. A particular kind of new possibility now arises, in which one generalises from Maxwell theory, which has a 2-index antisymmetric tensor field strength, to fields with a larger number of antisymmetric indices. It is really only when one is in higher dimensions that this gives genuinely "new" possibilities, because one can always dualise an n-index antisymmetric tensor field strength to one with $(D - n)$ indices in D dimensions, using Hodge duality. Thus in some sense it is really only in $D \geq 6$ that one encounters genuinely new possibilities for field strengths with $n \geq 3$ antisymmetric indices. It should be remarked, however, that there are certain subtleties about dualisation which mean that one cannot be too glib about replacing fields by their duals; we shall see more on this later.

Considering higher-dimensional theories, and theories with higher-degree antisymmetric tensor field strengths, is not just an idle academic exercise. Such theories are in fact precisely what arise in string theory and in M-theory, which are at present our best candidates for providing a unified description of all the fundamental forces in nature. For example, the effective low-energy limit of M-theory is an eleven-dimensional field theory whose bosonic sector comprises the metric tensor and a 4-index antisymmetric tensor field strength. The entire low-energy theory contains a fermionic field of spin $\frac{3}{2}$ as well, and together with the bosonic fields gives rise to the long-known theory of eleven-dimensional supergravity. If we concentrate just on the bosons, the equations of motion can be derived from the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\,\Big(R - \tfrac{1}{48} F_{MNPQ}\, F^{MNPQ}\Big) + \frac{1}{20736}\, \epsilon^{M_1\cdots M_{11}}\, F_{M_1\cdots M_4}\, F_{M_5\cdots M_8}\, A_{M_9\cdots M_{11}}\ , \quad (7.1)$$

where as a 4-form, $F = dA$. In terms of indices, $F_{MNPQ} = 4\partial_{[M}\, A_{NPQ]}$.

Two things are evident from the above discussion. Firstly, if the eleven-dimensional theory, or string theories in ten dimensions, are truly funamental, then we should be interested in all their predictions and consequences, including solutions in the higher dimensions. Secondly, especially if we hope that one day they may allow us to describe our four-dimensional world, we need to have a way of extracting four-dimensional physics from higher-dimensional theories. A satisfactory by-product of learning how to perform dimensional reduction is that we find that many of the lower-dimensional theories that we wish to consider are derivable from simpler theories in a higher dimension. For example, the four-dimensional $N = 8$ supergravity mentioned above can be derived by dimensional reduction from eleven-dimensional supergravity. Contrary to what one might have thought, things are immensely simpler in eleven dimensions than in four, and so this provides a very useful way of learning about the four-dimensional theory.

To begin, therefore, let us make a preliminary study of how dimensional reduction works. This will lead us on to a number of topics that will develop in various directions, including the study of complex manifolds and Kähler geometry, and a study of coset spaces and non-linear sigma models. Our first step, though, will be a relatively humble one, where we perform a dimensional reduction in which the spacetime dimension is reduced by 1. This is the original example considered by Kaluza and Klein, and although there have been many developments and advances since their days, the general procedure for dimensional reduction bears their names.

7.1 Kaluza-Klein reduction on S^1

The higher-dimensional theories that we shall consider will all be theories of gravity plus additional fields, and so a good starting point is to study how the dimensional reduction of gravity itself proceeds. In fact this is really the hardest part of the calculation, and so once this is done the rest will be comparitively simple.

Let us assume that we are starting from Einstein gravity in $(D+1)$ dimensions, described by the Einstein-Hilbert Lagrangian

$$\mathcal{L} = \sqrt{-\hat g}\hat R\ , \quad (7.2)$$

where as usual $\hat{R}$ is the Ricci scalar and $\hat{g}$ denotes the determinant of the metric tensor. We put hats on the fields to signify that they are in $(D+1)$ dimensions. Now suppose that we wish to reduce the theory to D dimensions, by "compactifying" one of the coordinates on a circle, S^1, of radius L. Let this coordinate be called z. In principle, we could simply now expand all the components of the $(D+1)$-dimensional metric tensor as Fourier series of the form

$$\hat{g}_{MN}(x,z) = \sum_n g^{(n)}_{MN}(x)\, e^{\mathrm{i}\, n\, z/L}\ , \tag{7.3}$$

where we use x to denote collectively the D coordinates of the lower-dimensional spacetime. If one does this, one gets an infinite number of fields in D dimensions, labelled by the Fourier mode number n.

It turns out that the modes with $n \neq 0$ are associated with massive fields, while those with $n = 0$ are massless. The basic reason for this can be seen by considering a simpler toy example, of a massless scalar field $\hat{\phi}$ in flat $(D+1)$-dimensional space. It satisfies

$$\hat{\Box}\,\hat{\phi} = 0\ , \tag{7.4}$$

where $\hat{\Box} = \partial^M\,\partial_M$. Now if we Fourier expand $\hat{\phi}$ after compactifying the coordinate z, so that

$$\hat{\phi}(x,z) = \sum_n \phi_n(x)\, e^{\mathrm{i}\, n\, z/L}\ , \tag{7.5}$$

then we immediately see that the lower-dimensional fields $\phi_n(x)$ will satisfy

$$\Box\phi_n - \frac{n^2}{L^2}\,\phi_n = 0\ . \tag{7.6}$$

This is the wave equation for a scalar field of mass $|n|/L$.

The usual Kaluza-Klein philosophy is to assume that the radius L of the compactifying circle is very small (otherwise we would see it!), in which case the masses of the the non-zero modes will be enormous. (By small, we mean that L is roughly speaking of order the Planck length, 10^{-33} centimetres, so that the non-zero modes will have masses of order the Planck mass, 10^{-5} grammes.) Thus unless we were working with accelerators way beyond even intergallactic scales, the energies of particles that we ever see would be way below the scales of the Kaluza-Klein massive modes, and they can safely be neglected. Thus usually, when one speaks of Kaluza-Klein reduction, one has in mind a compactification together with a truncation to the massless sector. At least in a case such as our compactification on S^1, this truncation is *consistent*, in a manner that we shall elaborate on later.

Our Kaluza-Klein reduction ansatz, then, will simply be to take $\hat{g}_{MN}(x,z)$ to be independent of z. The main point now is that from the D-dimensional point of view, the index

M, which runs over the $(D+1)$ values of the higher dimension, splits into a range lying in the D lower dimensions, or it takes the value associated with the compactified dimension z. Thus we may denote the compoonents of the metric $\hat{g}_{MN}$ by $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$. From the D-dimensional viewpoint these look like a 2-index symmetric tensor (the metric), a 1-form (a Maxwell potential) and a scalar field respectively.

We could simply define $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$ to be the D-dimensional fields $g_{\mu\nu}$, $\mathcal{A}_\mu$ and ϕ respectively. There is nothing logically wrong with doing this, and it would give perfectly correct lower-dimensional equations of motion. However, as a parameterisation this simple-looking choice is actually very unnatural, and the equations of motion that result look like a dog's breakfast. The reason is that this naive parameterisation pays no attention to the underlying symmetries of the theory. A much better way to parameterise things is as follows. We write the $(D+1)$ dimensional metric in terms of D-dimensional fields $g_{\mu\nu}$, $\mathcal{A}_\mu$ and ϕ as follows:

$$d\hat{s}^2 = e^{2\alpha\phi}\, ds^2 + e^{2\beta\phi}\, (dz + \mathcal{A})^2 \ , \tag{7.7}$$

where α and β are constants that we shall choose for convenience in a moment, and $\mathcal{A} = \mathcal{A}_\mu\, dx^\mu$. All the fields on the right-hand side are independent of z. Note that this ansatz means that the components of the higher-dimensional metric $\hat{g}_{MN}$ are given in terms of the lower-dimensional fields by

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi}\, g_{\mu\nu} + e^{2\beta\phi}\, \mathcal{A}_\mu\, \mathcal{A}_\nu \ , \qquad \hat{g}_{\mu z} = e^{2\beta\phi}\, \mathcal{A}_\mu \ , \qquad \hat{g}_{zz} = e^{2\beta\phi} \ . \tag{7.8}$$

Thus as long as we choose $\beta \neq 0$, this will adequately parameterise the higher-dimensional metric.

To proceed, we make a convenient choice of vielbein basis, namely

$$\hat{e}^a = e^{\alpha\phi}\, e^a \ , \qquad \hat{e}^z = e^{\beta\phi}\, (dz + \mathcal{A}) \ . \tag{7.9}$$

(One should pause here, to take note of exactly which is a vielbein, and which is an exponential! We are using latin letters a, b, *etc.* to denote tangent-space indices in D dimensions. The use of z as the index associated with the extra dimension wil not, hopefully, create too much confusion. Thus $\hat{e}^z$ here means the z component of the $(D+1)$-dimensional vielbein.) Notice, by the way, that if we had chosen the "naive" identification of D-dimensional fields mentioned above, we would have been hard-pressed to come up with any way of writing down a vielbein basis; it would be possible, of course, but it would have been messy.)

Using the formalism of chapter 2, it is now a mechanical, if slightly tedious, exercise to compute the spin connection, and then the curvature. Our goal is to express the $(D+$

1)-dimensional quantities in terms of the D-dimensional ones, so that eventually we can express the $(D+1)$-dimensional Einstein-Hilbert Lagrangian in terms of a D-dimensional Lagrangian. For the spin connection, one finds that

$$\begin{aligned}
\hat{\omega}^{ab} &= \omega^{ab} + \alpha\, e^{-\alpha\phi}\, (\partial^b\phi\, \hat{e}^a - \partial^a\phi\, \hat{e}^b) - \tfrac{1}{2}\mathcal{F}^{ab}\, e^{(\beta-2\alpha)\phi}\, \hat{e}^z\ , \\
\hat{\omega}^{az} &= -\hat{\omega}^{za} = -\beta\, e^{-\alpha\phi}\, \partial^a\phi\, \hat{e}^z - \tfrac{1}{2}\mathcal{F}^a{}_b\, e^{(\beta-2\alpha)\phi}\, \hat{e}^b\ ,
\end{aligned} \tag{7.10}$$

where $\partial_a\phi$ means $E^\mu_a\, \partial_\mu\phi$, and E^μ_a is the inverse of the D-dimensional vielbein $e^a = e^a_\mu\, dx^\mu$. Also, $\mathcal{F}_{ab}$ denotes the vielbein components of the D-dimensional field strength $\mathcal{F} = d\mathcal{A}$.

The calculation of the curvature 2-forms proceeds uneventfully. Rather than present all the formulae here, we shall just present the key results. Firstly, we can exploit our freedom to choose the values of the constants α and β in the metric ansatz in the following way. There are two things that we would like to achieve, one of which is to ensure that the dimensionally-reduced Lagrangian is of the Einstein-Hilbert form $\mathcal{L} = \sqrt{-g}\, R + \cdots$. If the values of α and β are left unfixed, we instead end up with $\mathcal{L} = e^{(\beta+(D-2)\alpha)\phi}\, \sqrt{-g}\, R + \cdots$. Thus we immediately see that we should choose $\beta = -(D-2)\alpha$. Provided we are not reducing down to $D = 2$ dimensions, this will not present any problem. The other thing that we would like is to ensure that the scalar field ϕ acquires a kinetic term with the canonical normalisation, meaning a term of the form $-\frac{1}{2}\sqrt{-g}\, (\partial\phi)^2$ in the Lagrangian. This determines the choice of overall scale, and it turns out that we should choose our constants as follows:

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}\ , \qquad \beta = -(D-2)\, \alpha\ . \tag{7.11}$$

With these choices for the constants in the metric ansatz, we can now present the results for the vielbein components of the Ricci tensor:

$$\begin{aligned}
\hat{R}_{ab} &= e^{-2\alpha\phi}\Big(R_{ab} - \tfrac{1}{2}\partial_a\phi\, \partial_b\phi - \alpha\, \eta_{ab}\, \square\, \phi\Big) - \tfrac{1}{2}e^{-2D\alpha\phi}\, \mathcal{F}_a{}^c\, \mathcal{F}_{bc}\ , \\
\hat{R}_{az} &= \hat{R}_{za} = \tfrac{1}{2}e^{(D-3)\alpha\phi}\, \nabla^b\Big(e^{-2(D-1)\alpha\phi}\, \mathcal{F}_{ab}\Big)\ , \\
\hat{R}_{zz} &= (D-2)\, a\, e^{-2\alpha\phi}\, \square\, \phi + \tfrac{1}{4}e^{-2D\alpha\phi}\, \mathcal{F}^2\ ,
\end{aligned} \tag{7.12}$$

where $\mathcal{F}^2$ means $\mathcal{F}_{ab}\, \mathcal{F}^{ab}$. From these, it follows that the Ricci scalar $\hat{R} = \eta^{AB}\, \hat{R}_{AB} = \eta^{ab}\, \hat{R}_{ab} + \hat{R}_{zz}$ is given by

$$\hat{R} = e^{-2\alpha\phi}\Big(R - \tfrac{1}{2}(\partial\phi)^2 + (D-3)\alpha\, \square\, \phi\Big) - \tfrac{1}{4}e^{-2D\alpha\phi}\, \mathcal{F}^2\ . \tag{7.13}$$

Now, finally, we calculate the determinant of the metric $\hat{g}$ in terms of the determinant of g, from the ansatz (7.7), finding

$$\sqrt{-\hat{g}} = e^{(\beta+D\alpha)\phi}\, \sqrt{-g} = e^{2\alpha\phi}\, \sqrt{-g}\ , \tag{7.14}$$

where the second equality follows using our relation between β and α given in (7.11). Putting all the results together, we see that the dimensional reduction of the higher-dimensional Einstein-Hilbert Lagrangian gives

$$\mathcal{L} = \sqrt{-\hat{g}}\,\hat{R} = \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\phi}\,\mathcal{F}^2\Big)\,, \tag{7.15}$$

where we have dropped the $\Box\,\phi$ term in (7.13) since it just gives a total derivative in $\mathcal{L}$, which therefore does not contribute to the field equations. In modern parlance, the scalar field ϕ is called a dilaton.

If the scalar field in (7.15) were set to zero, we would simply have the Einstein-Maxwell Lagrangian in D dimensions. This is in fact what Kaluza and Klein originally did (so one is told; no living soul has ever actually looked at their papers). They were interested, of course, in the idea that a unification of Einstein's theory of gravity and Maxwell's electrodynamics could be achieved by reformulating them as pure gravity in five dimensions. However, it is not actually allowed to set the scalar field to zero; this would be in conflict with the field equation for ϕ. To see this, and for general future reference, let us pause to work out the field equations coming from (7.15). They are

$$\begin{aligned}
R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} &= \tfrac{1}{2}\Big(\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}(\partial\phi)^2\,g_{\mu\nu}\Big) + \tfrac{1}{2}e^{-2(D-1)\alpha\phi}\,\Big(\mathcal{F}^2_{\mu\nu} - \tfrac{1}{4}\mathcal{F}^2\,g_{\mu\nu}\Big)\,,\\
\nabla^\mu\Big(e^{-2(D-1)\alpha\phi}\,\mathcal{F}_{\mu\nu}\Big) &= 0\,,\\
\Box\,\phi &= -\tfrac{1}{2}(D-1)\alpha\,e^{-2(D-1)\alpha\phi}\,\mathcal{F}^2\,,
\end{aligned} \tag{7.16}$$

where we have defined $\mathcal{F}^2_{\mu\nu} = \mathcal{F}_{\mu\rho}\,\mathcal{F}_\nu{}^\rho$. Actually, it is usually more convenient to eliminate the $-\frac{1}{2}R\,g_{\mu\nu}$ term in the Einstein equation, by subracting out the appropriate multiple of the trace, so that we get

$$R_{\mu\nu} = \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}e^{-2(D-1)\alpha\phi}\,\Big(\mathcal{F}^2_{\mu\nu} - \frac{1}{2(D-2)}\,\mathcal{F}^2\,g_{\mu\nu}\Big)\,. \tag{7.17}$$

We see from the last equation in (7.16) that one cannot in general set $\phi = 0$, since there is a source term on the right-hand side of the equation, involving $\mathcal{F}^2$. In other words, the details of the *interactions* between the various lower-dimensional fields prevent the truncation of the scalar ϕ. Thus it is an Einstein-Maxwell-Scalar system that comes from the consistent dimensional reduction of the higher-dimensional pure Einstein theory. One would not notice this subtlety if one simply made the ansatz (7.7) but with $\phi = 0$, and plugged the resulting Ricci scalar into the higher-dimensional Einstein-Hilbert Lagrangian. What one would be failing to notice is that such an ansatz would be inconsistent with the *higher-dimensional* equations of motion, specifically, with the $\hat{R}_{zz}$ component of the higher-dimensional Einstein equation. Neglecting some of the content of the higher-dimensional equations of motion is, from a modern viewpoint, a philosophically unattractive thing to do, since it would be denying the fundamental significance of the higher-dimensional theory. Thus the transgression of Kaluza and Klein is at least deserving of censure, even if it does not rise to the level of an impeachable offence.

After this little cautionary tale, one might wonder whether we ourselves might be guilty of exactly the same offence. Recall that early on, we set all the non-zero modes in the Fourier expansion (7.3) of the metric to zero. Suppose we had kept them instead, and eventually worked out the analogue of (7.15) with the entire infinite towers of massive as well as massless fields. Might we not have found that the equations of motion of the massive fields would forbid us from setting them to zero? The answer is that a little bit of (elementary) group theory saves us. The mode functions $e^{im\,n\,z/L}$ in the Fourier expansion (7.3) are representations of the $U(1)$ group of the circle S^1. The mode $n = 0$ is a singlet, while the non-zero modes are all doublets, in the sense that the modes with numbers n and $-n$ are complex conjugates of each other. When we truncated out all the non-zero modes, what we were doing was keeping all the group singlets, and throwing out all the non-singlets. This is guaranteed to be a consistent truncation, since no amount of multiplying group singlets together can ever generate non-singlets. To put it another way, the label n is like a $U(1)$ charge, and there is a charge-conservation law that must be obeyed. Each term in field equation for any particular field labelled by n will necessarily have net charge equal to n, and so at least one factor in each term in the equation must have non-zero charge whenever n is non-zero. Thus provided we truncate out *all* the non-zero modes, the consistency is guaranteed.

In more complicated Kaluza-Klein reductions, where the compactifying manifold is not simply a circle or a product of circles (a torus), the issue of the consistency of the truncation to the massless sector is a much more tricky one. It is a question that is usually ignored by those who do compactifications on K3 or Calabi-Yau manifolds, but there is always a lurking suspicion (or hope?) that one day their sins will catch up with them.

Having obtained the lower-dimensional theory described by the Lagrangian (7.15), we could go on to study spherically-symmetric black-hole solutions, and so on. This is actually a very important subject, but we shall postpone looking into it for a while. For now, let us continue with the easier part of a Kaluza-Klein dimensional reduction, where we see what happens when an antisymmetric tensor field strength is reduced from $(D+1)$ to D dimensions. Suppose we have an n-index field strength in the higher dimension, which we denote by $\hat{F}_{(n)}$. Suppose, furthermore, that this is given in terms of a potential $\hat{A}_{(n-1)}$, so that $\hat{F}_{(n)} = d\hat{A}_{(n-1)}$. In terms of indices, it is clear that after reduction on S^1 there will be to kinds of D-dimensional potentials, namely one with all $(n-1)$ indices lying in the D-dimensional spacetime, and the other with $(n-2)$ indices lying in the D-dimensional spacetime, and the remaining index being in the direction of the S^1. This is most easily expressed in terms of differential forms. Thus the ansatz for the reduction of the potential is

$$\hat{A}_{(n-1)}(x,z) = A_{(n-1)}(x) + A_{(n-2)}(x) \wedge dz \ . \tag{7.18}$$

Now, let us calculate the field strength. Clearly, we will have

$$\hat{F}_{(n)} = dA_{(n-1)} + dA_{(n-2)} \wedge dz \ . \tag{7.19}$$

One might naively be tempted to identify $dA_{(n-1)}$ and $dA_{(n-2)}$ as the lower-dimensional field strengths $F_{(n)}$ and $F_{(n-1)}$. There is nothing logically wrong with doing so, but it is not a very convenient choice. Much better is to add and subtract a term in (7.19), so that we get

$$\begin{aligned} \hat{F}_{(n)} &= dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} + dA_{(n-2)} \wedge (dz + \mathcal{A}_{(1)}) \ , \\ &\equiv F_{(n)} + F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)}) \ , \end{aligned} \tag{7.20}$$

where $\mathcal{A}_{(1)}$ is the Kaluza-Klein potential that we encountered in the metric reduction. We have appended a subscript $_{(1)}$ to it now, in keeping with our general notation to indicate the degrees of differential forms. Thus the D-dimensional field strengths are given by

$$F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} \ , \qquad F_{(n-1)} = dA_{(n-2)} \ . \tag{7.21}$$

This is in a sense a purely notational change from the "naive" choice mentioned above; it is entirely up to us to decide what particular combination of quantities will be dignified with the name $F_{(n)}$. The point is that the specific choice in (7.21) has a particular significance, which becomes apparent when we calculate the higher-dimensional kinetic term $\hat{F}_{(n)}^2$ in terms of the lower-dimensional fields.[2] The calculation is most easily done in the vielbein basis, since then the metric is just the diagonal one η_{AB}. Consequently, in view of the definition of the vielbeins in (7.9), the vielbein components of the $(n-1)$-form field strength in D dimensions will be the ones where the n'th index is a vielbein z index, not a coordinate z index, meaning that we should read off $F_{(n-1)}$ from $F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)})$, and not from

[2]The mathematicians have, curiously, attached the name "transgression" to the process by which these extra modifications to field strengths arise. The etymology is unclear, but here there is not meant to be any connotation even of a matter worthy of censure.

$F_{(n-1)} \wedge dz$. It is now easily seen from (7.9) and (7.21) that in terms of vielbein components we shall have

$$\begin{aligned}
\hat{F} &= \frac{1}{n!}\, \hat{F}_{A_1\cdots A_n}\, \hat{e}^{A_1} \wedge \cdots \wedge \hat{e}^{A_n} \\
&= \frac{e^{n\alpha\phi}}{n!}\, \hat{F}_{a_1\cdots a_n}\, e^{a_1} \wedge \cdots \wedge e^{a_n} + \frac{e^{((n-1)\alpha+\beta)\phi}}{(n-1)!}\, \hat{F}_{a_1\cdots a_{n-1} z}\, e^{a_1} \wedge \cdots \wedge e^{a_{n-1}} \wedge (dz + \mathcal{A}_{(1)})\ , \\
&\equiv \frac{1}{n!}\, F_{a_1\cdots a_n}\, e^{a_1} \wedge \cdots \wedge e^{a_n} + \frac{1}{(n-1)!}\, F_{a_1\cdots a_{n-1}}\, e^{a_1} \wedge \cdots \wedge e^{a_{n-1}} \wedge (dz + \mathcal{A}_{(1)})\ , \qquad (7.22)
\end{aligned}$$

implying that

$$\hat{F}_{a_1\cdots a_n} = e^{-n\alpha\phi}\, F_{a_1\cdots a_n}\ , \qquad \hat{F}_{a_1\cdots a_{n-1} z} = e^{(D-n-1)\alpha\phi}\, F_{a_1\cdots a_{n-1}}\ , \qquad (7.23)$$

where we have used (7.11) to express β in terms of α. It is now easy to see, bearing in mind the relation (7.14) between the determinants of the metrics in $(D+1)$ and D dimensions, that the kinetic term for the $(D+1)$-dimensional n-form field strength $\hat{F}_{(n)}$ will give, upon Kaluza-Klein reduction to D dimensions,

$$\mathcal{L} = -\tfrac{\sqrt{-\hat{g}}}{2\, n!}\, \hat{F}^2_{(n)} = -\tfrac{\sqrt{-g}}{2\, n!}\, e^{-2(n-1)\alpha\phi}\, F^2_{(n)} - \tfrac{\sqrt{-g}}{2\,(n-1)!}\, e^{2(D-n)\alpha\phi}\, F^2_{(n-1)}\ . \qquad (7.24)$$

At this point, let us pause for a moment in order to find a nicer way to present the Lagrangians that we are encountering. There are two reasons for doing so; firstly, on general aesthetic grouds, but also, and more importantly, to make the process of varying the Lagrangian to obtain the equations of motion as simple and straightforward as possible. The advantage of doing this is already evident if we consider what happens when we want to vary the reduced Lagrangian (7.24) with respect to the potential $A_{(n-2)}$. Not only does this potential appear in its "own" field strength $F_{(n-1)}$, but it also appears in the "transgression" term in $F_{(n)}$ (see equation (7.21)). Already in this example, therefore, it is apparent that getting the right signs, combinatoric factors, *etc.* when working out the equation of motion in index notation will be a tedious and wearisome business. It is highly preferrable to be able to work with the language of differential forms.

Recall that we define the Hodge dual of the basis for p-forms in D dimensions by

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) \equiv \frac{1}{q!}\, \epsilon_{\nu_1\cdots\nu_q}{}^{\mu_1\cdots\mu_p}\, dx^{\nu_1} \wedge dx^{\nu_q}\ , \qquad (7.25)$$

where $q = D - p$. Here, $\epsilon_{\mu_1\cdots\mu_D}$ is the totally antisymmetric Levi-Civita tensor, whose components are $\pm\sqrt{|g|}$ or 0, given by

$$\epsilon_{\mu_1\cdots\mu_D} = \sqrt{|g|}\, \varepsilon_{\mu_1\cdots\mu_D}\ , \qquad (7.26)$$

where $\varepsilon_{\mu_1\cdots\mu_D}$ is the totally antisymmetric Levi-Civita tensor *density*, with

$$\varepsilon_{\mu_1\cdots\mu_D} \equiv (+1,-1,0) \tag{7.27}$$

according to whether $\mu_1\cdots\mu_D$ is an *even* permutation of the canonically-ordered set of index values, an *odd* permutation, or no permutation at all. A natural canonical ordering of indices would be $0,1,2,\ldots$, but it is, of course, ultimatley a matter of pure convention. It is also sometimes useful to define a totally antisymmetric tensor density with upstairs indices, and components given numerically by

$$\varepsilon^{\mu_1\cdots\mu_D} \equiv (-1)^t\,\varepsilon_{\mu_1\cdots\mu_D}\ , \tag{7.28}$$

where t is the number of timelike coordinates. Note that this is the *one and only* time that we ever introduce a pair of objects for which we use the same symbol, but where the one with upstairs indices is not obtained by raising the indices on the one with downstairs indices using the metric. In terms of $\varepsilon^{\mu_1\cdots\mu_D}$, the Levi-Civita *tensor* with upstairs indices is given by

$$\epsilon^{\mu_1\cdots\mu_D} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{\mu_1\cdots\mu_D}\ . \tag{7.29}$$

This, of course, *is* obtained from $\epsilon_{\mu_1\cdots\mu_D}$ simply by raising the indices using the metric.

It is easy to see from the definition (7.25) that if we apply the Hodge dual to a p-form A, we get a $(D-p)$-form $B=*A$ with components given by

$$B_{\mu_1\cdots\mu_q} = \frac{1}{p!}\,\epsilon_{\mu_1\cdots\mu_q}{}^{\nu_1\cdots\nu_p}\,A_{\nu_1\cdots\nu_p}\ , \tag{7.30}$$

where $q\equiv D-p$. (Note the order in which the indices appear on the epsilon tensors in (7.25) and (7.30).) As a partiucular case, we see that the Hodge dual of the pure number 1 (a 0-form) is the D-form whose components are the Levi-Civita tensor, and thus we may write

$$\begin{aligned} *1 &= \epsilon = \tfrac{1}{D!}\,\epsilon_{\mu_1\cdots\mu_D}\,dx^{\mu_1}\cdots dx^{\mu_D}\ ,\\ &= \sqrt{|g|}\,dx^0\cdots dx^{D-1} = \sqrt{|g|}\,d^Dx\ . \end{aligned} \tag{7.31}$$

Thus $*1$ is nothing but the generally coordinate invariant volume element. Note that owing to the tiresome, but unavoidable, $(-1)^t$ factor in (7.28), we have

$$dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} = (-1)^t\,\varepsilon^{\mu_1\cdots\mu_D}\,d^Dx = (-1)^t\,\epsilon^{\mu_1\cdots\mu_D}\,\sqrt{|g|}\,d^Dx\ . \tag{7.32}$$

From the above definitions, the following results follow straightforwardly. If A and B are any two p-forms, then

$$*A\wedge B = *B\wedge A = \frac{1}{p!}\,|A\cdot B|\,\epsilon = \frac{1}{p!}\,|A.B|\,*1\ , \tag{7.33}$$

where

$$|A \cdot B| \equiv A_{\mu_1 \cdots \mu_p} B^{\mu_1 \cdots \mu_p} \ , \tag{7.34}$$

is the inner product of A and B. Also, applying $*$ twice, we have that if A is any p-form, then

$$* * A = (-1)^{pq+t} A \ , \tag{7.35}$$

where as usual we define $q \equiv D - p$.

A Lagrangian density $\mathcal{L}$ is something which is to be multiplied by $d^D x$ and then integrated over the spacetime manifold to get the action. For example, the Einstein-Hilbert Lagrangian density is $\sqrt{-g}\, R$, and this is integrated to give $\int R \sqrt{-g}\, d^D x$. From a differential-geometric point of view, it is really not 0-forms, but rather D-forms, that can be integrated over a D-dimensional manifold. Thus we can really think of the Einstein-Hilbert action as being obtained by integrating the D-form $R *1$ over the manifold. This is a convenient way to think of things, and so typically, from now on, when we speak of a Lagrangian we will mean the D-form whose integral gives the action.

It is now easily seen from the previous definitions that the D-form Lagrangian corresponding to the circle reduction of the Einstein-Hilbert Lagrangian, which we obtained in the "traditional" language in (7.15), is given by

$$\mathcal{L} = R *1 - \tfrac{1}{2} {*d\phi} \wedge d\phi - \tfrac{1}{2} e^{-2(D-1)\alpha\phi} *\mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} \ , \tag{7.36}$$

where we have put a $_{(2)}$ subscript on the Maxwell field strength to remind us that it is a 2-form. Similarly, we see that the Lagrangian (7.24) becomes, when written as a D-form,

$$\mathcal{L} = -\tfrac{1}{2} e^{-2(n-1)\alpha\phi} *F_{(n)} \wedge F_{(n)} - \tfrac{1}{2} e^{2(D-n)\alpha\phi} *F_{(n-1)} \wedge F_{(n-1)} \ . \tag{7.37}$$

Note that the previous $n!$ combinatoric denominator, associated with the kinetic term for an n-form field strength, is nicely eliminated in the Lagrangians written as differential forms.

It is now a completely straightforward matter to vary the Lagrangian for any gauge field, and to get the combinatorics and signs correct without headaches. The only rule one ever needs, apart fom the usual ones for carrying differential forms over each other, is that the variation of an expression of the form $X_{(p)} \wedge dA_{(q)}$ with respect to $A_{(q)}$ gives, after integration by parts, $-(-1)^p\, dX_{(p)} \wedge A_{(q)}$, when $X_{(p)}$ is a p-form. This is just the usual minus sign coming from integration by parts, accompanied by an additional $(-1)^p$ factor coming from the fact that the exterior derivative has to be taken over a p-form.

For example, if we look at the equations of motion coming from varying the Lagrangian (7.37) with respect to the potential $A_{(n-1)}$ we get

$$\delta\mathcal{L} = -e^{-2(n-1)\alpha\phi} {*F_{(n)}} \wedge d\delta A_{(n-1)} \longrightarrow (-1)^{D-n}\, d\Big(e^{-2(n-1)\alpha\phi} {*F_{(n)}}\Big) \wedge \delta A_{(n-1)}\ , \qquad (7.38)$$

where the arrow indicates that the result is obtained after integration by parts. Varying instead with respect to $A_{(n-2)}$ gives

$$\begin{aligned}\delta\mathcal{L} &= -e^{2(D-n)\alpha\phi} {*F_{(n-1)}} \wedge d\delta A_{(n-2)} + e^{-2(n-1)\alpha\phi} {*F_{(n)}} \wedge d\delta A_{(n-2)} \wedge \mathcal{A}_{(1)}\ , \qquad (7.39)\\ &\longrightarrow (-1)^{D-n+1}\, d\Big(e^{2(D-n)\alpha\phi} {*F_{(n-1)}}\Big) \wedge \delta A_{(n-2)} - (-1)^D\, d\Big(e^{-2(n-1)\alpha\phi} {*F_{(n)}} \wedge \mathcal{A}_{(1)}\Big)\, \delta A_{(n-2)}\ .\end{aligned}$$

The first lesson to note from this example is that when varying an expression such as $-\frac{1}{2}{*F_{(n)}} \wedge F_{(n)}$ that is quadratic in $F_{(n)}$, the terms coming from varying the potentials in each $F_{(n)}$ always simply add up, nicely removing the $\frac{1}{2}$ prefactor. The second lesson is that the chief remaining subtleties in varying Lagrangains are associated with the occurrence of the transgression terms in the various field strengths, as we have here in the definition of $F_{(n)}$ in (7.21). Having now got the variation expressed as $\delta\mathcal{L} = X \wedge \delta A$ for some X, one simply reads off the field equation as $X = 0$. In our example here, note that the field equation for $F_{(n)}$ can be used to simplify the field equation for $F_{(n-1)}$, leading simply to

$$\begin{aligned} d\Big(e^{-2(n-1)\alpha\phi} {*F_{(n)}}\Big) &= 0\ ,\\ d\Big(e^{2(D-n)\alpha\phi} {*F_{(n-1)}}\Big) + (-1)^D\, e^{-2(n-1)\alpha\phi} {*F_{(n)}} \wedge \mathcal{F}_{(2)} &= 0\ . \qquad (7.40)\end{aligned}$$

7.2 Lower-dimensional symmetries from the S^1 reduction

In the case where we started just from pure Einstein garvity in $(D+1)$ dimensions, we ended up with an Einstein-Maxwell-Scalar system in D dimensions. Thus the higher-dimensional theory had general coordinate covariance, while the lower-dimensional one has general coordinate covariance and the local $U(1)$ gauge invariance of the Maxwell field. In fact, as can be seen from (7.15), it also has another symmetry, namely a constant shift of the dilaton field ϕ, accompanied by an appropriate constant scaling of the Maxwell potential:

$$\phi \longrightarrow \phi + c\ , \qquad \mathcal{A}_\mu \longrightarrow e^{c(D-1)\alpha}\, \mathcal{A}_\mu\ . \qquad (7.41)$$

At first sight, therefore, one might think that the lower-dimensional theory had more symmetry than the higher-dimensional one. Of course this is not really the case; the point is that the local general coordinate symmetry in the higher dimension involves coordinate reparameterisations by arbitrary functions of $(D+1)$ coordinates, while the local general coordinate and $U(1)$ gauge transformations in the lower dimension involve arbitrary functions of only D coordinates. Thus in effect the symmetries of the D-dimensional theory really constitute only an infinitesimal residue of the $(D+1)$-dimensional general coordinate symmetries. We can understand this better by looking in detail at the Kaluza-Klein reduction ansatz (7.7) for the $(D+1)$-dimensional metric.

The original $(D+1)$-dimensional Einstein theory is invariant under general coordinate transformations, which can be written (see section 5.1) in infinitesimal form as

$$\delta\hat{x}^M = -\hat{\xi}^M\ , \qquad \delta\hat{g}_{MN} = \hat{\xi}^P\,\partial_P\,\hat{g}_{MN} + \hat{g}_{PN}\,\partial_M\,\hat{\xi}^P + \hat{g}_{MP}\,\partial_N\,\hat{\xi}^P\ . \tag{7.42}$$

As yet, the parameters $\hat{\xi}^M$ are arbitrary functions of all $(D+1)$ coordinates. Now, the form of the Kaluza-Klein ansatz (7.7) will not in general be preserved by such transformations. In fact, it is rather easy to see that the most general allowed form for transformations that preserve (7.7) will be

$$\hat{\xi}^\mu = \xi^\mu(x)\ , \qquad \hat{\xi}^z = c\,z + \lambda(x)\ , \tag{7.43}$$

where the $(D+1)$-dimensional index on $\hat{\xi}^M$ is split as $\hat{\xi}^\mu$ and $\hat{\xi}^z$, with μ a D-dimensional index. The coordinates $\hat{x}^M$ are split as (x^μ, z), and the x arguments on $\xi^\mu(x)$ and $\lambda(x)$ indicate that these functions depend only on the D-dimensional coordinates x^μ. The parameter c is a constant. Note that from (7.7) we have that the components of the $(D+1)$-dimensional metric $\hat{g}_{MN}$ are given in terms of the D-dimensional metric $g_{\mu\nu}$, gauge potential $\mathcal{A}_\mu$ and dilaton ϕ by

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi}\,g_{\mu\nu} + e^{2\beta\phi}\,\mathcal{A}_\mu\,\mathcal{A}_\nu\ , \qquad \hat{g}_{\mu z} = \hat{g}_{z\mu} = e^{2\beta\phi}\,\mathcal{A}_\mu\ , \qquad \hat{g}_{zz} = e^{2\beta\phi}\ , \tag{7.44}$$

where $\beta = -(D-2)\,\alpha$.

Let us look first at the local transformations, namely those parameterised by $\xi^\mu(x)$ and $\lambda(x)$ (so we take the constant $c=0$ for now). We shall see that these are the parameters of D-dimensional general coordinate transformations, and $U(1)$ gauge transformations, respectively. Under these transformations, we see first from (7.42) that

$$\delta\hat{g}_{zz} = \xi^\rho\,\partial_\rho\hat{g}_{zz} + \hat{g}_{zz}\ , \tag{7.45}$$

where we have dropped those terms that give zero by virtue either of the form of the metric ansatz (7.7), or by our assumption for now that c is zero. From (7.44), we thus deduce that

$$\delta\phi = \xi^\rho\,\partial_\rho\,\phi\ , \tag{7.46}$$

implying that ϕ is indeed transforming as a scalar under the D-dimensional general coordinate transformations parameterised by ξ^μ, and that is it inert (as it should be) under the $U(1)$ gauge transformations parameterised by λ.

Next, looking at the (μz) components in (7.42), we see that

$$\delta\hat{g}_{\mu z} = \xi^\rho\,\partial_\rho\,\hat{g}_{\mu z} + \hat{g}_{\rho z}\,\partial_\mu\,\xi^\rho\ . \tag{7.47}$$

Substituting from (7.44), what we already learned about the transformations of ϕ, we deduce that $\mathcal{A}_\mu$ transformas as

$$\delta\mathcal{A}_\mu = \xi^\rho\,\partial_\rho\,\mathcal{A}_\mu + \mathcal{A}_\rho\,\partial_\mu\,\xi^\rho + \partial_\mu\,\lambda\ . \tag{7.48}$$

This shows that $\mathcal{A}_\mu$ transformas properly as a covector under general coordinate transformations ξ^ρ, and that it has the usual gauge transformation of a $U(1)$ gauge field, under the parameter λ.

Finally, looking at the $(\mu\nu)$ components in (7.42), we have

$$\delta\hat{g}_{\mu\nu} = \xi^\rho\,\partial_\rho\,\hat{g}_{\mu\nu} + \hat{g}_{\rho\nu}\,\partial_\mu\,\xi^\rho + \hat{g}_{\mu\rho}\,\partial_\nu\,\xi^\rho + \hat{g}_{z\nu}\,\partial_\mu\,\xi^z + \hat{g}_{\mu z}\,\partial_\nu\,\xi^z\ . \tag{7.49}$$

Using what we have now learned about the transformation rules for ϕ and $\mathcal{A}_\mu$, we find, after substituting from (7.44) that

$$\delta g_{\mu\nu} = \xi^\rho\,\partial_\rho\,g_{\mu\nu} + g_{\rho\nu}\,\partial_\mu\,\xi^\rho + g_{\mu\rho}\,\partial_\nu\,\xi^\rho\ , \tag{7.50}$$

showing that the D-dimensional metric indeed has the proper transformation properties under general coordinate transformations ξ^ρ, and that it is inert, as it should be, under the $U(1)$ gauge transformations λ.

We have now taken care of the local parameters in (7.43). We have seen that the subset of the original $(D+1)$-dimensional general coordinate transformations $\hat{\xi}^M$ that preserve the form of the Kaluza-Klein metric ansatz (7.7) include the D-dimensional general coordinate transformations ξ^μ, and the D-dimensional $U(1)$ local gauge transformations of the Kaluza-Klein vector potential $\mathcal{A}_\mu$. The remaining parameter to consider is the constant c in (7.43). This is associated with the constant shift symmetry of the dilaton ϕ, given in (7.41). To see how this symmetry comes out of (7.43), we have to introduce one further ingredient in the discussion.

The higher-dimensional equations of motion, namely the Einstein equations $\hat{R}_{MN} - \frac{1}{2}\hat{R}\,\hat{g}_{MN} = 0$, actually have an additional global symmetry in addition to the local general coordinate transformations. This is a symmetry under which the metric is scaled by a constant factor, $\hat{g}_{MN} \longrightarrow k^2\, \hat{g}_{MN}$. From the definitions in chapter 2, it is easily seen that the various curvature tensors transform under this constant scaling as

$$\hat{R}^M{}_{NPQ} \longrightarrow \hat{R}^M{}_{NPQ}\ , \qquad \hat{R}_{MN} \longrightarrow \hat{R}_{MN}\ , \qquad \hat{R} \longrightarrow k^{-2}\, \hat{R}\ . \tag{7.51}$$

In other words, the Riemann tensor with its coordinate indices in their "natural" positions is inert. No metric is needed in order then to construct the Ricci tensor, $\hat{R}_{MN} = \hat{R}^P{}_{MPN}$, and so it too is inert. However, the construction of the Ricci scalar then requires the use of the inverse metric, $\hat{R} = \hat{g}^{MN}\, \hat{R}_{MN}$, and so it acquires the scaling given above in (7.51). The upshot is that the Einstein equation is actually invariant under the scaling.

The reason for discussing this scaling symmetry in terms of the equations of motion is that, as is easily seen, it is not a symmetry of the Lagrangian itself. Clearly, we will have $\sqrt{-\hat{g}} \longrightarrow k^{D+1}\sqrt{-\hat{g}}$ in $(D+1)$ dimensions, and hence the Einstein-Hilbert Lagrangian will scale as $\sqrt{-\hat{g}}\,\hat{R} \longrightarrow k^{D-1}\sqrt{-\hat{g}}\,\hat{R}$. The crucial point is, however, that this is a uniform constant scaling of the Lagrangian. Now, the equations of motion that follow from two Lagrangians that are related by a constant scale factor are the same, and hence we can understand the invariance of the equations of motion from this viewpoint too. In certain less trivial examples, notably elven-dimensional supergravity, on also finds that there is such a uniform scaling symmetry of the Lagrangian, and hence a scale-invariance of the equations of motion. It is less trivial in this example, because the various terms in the Lagrangian (7.1) must all conspire to scale the same way.

Returning now to our discussion of the symmetries of teh Kaluza-Klein reduction of $(D+1)$-dimensional Einstein theory, we have learned that there is the additional symmetry $\hat{G}_{MN} \longrightarrow k^2\, \hat{g}_{MN}$ in the original $(D+1)$-dimensional theory, where k is a constant. In infinitesimal form, this translates into the statement that $\delta\hat{g}_{MN} = 2a\, \hat{g}_{MN}$, where a is an infinitesimal constant parameter. Thus if we write out the residual general-coordinate transformations (7.43), specialised to include just the constant parameter c, and include also the scaling symmetry, we will have the following infinitesimal global symmetry:

$$\delta\hat{g}_{MN} = c\,\delta^z_M\,\hat{g}_{zN} + c\,\delta^z_N\,\hat{g}_{Mz} + 2a\,\hat{g}_{MN}\ . \tag{7.52}$$

Note that the δ symbols on the right-hand side are Kronecker deltas, non-vanishing only when the m or N index takes the $(D+1)$'th value z.

Plugging in the form of the metric ansatz (7.44), and taking (MN) to be (zz), $(z\mu)$ and $\mu\nu)$ successively, we can read off the transformation rules for ϕ, $\mathcal{A}_\mu$ and $g_{\mu\nu}$, finding

$$\beta\,\delta\phi = a + c\ , \qquad \delta\mathcal{A}_\mu = -c\,\mathcal{A}_\mu\ , \qquad \delta g - \mu\nu = 2a\, g_{\mu\nu} - 2\alpha\, g_{\mu\nu}\,\delta\phi\ . \tag{7.53}$$

It is now evident that we can use the scaling transformation a as a compensator for the dilaton-shift transformation c, in such a way that under the appropriate combined transformation the metric $g_{\mu\nu}$ is inert, i.e. $\delta g_{\mu\nu} = 0$. Clearly to to this, we should choose

$$a = -\frac{c}{D-1}\ , \tag{7.54}$$

bearing in mind that the constants α and β in the Kaluza-Klein ansatz (7.7) were chosen so that $\beta = -(D-2)\,\alpha$. Thus we arrive at the global transformation

$$\delta\phi = -\frac{c}{\alpha\,(D-1)}\ , \qquad \delta\mathcal{A}_\mu = -c\,\mathcal{A}_\mu\ , \qquad \delta g_{\mu\nu} = 0\ . \tag{7.55}$$

After a constant scaling redefinition of the parameter c, this can be seen to be precisely the dilaton shift symmetry given in (7.41).

Of course since we have just made use of a particular linear combination of the original two global symmetries, with parameters a and c related by (7.54), it follows that the "orthogonal" combination is still also a symmetry of the D-dimensional theory. This other combination is nothing but a uniform scaling symmetry of the entire D dimensional theory. What we have done by taking combinations of the a and c transformations is to diagonalise the two symmetries, one of which, given by (7.55), is a purely *internal* symmetry that leaves the lower-dimensional metric invariant and acts only on the other fields. The other combination is a scaling symmetry that acts on all fields that carry indices; in this case, on $g_{\mu\nu}$ and $\mathcal{A}_\mu$. In fact the general rule for the scaling symmetries, if they are present in a particular theory, is that each fundamental field is scaled according to the number of indices it carries:

$$g_{\mu\nu} \longrightarrow k^2\, g_{\mu\nu}\ , \qquad A_{\mu_1\cdots\mu_n} \longrightarrow k^n\, A_{\mu_1\cdots\mu_n}\ . \tag{7.56}$$

Thus in our example of the D-dimensional Lagrangian (7.15), one can easily verify that it is invariant under

$$g_{\mu\nu} \longrightarrow k^2\, g_{\mu\nu}\ , \qquad \mathcal{A}_\mu \longrightarrow k\,\mathcal{A}_\mu\ . \tag{7.57}$$

Furthermore, it is easily established from the combined transformations (7.53) that we can indeed find a combination of the parameters, namely $a = -c$, that gives (7.57) in its infinitesimal form. This is precisely the combination that leaves ϕ invariant, which is consistent with the general rule (7.56) since ϕ has no indices. These kinds of scaling transformations have been referred to as "trombone" symmetries.

To complete the story of S^1 reductions, let us consider the dimensional reduction of $D = 11$ supergravity down to $D = 10$. In our new, improved notation, the eleven-dimensional

Lagrangian can be written as the 11-form

$$\mathcal{L}_{11} = R\,{*1} - \tfrac{1}{2}{*F_{(4)}} \wedge F_{(4)} + \tfrac{1}{6} dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)} \ . \tag{7.58}$$

Substituting all the previous results, we find that we can write $\mathcal{L}_{11} = \mathcal{L}_{10} \wedge dz$, with the ten-dimensional Lagrangian given by

$$\begin{aligned}\mathcal{L}_{10} &= R\,{*1} - \tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2} e^{\frac{3}{2}\phi}\,{*\mathcal{F}_{(2)}} \wedge \mathcal{F}_{(2)} \\ &\quad - \tfrac{1}{2} e^{\frac{1}{2}\phi}\,{*F_{(4)}} \wedge F_{(4)} - \tfrac{1}{2} e^{-\phi}\,{*F_{(3)}} \wedge F_{(3)} + \tfrac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)} \ ,\end{aligned} \tag{7.59}$$

with $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ being the Kaluza-Klein Maxwell field, and $F_{(3)} = dA_{(2)}$ and $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}$ being the two field strengths coming from the 4-form $F_{(4)}$ in $D = 11$. Note that the final term in the ten-dimensional Lagrangian comes from the cubic term $dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}$ in $D = 11$, and that this requires no metric in its construction. This ten-dimensional theory is the bosonic sector of the type IIA supergravity theory, which is the low-energy limit of the type IIA string.

Note that the eleven-dimensional theory has the "trombone" symmetry described above, namely a symmetry under the constant rescaling $g_{\mu\nu} \longrightarrow k^2\, g_{\mu\nu}$ and $A_{\mu\nu\rho} \longrightarrow k^3\, A_{\mu\nu\rho}$. Consequently, the ten-dimensional theory has the global internal symmetry $\phi \longrightarrow \phi + c$, together with

$$\mathcal{A}_{(1)} \longrightarrow e^{-\frac{3}{4}c}\, \mathcal{A}_{(1)} \ , \qquad A_{(3)} \longrightarrow e^{-\frac{1}{4}c}\, A_{(3)} \ , \qquad A_{(2)} \longrightarrow e^{\frac{1}{2}c}\, A_{(2)} \ . \tag{7.60}$$

7.3 Kaluza-Klein Reduction of $D = 11$ supergravity on T^n

It is clear that having established the procedure for performing a Kaluza-Klein reduction from $D + 1$ dimensions to D dimensions on the circle S^1, the process can be repeated for a succession of circles. Thus we may consider a reduction from $D + n$ dimensions to D dimensions on the n-torus $T^n = S^1 \times \cdots \times S^1$. At each successive step, for example the i'th reduction step, one generates a Kaluza-Klein vector potential $\mathcal{A}^i_{(1)}$, and a dilaton ϕ_i from the reduction of the metric. In addition, p-form potential already present in $D + i$ dimensions will descend to give a p-form and a $(p - 1)$-form potential, by the mechanism that we have already studied. As a result, one obtains a rapidly-proliferating number of fields as one descends through the dimensions.

Let us consider an example where we again begin with $D = 11$ supergravity, and now reduce it to D dimensions on the $n = (11 - D)$ torus, with coordinates z^i. As well as the set of Kaluza-Klein vectors $\mathcal{A}^i_{(1)}$ and dilatons ϕ_i, we will have 0-form potentials or "axions"

$\mathcal{A}^i_{(0)j}$ coming from the further reduction of the Kaluza-Klein vectors. Since such an axion cannot be generated until the Kaluza-Klein vector $\mathcal{A}^i_{(1)}$ has first been generated at a previous reduction step, we see that the axions $\mathcal{A}^i_{(0)j}$ will necessarily have $i < j$. In addition, the potential $A_{(3)}$ $A_{(3)}$ in $D = 11$ will give, upon reduction, the potentials $A_{(3)}$, $A_{(2)i}$, $A_{(1)ij}$ and $A_{(0)ijk}$. Here, the $i, j, \ldots$ indices are essentially internal coordinate indices corresponding to the torus directions. Thus these indices are antisymmetrised.

We will not labour too much over the details of the calculation of the torus reduction. It is clear that one just has to apply the previously-derived formulae for the single-step reduction of the Einstein-Hilbert and gauge-field actions repeatedly, until the required lower dimension $D = 11 - n$ is reached. If one does this, one obtains the following Lagrangian in D dimensions (see hep-th/9512012, hep-th/9710119):

$$\begin{aligned}
\mathcal{L} &= R{*\mathbb{1}} - \tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}e^{\vec{a}\cdot\vec{\phi}}\,{*F_{(4)}} \wedge F_{(4)} - \tfrac{1}{2}\sum_i e^{\vec{a}_i\cdot\vec{\phi}}\,{*F_{(3)i}} \wedge F_{(3)i} \\
&\quad -\tfrac{1}{2}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\,{*F_{(2)ij}} \wedge F_{(2)ij} - \tfrac{1}{2}\sum_i e^{\vec{b}_i\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(2)}} \wedge \mathcal{F}^i_{(2)} - \tfrac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,{*F_{(1)ijk}} \wedge F_{(1)ijk} \\
&\quad -\tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(1)j}} \wedge \mathcal{F}^i_{(1)j} + \mathcal{L}_{FFA}\,.
\end{aligned} \tag{7.61}$$

where the "dilaton vectors" $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$, $\vec{a}_{ijk}$, $\vec{b}_i$, $\vec{b}_{ij}$ are constants that characterise the couplings of the dilatonic scalars $\vec{\phi}$ to the various gauge fields. They are given by

$$\begin{array}{lll}
 & F_{MNPQ} & \text{vielbein} \\
4-\text{form}: & \vec{a} = -\vec{g}\,, & \\
3-\text{forms}: & \vec{a}_i = \vec{f}_i - \vec{g}\,, & \\
2-\text{forms}: & \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}\,, & \vec{b}_i = -\vec{f}_i\,, \\
1-\text{forms}: & \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}\,, & \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j\,,
\end{array} \tag{7.62}$$

where the vectors $\vec{g}$ and $\vec{f}_i$ have $(11 - D)$ components in D dimensions, and are given by

$$\begin{aligned}
\vec{g} &= 3(s_1, s_2, \ldots, s_{11-D})\,, \\
\vec{f}_i &= \Big(\underbrace{0, 0, \ldots, 0}_{i-1}, (10-i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{11-D}\Big)\,,
\end{aligned} \tag{7.63}$$

where $s_i = \sqrt{2/((10-i)(9-i))}$. It is easy to see that they satisfy

$$\vec{g}\cdot\vec{g} = \tfrac{2(11-D)}{D-2}, \qquad \vec{g}\cdot\vec{f}_i = \tfrac{6}{D-2}\,, \qquad \vec{f}_i\cdot\vec{f}_j = 2\delta_{ij} + \tfrac{2}{D-2}\,. \tag{7.64}$$

Note also that

$$\sum_i \vec{f}_i = 3\,\vec{g}\,. \tag{7.65}$$

Note that the D-dimensional metric is related to the eleven-dimensional one by

$$ds_{11}^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}}\, ds_D^2 + \sum_i e^{2\vec{\gamma}_i\cdot\vec{\phi}}\, (h^i)^2 \ , \tag{7.66}$$

where $\vec{\gamma}_i = \frac{1}{6}\vec{g} - \frac{1}{2}\vec{f}_i$, and

$$h^i = dz^i + \mathcal{A}_1^i + \mathcal{A}_{0j}^i\, dz^j \ . \tag{7.67}$$

There are, of course, a number of subtleties that have been sneaked into the formulae presented above. First of all, as we already saw from the single-step reduction from $D+1$ to D dimensions, one acquires transgression terms that modify the leading-order expressions $F_{(n)} = dA_{(n-1)} + \cdots$ for the lower-dimensional field strengths. This can all be handled in a fairly mechanical, although somewhat involved, manner. After a certain amount of algebra, one can show that the various field strengths are given by

$$\begin{aligned}
F_{(4)} &= \tilde{F}_{(4)} - \gamma^i{}_j\, \tilde{F}_{(3)i} \wedge \mathcal{A}_{(1)}^j + \tfrac{1}{2}\gamma^i{}_k\, \gamma^j{}_\ell\, \tilde{F}_{(2)ij} \wedge \mathcal{A}_{(1)}^k \wedge \mathcal{A}_{(1)}^\ell \\
&\quad -\tfrac{1}{6}\gamma^i{}_\ell\, \gamma^j{}_m\, \gamma^k{}_n\, \tilde{F}_{(1)ijk} \wedge \mathcal{A}_{(1)}^\ell \wedge \mathcal{A}_{(1)}^m \wedge \mathcal{A}_{(1)}^n \ , \\
F_{(3)i} &= \gamma^j{}_i\, \tilde{F}_{(3)j} + \gamma^j{}_i\, \gamma^k{}_\ell\, \tilde{F}_{(2)jk} \wedge \mathcal{A}_{(1)}^\ell + \tfrac{1}{2}\gamma^j{}_i\, \gamma^k{}_m\, \gamma^\ell{}_n\, \tilde{F}_{(1)jk\ell} \wedge \mathcal{A}_{(1)}^m \wedge \mathcal{A}_{(1)}^n \ , \\
F_{(2)ij} &= \gamma^k{}_i\, \gamma^\ell{}_j\, \tilde{F}_{(2)k\ell} - \gamma^k{}_i\, \gamma^\ell{}_j\, \gamma^m{}_n\, \tilde{F}_{(1)k\ell m} \wedge \mathcal{A}_{(1)}^n \ , \\
F_{(1)ijk} &= \gamma^\ell{}_i\, \gamma^m{}_j\, \gamma^n{}_k\, \tilde{F}_{(1)\ell mn} \ , \\
\mathcal{F}_{(2)}^i &= \tilde{\mathcal{F}}_{(2)}^i - \gamma^j{}_k\, \tilde{\mathcal{F}}_{(1)j}^i \wedge \mathcal{A}_{(1)}^k \ , \\
\mathcal{F}_{(1)j}^i &= \gamma^k{}_j\, \tilde{\mathcal{F}}_{(1)k}^i \ ,
\end{aligned} \tag{7.68}$$

where the tilded quantities represent the unmodified pure exterior derivatives of the corresponding potentials, $\tilde{F}_{(n)} \equiv dA_{(n-1)}$, and $\gamma^i{}_j$ is defined by

$$\gamma^i{}_j = [(1+\mathcal{A}_0)^{-1}]^i{}_j = \delta^i_j - \mathcal{A}_{(0)j}^i + \mathcal{A}_{(0)k}^i\, \mathcal{A}_{(0)j}^k + \cdots \ . \tag{7.69}$$

Recalling that $\mathcal{A}_{(0)j}^i$ is defined only for $j > i$ (and vanishes if $j \le i$), we see that the series terminates after a finite number of terms. We also define here the inverse of $\gamma^i{}_j$, namely $\tilde{\gamma}^i{}_j$ given by

$$\tilde{\gamma}^i{}_j = \delta^i_j + \mathcal{A}_{(0)j}^i \ . \tag{7.70}$$

Another point still requiring explanation is the term denoted by $\mathcal{L}_{FFA}$ in (7.61). This is the D-dimensional descendant of the term $\frac{1}{6} dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}$. Again, the calculations are purely mechanical, and we can just present the results:

$$\begin{aligned}
D = 10: &\quad \tfrac{1}{2}\tilde{F}_{(4)} \wedge \tilde{F}_{(4)} \wedge A_{(2)} \ , \\
D = 9: &\quad \Big(\tfrac{1}{4}\tilde{F}_{(4)} \wedge \tilde{F}_{(4)} \wedge A_{(1)ij} - \tfrac{1}{2}\tilde{F}_{(3)i} \wedge \tilde{F}_{(3)j} \wedge A_{(3)}\Big)\epsilon^{ij} \ ,
\end{aligned}$$

$$
\begin{aligned}
D=8: &\quad \Big(\tfrac{1}{12}\tilde F_{(4)}\wedge\tilde F_{(4)}A_{(0)ijk}-\tfrac16\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(2)k}-\tfrac12\tilde F_{(4)}\wedge\tilde F_{(3)i}\wedge A_{(1)jk}\Big)\epsilon^{ijk}\ ,\\
D=7: &\quad \Big(\tfrac16\tilde F_{(4)}\wedge\tilde F_{(3)i}A_{(0)jkl}-\tfrac14\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(1)kl}+\tfrac18\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(3)}\Big)\epsilon^{ijkl}\ ,\\
D=6: &\quad \Big(\tfrac{1}{12}\tilde F_{(4)}\wedge\tilde F_{(2)ij}A_{(0)klm}-\tfrac{1}{12}\tilde F_{(3)i}\wedge\tilde F_{(3)j}A_{(0)klm}+\tfrac18\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(2)m}\Big)\epsilon^{ijklm}\ ,\\
D=5: &\quad \Big(\tfrac{1}{12}\tilde F_{(3)i}\wedge\tilde F_{(2)jk}A_{(0)lmn}+\tfrac{1}{48}\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(1)mn} \qquad\qquad (7.71)\\
&\qquad -\tfrac{1}{72}\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(3)}\Big)\epsilon^{ijklmn}\ ,\\
D=4: &\quad \Big(\tfrac{1}{48}\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}A_{(0)mnp}-\tfrac{1}{72}\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(2)p}\Big)\epsilon^{ijklmnp}\ ,\\
D=3: &\quad -\tfrac{1}{144}\,\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(1)pq}\,\epsilon^{ijklmnpq}\ ,\\
D=2: &\quad -\tfrac{1}{1296}\,\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}A_{(0)pqr}\,\epsilon^{ijklmnpqr}\ .
\end{aligned}
$$

We may now ask the analogous question to the one we considered in the single-step S^1 reduction, namely what are the symmetries of the dimensionally-reduced theory, and how do they arise from the original higher-dimensional symmetries. Although the discussion above was aimed at the specific example of the T^n reduction of $D = 11$ supergravity,. it is obvious that much of the general structure, for example in the reduction of the Einstein-Hilbert term, is applicable to any starting dimension.

Let us consider the higher-dimensional general coordinate transformations, which, in infinitesimal form, are paramameterised in terms of the vector $\hat\xi^M$ as before: $\delta\hat x^M = -\hat\xi^M(\hat x)$. The difference now is that we have n reduction coordinates z^i, and so the higher-dimensional coordinates $\hat x^M$ are split as $\hat x^M = (x^\mu, z^i)$. As in the S^1 reduction, we must first identify the subset of these higher-dimensional general coordinate transformations that leaves the *structure* of the dimensional-reduction ansatz (7.66) invariant. (In other words, we need to find the transformations which allow the metric still to be written in the same form (7.66), but with, in general, transformed lower-dimensional fields $g_{\mu\nu}$, $\mathcal{A}^i_{(1)}$, $\mathcal{A}^i_{(0)j}$ and $\vec\phi$. The crucial point is that only those higher-dimensional general coordinate transformations that preserve the z^i-independence of the lower-dimensional fields are allowed.)

It is not hard to see, using the expression (7.42) for the infinitesimal general coordinate transformations of $\hat g_{MN}$, that the subset that preserves the structure of (7.66) is

$$
\hat\xi^(x,z) = \xi^\mu(x)\ , \qquad \hat\xi^i(x,z) = \Lambda^i{}_j\, z^j + \xi^i(x)\ , \tag{7.72}
$$

where the quantities $\Lambda^i{}_j$ are constants. This generalises the expression (7.43) that we obtained in the case of the S^1 reduction. Clearly, we can expect that $\xi^\mu(x)$ will again describe the general coordinate transformations of the lower-dimensional theory. The n local parameters $\xi^i(x)$, which generalise the single local parameter $\lambda(x)$ of the S^1-reduction case, will now describe the local $U(1)$ gauge invariances of the n Kaluza-Klein vector fields $\mathcal{A}^i\mu$.

This leaves only the global transformations, parameterised by the constants $\Lambda^i{}_j$ to interpret. These generalise the single constant c of the S^1 reduction example. In that case, we saw that after taking into account the additional scaling symmetry of the higher-dimensional equations of motion, which could be used as a compensating transformation, we could extract a symmetry in the lower dimension that left the metric invariant, and described a constant shift of the dilaton, combined with appropriate constant rescalings of the gauge fields. In group-theoretic terms, that was an $\mathrm{I\!R}$ transformation; the group parameter c took values anywhere on the real line.

In our present case with a reduction on the torus T^n, we have n^2 constant parameters $\Lambda^i{}_j$ appearing in (7.72). They act by matrix multiplication on the "column vector" composed of the internal coordinates z^i on the torus,

$$\delta z^i = -\Lambda^i{}_j\, z^j\ . \tag{7.73}$$

The matrix $\Lambda^i{}_j$ is unrestricted; it just has n^2 real components. This is the general linear group of real $n\times n$ matrices, denoted by $GL(n,\mathrm{I\!R})$. There is, of course, again also the uniform scaling symmetry of the higher-dimensional equations of motion. One can use this as a "compensator," to allow all of the $\Lambda^i{}_j$ transformations to become purely internal symmetries, which act on the various lower-dimensional potentials and dilatons, but which leave the lower-dimensional metric invariant. This can be seen by calculations that are precisely analogous to the ones for the S^1 reduction in the previous section.

The conclusion, therefore, from the above discussion is that when the Einstein-Hilbert action is dimensionally reduced on the n-dimensional torus T^n, it gives rise to a theory in the lower dimension that has a $GL(n,\mathrm{I\!R})$ global symmetry, in addition to the local general coordinate and gauge symmetries generated by $\xi^\mu(x)$ and $\xi^i(x)$. In fact, the $GL(n,\mathrm{I\!R})$ transformations are also symmetries of the theory that we get when we include the other terms in the eleven-dimensional sueprgravity Lagrangian. This is a rather general feature; any theory with gravity coupled to other matter fields will, upon dimensional reduction on T^n, give rise to a theory with a $GL(n,\mathrm{I\!R})$ global symmetry. (Strictly speaking, one can only be sure of $SL(n,\mathrm{I\!R})$ as an *internal* symmetry that leaves the metric invariant; getting the full $GL(n,\mathrm{I\!R})$ depends on having the extra homogeneous scaling symmetry of the higher-dimensional equations of motion; note that $GL(n,\mathrm{I\!R})\sim SL(n,\mathrm{I\!R})\times\mathrm{I\!R}$.)

Actually, as we shall see later, the reduction of eleven-dimensional supergravity on T^n actually typically gives a *bigger* global symmetry than $GL(n,\mathrm{I\!R})$. The reason for this is that there is actually a "conspiracy" between the metric and the 3-form potential of $D = 11$, and between them they create a lower-dimensional system that has an enlarged global symmetry. The phenomenon first sets in when one descends down to eight dimensions on the 3-torus, for which the global symmetry is $SL(2,\mathrm{I\!R}) \times SL(3,\mathrm{I\!R})$, rather than the naively-expected $GL(3,\mathrm{I\!R})$. By the time one considers a reduction from $D = 11$ to $D = 3$ on the 8-torus, the naively-expected $GL(8,\mathrm{I\!R})$ is enlarged to an impressive E_8. We won't study all the deatils of how these enlargements occur, but we will look at some of the elements in the mechanism. First, let us consider the simplest non-trivial example of a global symmetry, which arises in a reduction of pure gravity on a 2-torus.

7.4 $SL(2,\mathrm{I\!R})$ and the 2-torus

Let us consider pure gravity in $D + 2$ dimensions, reduced to D dimensions on T^2. From the earlier discussions it is clear that we will get the following fields in the dimensionally-reduced theory: $(g_{\mu\nu}, \mathcal{A}^i_{(1)}, \mathcal{A}^1_{(0)2}, \vec\phi)$. The notation is a little ugly-looking here, so let us just review what we have. There are two Kaluza-Klein gauge potentials $\mathcal{A}^i_{(1)}$, and then there is the 0-form potential, or axion, $A^1_{(0)2}$. This is what comes from the dimensional reduction of the first of the two Kaluza-Klein vectors, $\mathcal{A}^1_{(1)}$, which, at the second reduction step gives not only a vector, but also the axion. We can make things look nicer by using the symbol χ to represent $A^1_{(0)2}$. From the previous results, it is not hard to see that the dimensionally-reduced Lagrangian is

$$\mathcal{L} = R\,{*1} - \tfrac12 {*d\vec\phi} \wedge d\vec\phi - \tfrac12 \sum_i e^{\vec c_i\cdot\vec\phi}\, {*F_{(2)i}} \wedge F_{(2)i} - \tfrac12 e^{\vec c\cdot\vec\phi}\, {*d\chi} \wedge d\chi\ , \tag{7.74}$$

where the dilaton vectors are given by

$$\begin{aligned} \vec c_1 &= \Big(-\sqrt{\tfrac{2D}{D-1}}, -\sqrt{\tfrac{2}{(D-1)(D-2)}}\Big)\ , \qquad \vec c_2 = \Big(0, -\sqrt{\tfrac{2(D-1)}{D-2}}\Big)\ , \\ \vec c &= \Big(-\sqrt{\tfrac{2D}{D-1}}, \sqrt{\tfrac{2(D-2)}{D-1}}\Big)\ . \end{aligned} \tag{7.75}$$

The field strengths are given by

$$\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} - d\chi \wedge \mathcal{A}^2_{(1)}\ , \qquad \mathcal{F}_{(2)} = d\mathcal{A}^2_{(1)}\ . \tag{7.76}$$

Things simplify a lot if we rotate the basis for the two dilatons $\vec\phi = (\phi_1, \phi_2)$. Make the orthogonal transformation to two new dilaton combinations, which we may call ϕ and φ:

$$\phi = -\tfrac12\sqrt{\tfrac{2D}{D-1}}\,\phi_1 + \tfrac12\sqrt{\tfrac{2(D-2)}{D-1}}\,\phi_2\ , \qquad \varphi = -\tfrac12\sqrt{\tfrac{2(D-2)}{D-1}}\,\phi_1 - \tfrac12\sqrt{\tfrac{2D}{D-1}}\,\phi_2\ . \tag{7.77}$$

After a little algebra, the Lagrangian (7.74) can be seen to become

$$\mathcal{L} = R\,{*1} - \tfrac12 {*d\varphi}\wedge d\varphi - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 e^{\phi+q\varphi}\,{*\mathcal{F}^1_{(2)}}\wedge \mathcal{F}^1_{(2)} - \tfrac12 e^{-\phi+q\varphi}\,{*\mathcal{F}^2_{(2)}}\wedge \mathcal{F}^2_{(2)} - \tfrac12 e^{2\phi}\,{*d\chi}\wedge d\chi\,, \tag{7.78}$$

where $q = \sqrt{D/(D-2)}$.

Note also that from the expression (7.66) for the dimensionally-reduced metric, we have

$$ds^2_{D+2} = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}\, ds^2_D + e^{\sqrt{(D-2)/D}\,\varphi}\,\Big(e^{\phi}\,(dz_1 + \mathcal{A}^1_{(1)} + \chi\, dz_2)^2 + e^{-\phi}\,(dz_2 + \mathcal{A}^2_{(2)})^2\Big)\,. \tag{7.79}$$

This shows that the scalar φ has the interpretation of parameterising the volume of the 2-torus, since it occurs in an overall multiplicative factor of the internal compactifying metric, while ϕ parameterises a shape-changing mode of the torus, since it scales the lengths of the two circles of the torus in opposite directions. In fact ϕ and χ completely characterise the *moduli* of the torus. The moduli are parameters that change the shape of the torus, at fixed volume, while keeping it flat. One can see that as ϕ varies, the relative radii of the two circles change, while as χ varies, the angle between the two circles changes.

Let us now look at the scalars in the Lagrangian (7.78), namely ϕ, φ and χ, descrined by the scalar Lagrangian

$$\mathcal{L}_{\rm scal} = -\tfrac12 (\partial\varphi)^2 - \tfrac12 (\partial\phi)^2 - \tfrac12 e^{2\phi}\,(\partial\chi)^2\,. \tag{7.80}$$

It is evident that φ is decoupled from the others. It has a global shift symmetry, $\varphi \longrightarrow \varphi + k$. This gives an $\mathbb{R}$ factor in the global symmetry group. Now look at the dilaton-axion system (ϕ, χ). This is best analysed by defining a complex field $\tau = \chi + {\rm i}\, e^{-\phi}$. The Lagrangian for ϕ and χ can then be written as

$$\mathcal{L}_{(\phi,\chi)} \equiv -\tfrac12 (\partial\phi)^2 - \tfrac12 e^{2\phi}\,(\partial\chi)^2 = -\frac{\partial\tau\cdot\partial\bar\tau}{2\,\tau_2^2}\,, \tag{7.81}$$

where τ_2 means the imaginary part of τ; one commonly writes $\tau = \tau_1 + {\rm i}\,\tau_2$. Now, it is not hard to see that if τ is subjected to the following *fractional linear transformation*,

$$\tau \longrightarrow \frac{a\,\tau + b}{c\,\tau + d}\,, \tag{7.82}$$

where a, b, c and d are constants that satisfy

$$a\,d - b\,c = 1\,, \tag{7.83}$$

then the Lagrangian (7.81) is left invariant. But we can write the constants in a 2×2 matrix,

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{7.84}$$

with the condition (7.83) now restated as $\det\Lambda = 1$. What we have here is real 2×2 matrices of unit determinant. They therefore form the group $SL(2,\mathbb{R})$. This $SL(2,\mathbb{R})$ is a symmetry that acts non-linearly on the complex scalar field τ, as in (7.82).

Thus we have seen that the scalar Lagrangian (7.80) has in total an $\mathbb{R}\times SL(2,\mathbb{R})$ global symmetry. This makes the $GL(2,\mathbb{R})$ symmetry that was promised in the previous section. Note that the $SL(2,\mathbb{R})$ transformation (7.82) can be expressed directly on the dilaton and axion, where it becomes

$$\begin{aligned} e^{\phi} &\longrightarrow e^{\phi'} = (c\,\chi + d)^2\, e^{\phi} + c^2\, e^{-\phi}\ ,\\ \chi\, e^{\phi} &\longrightarrow \chi'\, e^{\phi'} = (a\,\chi + b)(c\,\chi + d)\, e^{\phi} + a\,c\,e^{-\phi}\ . \end{aligned} \tag{7.85}$$

To complete the story, we should go back to analyse the full Lagrangian (7.78) that includes the gauge fields $\mathcal{F}^i_{(2)}$. First of all, it is helpful to make a field redefinition $\mathcal{A}^1_{(1)} \longrightarrow \mathcal{A}^1_{(1)} + \chi\, \mathcal{A}^2_{(1)}$, which has the effect of changing the expression for the field strength $\mathcal{F}^1_{(2)}$, so that instead of (7.76) we have

$$\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} + \chi\, d\mathcal{A}^2_{(1)}\ , \qquad \mathcal{F}_{(2)} = d\mathcal{A}^2_{(1)}\ . \tag{7.86}$$

In other words, the derivative has bee shifted off χ, and onto $\mathcal{A}^2_{(1)}$ instead. The statement of how the $SL(2,\mathbb{R})$ transformations act on the gauge fields now becomes very simple; it is

$$\begin{pmatrix} \mathcal{A}^2_{(1)} \\ \mathcal{A}^1_{(1)} \end{pmatrix} \longrightarrow (\Lambda^T)^{-1} \begin{pmatrix} \mathcal{A}^2_{(1)} \\ \mathcal{A}^1_{(1)} \end{pmatrix}\ , \tag{7.87}$$

where Λ was defined in (7.84). This transformation on the potentials is to be performed at the same time as the transformation (7.85) is performed on the scalars. (If one spots the right way to do this calculation, the proof is not too difficult.) Note that while the scalars transform non-linearly under $SL(2,\mathbb{R})$, the two gauge potentials transform linearly, as a doublet. In other words, they just transform by matrix multiplication of $(\Lambda^T)^{-1}$ on the column vector formed from the two potentials.

7.5 Scalar coset Lagrangians

Many of the features of the 2-torus reduction that we saw in the previous section are rather general in all the toroidal dimensional reductions. In particular, one thing that we encountered was that the global suymmmetry of the lower-dimensional Lagrangian was already established by looking just at the scalar fields, and their symmetry transformations. Showing that the full Lagrangian had the symmetry was then a matter of showing that the terms in the full lower-dimensional Lagrangian that involve the higher-rank potentials (the two

1-form gauge potentials, in our 2-torus reduction example) also share the same symmetry. It is in fact essentially true in general that the extension of the global symmetry to the entire Lagrangian is "guaranteed," once it is eastablished as a symmetry of the scalar sector. Furthermore, the higher-rank potentials always transform in linear representations of the global symmetry group, while the scalars transform non-linearly. One can, for example, show without too much further trouble that if one reduces $D=11$ supergravity on the 2-torus, so that now the 3-form gauge potential is included also, the resulting additional gauge potentials in $D=9$ will again transform linearly under the $GL(2,\mathbb{R})$ global symmetry. These additional gauge potentials will comprise $A_{(3)}$, transforming as a singlet under the $SL(2,\mathbb{R})$ subgroup, two 2-forms $A_{(2)i}$, transforming as a doublet, and one 1-form, $A_{(1)12}$, transforming as a singlet. Under the $\mathbb{R}$ factor of $GL(2,\mathbb{R})$, which corresponds to the constant shift symmetry of the other dilaton φ, all the potentials will transform by appropriate constant scaling factors.

To understand the structure of the global symmetries better, we need to study the nature of the scalar Lagrangians that arise from the dimensional resuction. This is instructive not only in its own right, but also because it leads us into the subject of non-linear sigms models, and coset spaces, which are of importance in many other areas of physics too. Let us begin by considering the $SL(2,\mathbb{R})$ example from the previous section. It exhibits many of the general features that one encounters in non-linear sigma models, while having the merit of being rather simple and easy to calculate explicitly.

The group $SL(2,\mathbb{R})$ is the non-compact version of $SU(2)$, and consequently, its associated Lie algebra (the elements infinitesimally close to the identity) is essentially the same as that of $SU(2)$. Thus we have the generators (H, E_+, E_-), staisfying the Lie algebra

$$[H, E_\pm] = 2\,E_\pm \ , \qquad [E_+, E_-] = H \ . \tag{7.88}$$

H is the Cartan subalgebra generator, while $E_\pm$ are the raising and lowering operators. A convenient representation for the generators is in terms of 2×2 matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{7.89}$$

(So $H=\tau_3$, $E_\pm = 1/2(\tau_1 \pm \mathrm{i}\,\tau_2$, where τ_i are the Pauli matrices.)

Consider now the exponentiation of the H and E_+, and define

$$\mathcal{V} = e^{\frac{1}{2}\phi\, H}\, e^{\chi\, E_+} \ , \tag{7.90}$$

where ϕ and χ are thought of as fields depending on the coordinates of a D-dimensional spacetime. A simple calculation shows that

$$\mathcal{V} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi\, e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} . \tag{7.91}$$

We now compute the exterior derivative, to find

$$d\mathcal{V}\,\mathcal{V}^{-1} = \begin{pmatrix} \frac{1}{2}d\phi & e^{\phi}\, d\chi \\ 0 - \frac{1}{2}d\phi & \end{pmatrix} = \tfrac{1}{2}d\phi\, H + e^{\phi}\, d\chi\, E_+ \, . \tag{7.92}$$

Let us define also the matrix $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$. It is easy to see from (7.91) that we have

$$\mathcal{M} = \begin{pmatrix} e^{\phi} & \chi\, e^{\phi} \\ \chi\, e^{\phi} & e^{-\phi} + e^{\phi}\,\chi^2 \end{pmatrix} , \qquad \mathcal{M}^{-1} = \begin{pmatrix} e^{-\phi} + e^{\phi}\,\chi^2 & -\chi\, e^{\phi} \\ -\chi\, e^{\phi} & e^{\phi} \end{pmatrix} . \tag{7.93}$$

Thus we see that we may write a scalar Lagrangian as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M}\Big) = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}\,(\partial\chi)^2 \, . \tag{7.94}$$

This is nothing but the $SL(2,\mathrm{I\!R})$-invariant scalar Lagrangian that we encountered in the previous section. The advantage now is that we have a very nice way to see why it is $SL(2,\mathrm{I\!R})$ invariant.

To do this, observe that if we introduce an arbitrary constant $SL(2,\mathrm{I\!R})$ matrix Λ, given by

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad a\,d - b\,c = 1 \, , \tag{7.95}$$

then if we send $\mathcal{V} \longrightarrow \mathcal{V}'' = \mathcal{V}\,\Lambda$, we get $\mathcal{M} \longrightarrow (\mathcal{V}'')^T\,\mathcal{V}'' = \Lambda^T$
$v^T\,\mathcal{V}\,\Lambda = \Lambda^T\,\mathcal{M}\,\Lambda$, which manifestly leaves $\mathcal{L}$ invariant:

$$\mathcal{L} \longrightarrow \tfrac{1}{4}\mathrm{tr}\Big(\Lambda^{-1}\,\partial\mathcal{M}^{-1}\,(\Lambda^T)^{-1}\,\Lambda^T\,\partial\mathcal{M}\,\Lambda\Big) = \tfrac{1}{4}\mathrm{tr}\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M}\Big) \, . \tag{7.96}$$

The only trouble with this transformation is that when we sent $\mathcal{V} \longrightarrow \mathcal{V}'' = \mathcal{V}\,\Lambda$ we actually did something improper, because in general the transformed matrix $\mathcal{V}''$ is *not* of the upper-triangular form that the original matrix $\mathcal{V}$ given in (7.91 is. Thus by acting with Λ, we have done something that cannot, as it stands, be expressed as a transformation on the fields ϕ and χ. Happily, there is a simple remedy for this. What we must do is make a compensating *local* transformation $\mathcal{O}$ that acts on $\mathcal{V}$ from the left, at the same time as we multiply by the constant $SL(2,\mathrm{I\!R})$ matrix from the right. Thus we define a transformed matrix $\mathcal{V}'$ by

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda \, , \tag{7.97}$$

where, by definition, $\mathcal{O}$ is the matrix that does the job of restoring $\mathcal{V}'$ to the upper-triangular gauge. There is a unique orthogonal matrix that does the job, and after a little algebra, one finds that it is

$$\mathcal{O} = (c^2 + e^{2\phi}\,(c\,\chi + a)^2)^{-1/2} \begin{pmatrix} e^{\phi}\,(c\,\chi + a) & c \\ -c & e^{\phi}\,(c\,\chi + a) \end{pmatrix} . \tag{7.98}$$

The matrix $\mathcal{O}$ that we have just constructed does the job of restoring the $SL(2,\mathbb{R})$-transformed matrix $\mathcal{V}$ to the upper-triangular gauge of (7.91), which means that we can now interpret the action of $SL(2,\mathbb{R})$ in terms of transformations on ϕ and χ. But does it give us an invariance of the Lagrangian (7.94)? The answer is yes, and this is easily seen. The matrix $\mathcal{O}$ is the specific one thta does the job of compensating for the $SL(2,\mathbb{R})$ transformation with constant paraneters a, b, c and d. It is itself local, since it depends not only on the constant $SL(2,\mathbb{R})$ parameters but also on the fields ϕ and χ themselves. This does not cause trouble, howvere, because, crucially, $\mathcal{O}$ is an *orthogonal* matrix. This means that when we calculate how $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$ transforms, we find

$$\mathcal{M} \longrightarrow \mathcal{M}' = (\mathcal{V}')^T\,\mathcal{V}' = \Lambda^T\,\mathcal{V}^T\,\mathcal{O}^T\,\mathcal{O}\,\mathcal{V}\,\Lambda = \Lambda^T\,\mathcal{V}^T\,\mathcal{V}\,\Lambda = \Lambda^T\,\mathcal{M}\,\Lambda\,. \tag{7.99}$$

Thus the local compensating transformation cancels out when the transformed $\mathcal{M}$ matrix is calculated, and hence the previous calculation (7.96) demonstrating the invariance of the Lagrangian goes through without modification.

After a little algebra, it is not hard to see that the transformed fields ϕ' and χ', defined by (7.97), are precisely the ones that we obtained in the previous section, given in (7.85). It is not hard to see that at a given spacetime point (i.e. for fixed values of ϕ and χ), we can use the $SL(2,\mathbb{R})$ transformation to get from *any* pair of values for ϕ and χ to any other pair of values. This means that $SL(2,\mathbb{R})$ acts *transitively* on the *scalar manifold*, which is the manifold where the fields ϕ and χ take their values.

Let us take stock of what we have found. We have parameterised points in the scalar manifold in terms of the matrix $\mathcal{V}$ in (7.91). We have seen that acting from the right with an $SL(2,\mathbb{R})$ matrix Λ, we can get to any other point in the scalar manifold. But we must, in general, make a compensating $O(2)$ transformation as we do so, to make sure that we stay withing our original parameterisation scheme in terms of the upper-triangular matrices $\mathcal{V}$. Thus we may specify points in the scalar by the *coset* $SL(2,\mathbb{R})/O(2)$, consisting of $SL(2,\mathbb{R})$ motions modulo the appropriate $O(2)$ compensators. Thus we may say that the scalar manifold for the (ϕ,χ) dilaton/axion system is the coset space $SL(2,\mathbb{R})/O(2)$, and that it has $SL(2,\mathbb{R})$ as its global symmetry group.

In this example, the points in the $SL(2,\mathbb{R})/O(2)$ coset were parameterised by the *coset representative* $\mathcal{V}$, given in (7.91). We obtained this by exponentiating just two of the $SL(2,\mathbb{R})$ generators, namely the Cartan generator H and the raising operator E_+. Things don't always go quite so smoothly and easily as this, but in the case of the various scalar coset manifolds that arise in the toroidal compactifications of eleven-dimensional supergravity they do. Let us, therefore, pursue these examples a bit further.

Our discussion above was for the reduction of the Einstein-Hilbert action on T^2, staring in any dimension $D+2$ and ending up in D dimensions. We could generalise this to include some additional antisymmetric tensors in $D+2$ dimensions, and we would find in general that they give rise to sets of fields in D dimensions that transform linearly under $SL(2,\mathbb{R})$. In the case where we start with supergravity in $D=11$, we would have an additional 3-form potential, therefore. After reduction to $D=9$ on T^2, we would get the fields discussed above in from the gravity sector, together with fields $A_{(3)}$, $A_{(2)i}$ and $A_{(1)12}$ that descend from $A_{(3)}$. One finds that $A_{(3)}$ is a singlet under $SL(2,\mathbb{R})$, the two $A_{(2)i}$ form a doublet, and $A_{(1)12}$ is again a singlet.

The situation changes if we descend from $D=11$ on a higher-dimensional torus. The reason is that we now start to get additional axionic scalar fields from the descendants of $A_{(3)}$, over and above the scalars that come from the eleven-dimensional metric. For example, if we descend on T^3 to $D=8$, we now have not only the three dilatons $\vec{\phi}$, and three axions $\mathcal{A}^i_{(0)j}$, but also one additional axion $A_{(0)123}$. Now the scalars $\vec{\phi}$ and $\mathcal{A}^i_{(0)j}$ have a Lagrangian with the "expected" $GL(3,\mathbb{R})$ global symmetry. In fact, they parameterise points in the six-dimensional coset manifold $GL(3,\mathbb{R})/O(3)$. But what happens with the symmetry is the following. We saw in $D=9$, in the T^2 reduction, that the $\mathbb{R}$ factor in the $GL(2,\mathbb{R})$ symmetry "factored off" from the rest of the $SL(2,\mathbb{R})$. The same thing happens here, and there is one dilaton which contributes the $\mathbb{R}$ factor in $GL(3,\mathbb{R})$, and which is decoupled from the remaining five scalars that form the $SL(3,\mathbb{R})/O(3)$ coset. It does, however, couple to the the additional axion, $A_{(0)123}$, coming from the reduction of $A_{(3)}$. In fact they form a dilaton/axion system with an $SKL(2,\mathbb{R})$ global symmetry, working just like the $SL(2,\mathbb{R})$ that we saw in the T^2 reduction. Thus the final conclusion is that the reduction of $D=11$ supergravity on T^3 to $D=8$ gives a theory whose scalars parameterise the coset

$$\frac{SL(3,\mathbb{R})}{O(3)}\times\frac{SL(2,\mathbb{R})}{O(2)}\,, \tag{7.100}$$

and so there is an $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$ global symmetry.

To see the details in this eight-dimensional example, let us consider just the scalar sector of the dimensionally-reduced theory. From (7.61), we will have

$$\mathcal{L}_8 = -\tfrac{1}{2}{*d\vec\phi}\wedge d\vec\phi - \tfrac{1}{2}\sum_{i<j} e^{\vec b_{ij}\cdot\vec\phi}\, {*\mathcal{F}^i_{(1)j}}\wedge \mathcal{F}^i_{(1)j} - \tfrac{1}{2} e^{\vec a_{123}\cdot\vec\phi}\, {*F_{(1)123}}\wedge F_{(1)123}\ , \tag{7.101}$$

where

$$\mathcal{F}^1_{(1)2} = d\mathcal{A}^1_{(0)2}\ ,\qquad \mathcal{F}^2_{(1)3} = d\mathcal{A}^2_{(0)3}\ ,\qquad \mathcal{F}^1_{(1)3} = d\mathcal{A}^1_{(0)3} - \mathcal{A}^2_{(0)3}\, d\mathcal{A}^1_{(0)2}\ ,\qquad F_{(1)123} = dA_{(0)123}\ . \tag{7.102}$$

From the general results for the dilaton vectors, it is not hard to see that after performing an orthogonal transformation to make things look nicer, we can make the dilaton vectors become

$$\begin{aligned} &\vec b_{12} = (0,1,\sqrt{3})\ ,\qquad \vec b_{23} = (0,1,-\sqrt{3})\ ,\qquad \vec b_{13} = (0,2,0)\ ,\\ &\vec a_{123} = (2,0,0)\ . \end{aligned} \tag{7.103}$$

Thus we see that indeed the axion $A_{(0)123}$ and the dilaton ϕ_1 form an independent $SL(2,\mathbb{R})/O(2)$ scalar coset, which is decoupled from the rest of the scalar sector.

This leaves the $SL(3,\mathbb{R})$ part of the scalar coset still to understand. Perhaps the easiest way to see what's happening here is to recall a couple of facts about group theory. The generators of a Lie algebra $\mathcal{G}$ can be organised into Cartan generators, $\vec H$, which mutually commmute with each other, and raising and lowering operators $E_{\vec\alpha}$. If the rank of the algebra is n, then there are n Cartan generators, $\vec H = (H_1,\ldots,H_n)$. The raising and lowering operators have the commutation relations

$$[\vec H, E_{\vec\alpha}] = \vec\alpha\, E_{\vec\alpha} \tag{7.104}$$

with the Cartan generators, where $\vec\alpha$ are called the root vectors associated with the generators $E_{\vec\alpha}$. One sets up a scheme for defining root vectors to be positive or negative. The standard way to do this is to look at the components of the root vector $\vec\alpha = (\alpha_1,\ldots,\alpha_n)$, working from the left to the right. The *sign* of the root vector is defined to be the sign of the first non-zero component that is encountered. Generators with positive root vectors are called raising operators, and those with negative roots are called lowering operators. It is easily seen from (7.104) that if the commutator of two non-zero-root generators $E_{\vec\alpha}$ and $E_{\vec\beta}$ is non-vanishing, then it will be a generator with root vector $\vec\alpha+\vec\beta$. Thus in general we have

$$[E_{\vec\alpha}, E_{\vec\beta}] = N(\alpha,\beta)\, E_{\vec\alpha+\vec\beta}\ , \tag{7.105}$$

for some constant (possibly zero) $N(\alpha,\beta)$.

The classification of all the possible Lie algebras is quite straightforward, but it is a lengthy business, and we shall not stray into it here. Suffice it to say that it turns out that the Lie algebras can be classified by classifying all the possible root systems, which means determining all the possible sets of roots that satisfy certain consistency requirements. In turn, these root systems can be characterised in terms of the *simple roots.* These are defined to be the subset of the positive roots that allow one to express *any* positive root in the system as a linear combination of the simple roots with non-negative integer coefficients. One can show that the number of simple roots is equal to the rank of the algebra. In other words, there are as many simple roots as there are components to the root vectors.

In the example of $SL(2,\mathbb{R})$, which has rank 1, we had the single Cartan generator H, and the single positive-root generator E_+, with the single-component "root vector" 2, as in (7.88). In general, $SL(n+1,\mathbb{R})$ has rank n, and so for $SL(3,\mathbb{R})$ we have rank 2. Thus we expect two Cartan generators $\vec{H}$, and 2-component root vectors. In fact this is just what we are seeing in our eighet-dimensional scalar Lagrangian. Forgetting now about the $SL(2,\mathbb{R})$ part, which, as we have seen, factors off from the rest, we have two dilatons $\vec{\phi}=(\phi_2,\phi_3)$, and 2-component dilaton vectors

$$\vec{b}_{12}=(1,\sqrt{3})\ ,\qquad \vec{b}_{23}=(1,-\sqrt{3})\ ,\qquad \vec{b}_{13}=(2,0)\ . \tag{7.106}$$

(These follow from (7.103) by dropping the first component of each dilaton vector; i.e. the component associated with the decoupled $SL(2,\mathbb{R})$ part.) We can recognise the $\vec{b}_{ij}$ dilaton vectors as the positive roots of $SL(3,\mathbb{R})$, with $\vec{b}_{12}$ and $\vec{b}_{23}$ as the two simple roots, and $\vec{b}_{13}=\vec{b}_{12}+\vec{b}_{23}$. We may introduce positive-root generators ${E_i}^j$, defined for $i<j$, associated with the root-vectors $\vec{b}_{ij}$, and Cartan generators $\vec{H}$, with the commutation relations

$$[\vec{H},{E_i}^j]=\vec{b}_{ij}\,{E_i}^j\ ,\qquad [{E_i}^j,{E_k}^\ell]=\delta^j_k\,{E_i}^\ell-\delta^\ell_i\,{E_k}^j\ . \tag{7.107}$$

Note that the only non-zero commutator among the positive-root generators here is $[{E_1}^2,{E_2}^3]={E_1}^3$.

One can represent the various generators here in terms of 3×3 matrices. For ${E_i}^j$, we define it to be the matrix with zeroes everywhere except for a 1 at the position of row i and column j, and so

$${E_1}^2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\ ,\qquad {E_2}^3=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\ ,\qquad {E_1}^3=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\ . \tag{7.108}$$

The two Cartan generators $\vec{H}=(H_1,H_2)$ are then diagonal, with

$$H_1=\mathrm{diag}\,(1,0,-1)\ ,\qquad H_2=\tfrac{1}{\sqrt{3}}\,\mathrm{diag}\,(1,-2,1)\ . \tag{7.109}$$

The strategy for constructing the $SL(3,\mathrm{I\!R})/O(3)$ coset Lagrangian is now to follow the same path that we used for $SL(2,\mathrm{I\!R})$. We write down a coset representative $\mathcal{V}$, by exponentiating the Cartan and positive-root generators of $SL(3,\mathrm{I\!R})$, with the dilatons and axions as coefficients. We do this in the following way:

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}\, e^{\mathcal{A}^2_{(0)3}\,{E_2}^3}\, e^{\mathcal{A}^1_{(0)3}\,{E_1}^3}\, e^{\mathcal{A}^1_{(0)2}\,{E_1}^2}\ . \tag{7.110}$$

Note that there are obviously many different ways that one could organise this exponentiation; here, we exponentiate each generator separately, and then multiply the results together. An alternative would be to exponentiate the sum of generators times fields. This would, in general, give a slightly different expression for $\mathcal{V}$, since if A and B are two matrices that do not commute, then $e^A\, e^B \ne e^{A+B}$. (One can use the Baker-Campbell-Hausdorf formula to relate them.) The different possibilities correspond to making different choices for exactly how to parameterise points in the coset space, and eventually one choice is related to any other by making redefinitions of the fields. Thus any choice is equally as "good" as any other. The choice we are making here happens to be convenient, because it happens to correspond exactly to the choice of field variables in our eight-dimensional Lagrangian.

It is not hard to establish that with the coset representative $\mathcal{V}$ defined as in (7.110) above, one has

$$d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i_{(1)j}\, {E_i}^j\ , \tag{7.111}$$

where the 1-form field strengths $\mathcal{F}^i_{(1)j}$ are given in (7.102). In particular, the transgression term in $\mathcal{F}^1_{(1)3}$ comes from the fact that the commutator of ${E_1}^2$ and ${E_2}^3$ is non-zero, as given in (7.107). (One needs to use the following matrix relations in order to derive the result:

$$\begin{aligned} de^X\, e^{-X} &= dX + \tfrac{1}{2}[X,dX] + \tfrac{1}{6}[X,[X,dX]] + \cdots\ ,\\ e^X\, Y\, e^{-X} &= Y + [X,Y] + \tfrac{1}{2}[X,[X,Y]] + \cdots\ . \end{aligned} \tag{7.112}$$

Only the first couple of terms in these expansions are ever needed, since the multiple commutators of positive-root generators rapidly expire.)

It is also straightforward to calculate $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$, and hence the Lagrangian

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M}\Big)\ . \tag{7.113}$$

(In practice, Mathematica is handy for this sort of calculation.) After a little algebra, one finds that it is given by

$$\mathcal{L} = -\tfrac{1}{2}{*d\vec{\phi}}\wedge d\vec{\phi} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\, {*\mathcal{F}^i_{(1)j}}\wedge \mathcal{F}^i_{(1)j}\ . \tag{7.114}$$

In other words, we have succeeded in writing the part of the eight-dimensional scalar Lagrangian (7.101) in a manifestly $SL(3,\mathbb{R})$-invariant fashion.

To make the $SL(3,\mathbb{R})$ symmetry fully explicit, we should really repeat the steps that we followed in the case of the $SL(2,\mathbb{R})$ example. Namely, we should consider a general global $SL(3,\mathbb{R})$ transformation Λ acting *via* right-multiplication on the coset representative $\mathcal{V}$. This will in general take us out of the upper-triangular gauge of (7.110), and so we should then show that there exists a local, field-dependent, compensating $O(3)$ transformation $\mathcal{O}$, such that

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda \tag{7.115}$$

is back in the upper-triangular gauge. This means that one can then interpret $\mathcal{V}'$, *via* the definition (7.110), as the coset representative for a different point in the coset manifold, corresponding to the transformed fields with primes on them. The matrix $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$ that is used to construct the scalar Lagrangian (7.113) then transforms nicely as $\mathcal{M} \longrightarrow \mathcal{M}' = \Lambda^T\,\mathcal{M}\,\Lambda$, hence implying the invariance of the Lagrangian under global $SL(3,\mathbb{R})$ transformations.

In this particular case, it is perfectly possible to do this calculation explicitly, and to exhibit the required $O(3)$ compensator (again, Mathematica can be handy here). However, it is clear that in more complicated examples it would become increasingly burdensome to construct the compensator $\mathcal{O}$. Furthermore, we don't actually really need to know what it is; all we really need is to know that it exists. Luckily, there is a general theorem in the theory of Lie algebras, which does the job for us. It is known as the *Iwasawa Decomposition*, and it goes as follows. The claim is that every element g in the Lie group G obtained by exponentiating the Lie algebra $\mathcal{G}$ can be uniquely expressed as the following product:

$$g = g_K\, g_H\, g_N\ . \tag{7.116}$$

Here g_K is in the maximal compact subgroup K of G, g_H is in the Cartan subalgebra of G, and g_N is in the exponentiation of the positive-root part of the algebra $\mathcal{G}$.[3]

This is precisely what is needed for the discussion of the cosets that arise in these supergravity reductions. Our coset representative $\mathcal{V}$ is constructed by exponentiating the Cartan generators, and the full set of positive-root generators (see (7.90) for $SL(2,\mathbb{R})$, and (7.110) for $SL(3,\mathbb{R})$). Thus our coset representative is written as $\mathcal{V} = g_H\, g_N$. Now, we act

[3]Actually, as we shall see later, this statement of the Iwasawa decomposition is appropriate only in the rather special circumstance we have here, where G is maximally non-compact. We shall give a more general statement later.

by right-multiplication with a general group element Λ in G. This means that $\mathcal{V}\,\Lambda$ is *some* element of the group G. Now, we invoke the Iwasawa decomposition (7.116), which tells us that we must be able to write the group element $\mathcal{V}\,\Lambda$ in the form $g_K\,\mathcal{V}'$, where $\mathcal{V}'$ itself is of the form $g'_H\,g'_N$. This does what we wanted; it assures us that there exists a way of pulling out an element $\mathcal{O}$ of the maximal compact subgroup K of G on the left-hand side, such that we can write $\mathcal{V}\,\Lambda$ as $\mathcal{O}\,\mathcal{V}'$.

We are now in a position to proceed to the lower-dimensional theories obtained by compactifying eleven-dimensional supergravity on torii of higher dimensions. We can benefit from the lessons of the previous examples, and home in directly on the key points. Let us first, for reasons that will become clear later, consider the cases where the n-torus has $n \le 5$, meaning that we end up in dimensions $= 11 - n \ge 6$. The full set of axionic scalars will be $\mathcal{A}^i_{(0)j}$ and $A_{(0)ijk}$ in each dimension. From our T^2 and T^3 examples, we have seen that the dilaton vectors $\vec b_{ij}$ and $\vec a_{ijk}$ for these axions form the positive roots of a Lie algebra, and that by exponentiating the associated positive-root generators, with the axions as coefficients, and exponentiatiing the Cartan generators, with the dilatons as coefficients, we constructed a coset representative $\mathcal{V}$ for G/K, where G is the Lie group associated with the Lie algebra, and K is its maximal compact subgroup.

How do we identify what the group G is in each dimension? If we can identify the subset of the dilaton vectors that corresponds to the simple roots of the Lie algebra then we will have solved the problem. Bu this is easy; we just need to find what subset of the dilaton vectors $\vec b_{ij}$ and $\vec a_{ijk}$ allows us to express *all* of the dilaton vectors as linear combinations of the simple roots, with non-negative integer coefficients. The answer is very straightfoward; the simple roots are given by

$$\vec b_{i,i+1}\ ,\quad \text{for}\ \ 1 \le i \le n-1\ ,\qquad \text{and}\ \ \vec a_{123}\ . \tag{7.117}$$

To check that this is correct, it is only necessary to look at the results in (7.63)-(7.65). It is manifest from the fact that $\vec b_{ij} = -\vec f_i + \vec f_j$ that any $\vec b_{ij}$ can be expressed as multiples of the $\vec b_{i,i+1}$, with non-negative integer coefficients. It is also clear that by adding appropriate integer multiples of the $\vec b_{i,i+1}$ to $\vec a_{123}$, all of the $\vec a_{ijk}$ can be constructed.

Having found the simple roots, it is easy to determine what the Lie algebra is. All the Lie algebras are classified in terms of their Dynkin diagrams, which encode the information about the lengths of the simple roots, and the angles between them. The notation is as follows. The angle between any two simple roots can be only one out of four possibilities, namely 90, 120, 135 or 150 degrees. The simple roots are denoted by dots in the Dynkin diagram, and the angle between two roots is indicated by the number of lines joining the corresponding dots. The rule is no line, 1 line, 2 lines or 3 lines, corresponding to 90, 120, 135 or 150 degrees. The lengths of the simple roots are either all equal (such groups are called *simply laced*), or they have exactly two different lengths, in groups that are called, unimaginitively, *non-simply-laced*. In this latter case, the dots in the Dynkin diagram are filled-in to denote the shorter roots, and unfilled for the longer roots. In our case, it turns out that the roots are all of the same length. From the expressions in (7.64), it is easily seen that our simple roots are characterised by the Dynkin diagram

$$
\begin{array}{ccccccccccccc}
\vec b_{12} & & \vec b_{23} & & \vec b_{34} & & \vec b_{45} & & \vec b_{56} & & \vec b_{67} & & \vec b_{78} \\
\mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} \\
 & & & & | & & & & & & & & \\
 & & & & \mathrm{o} & & & & & & & & \\
 & & & & \vec a_{123} & & & & & & & &
\end{array}
$$

This diagram is telling us that all the angles that are not 90 degrees are 120 degrees, and that all the simple roots have equal lengths. The understanding is that in a given dimension $D = 11 - n$, only those dilaton vectors which are defined for $i \le n$ arise. Those familiar with group theory and Dynkin diagrams will be able to recognise the diagrams for the various n values as follows. For $n = 2$, we have just $\vec b_{12}$, and the algebra is $SL(2, \mathrm{I\!R})$. For $n = 3$, we have $(\vec b_{12}, \vec b_{23}, \vec a_{123})$, and the algebra is $SL(3, \mathrm{I\!R}) \times SL(2, \mathrm{I\!R})$. These are the two cases that we have already studied in detail. For $n = 4$, we have $(\vec b_{12}, \vec b_{23}, \vec b_{34}, \vec a_{123})$, and the Dynkin diagram is that of $SL(5, \mathrm{I\!R})$. For $n = 5$, we have $(\vec b_{12}, \vec b_{23}, \vec b_{34}, \vec b_{45}, \vec a_{123})$, and the Dynkin diagram is that of D_5, or $O(5, 5)$. We shall postpone the discussion of $n \ge 6$ for a while.

From our previous discussion of the T^2 and T^3 reductions, we expect now that we should introduce the appropriate positive-root generators associated with each of the dilaton vectors $\vec b_{ij}$ and $\vec a_{ijk}$. For the $\vec b_{ij}$, we just use the same notation as before, with generators ${E_i}^j$, except that now the range of the i and j indices is extended to $1 \le i < j \le n$. For the $\vec a_{ijk}$, we introduce generators E^{ijk}. The commutation relations for these, and the Cartan generators $\vec H$, will be

$$[\vec H, {E_i}^j] = \vec b_{ij}\, {E_i}^j\ , \qquad [\vec H, E^{ijk}] = \vec a_{ijk}\, E^{ijk} \qquad \text{no sum} \tag{7.118}$$

$$[{E_i}^j, {E_k}^\ell] = \delta_k^j\, {E_i}^\ell - \delta_i^\ell\, {E_k}^j\ , \tag{7.119}$$

$$[{E_\ell}^m, E^{ijk},] = -3\delta_\ell^{[i}\, E^{|m|jk]}\ , \tag{7.120}$$

$$[E^{ijk}, E^{\ell mn}] = 0 \,, \tag{7.121}$$

We can recognise the commutation relations for the $\vec{H}$ and the $E_i{}^j$ as being precisely those of the Lie algebra $SL(n,\mathbb{R})$. This is reasonable on two counts. Firstly, since these are the generators associated with the fields coming from the reduction of pure gravity, namely $\vec{\phi}$ and $\mathcal{A}^i_{(0)j}$, we already expected to find a $GL(n,\mathbb{R})$ symmetry after reduction on the n-torus. (One never really sees the extra $\mathbb{R}$ factor of $GL(n,\mathbb{R}) \sim \mathbb{R}\times SL(n,\mathbb{R})$ in the Dynkin diagrams; it is associated with the fact that there is one extra Cartan generator over and above the $(n-1)$ that are needed for $SL(n,\mathbb{R})$.) Another way of seeing why this $SL(n,\mathbb{R})$ subgroup is reasonable is by looking at the Dynkin diagram above; if we delete the simple root $\vec{a}_{123}$, then the remaining simple roots $\vec{b}_{i,i+1}$ do indeed precisely give us the Dynkin diagram of $SL(n,\mathbb{R})$.

The extra commutation relations involving E^{ijk} extend the algebras from $SL(n,\mathbb{R})$ to the larger ones discussed above. Thus in addition to the $D = 9$ and $D = 8$ cases discussed previously, in $D = 7$ we will have the scalar coset $SL(5,\mathbb{R})/O(5)$, and in $D = 6$ we will have $O(5,5)/(O(5)\times O(5))$. In each case, in accordance with our discussion of the Iwasawa decomposition, the denominator group in the coset is the maximal compact subgroup of the numerator. The coset representatives in all cases $n \le 5$ are constructed as follows:

$$\mathcal{V} = e^{\frac12\vec{\phi}\cdot\vec{H}}\Big(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}\,E_i{}^j}\Big)\,\exp\Big(\sum_{i<j<k} A_{(0)ijk}\,E^{ijk}\Big)\,, \tag{7.122}$$

where the ordering of terms is *anti-lexigraphical*, i.e. $\cdots(24)(23)\cdots(14)(13)(12)$, in the product. With this specific way of organising the exponentiation, it turns out that, with the commutation relations given above, one has

$$d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac12 d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac12\vec{b}_{ij}\cdot\vec{\phi}}\,\mathcal{F}^i{}_{(1)j}\,E_i{}^j + \sum_{i<j<k} e^{\frac12\vec{a}_{ijk}\cdot\vec{\phi}}\,F_{(1)ijk}\,E^{ijk}\,, \tag{7.123}$$

where the various 1-form field strengths, with all their transgression terms, are precisely as given in equation (7.68). (It is quite an involved calculation to show this!) In all the cases with $n \le 5$, one can define the matrix $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$, and it will follow that the scalar Lagrangian can be written as $\mathcal{L} = \frac14\mathrm{tr}\,(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})$.

7.6 Scalar cosets in $D = 5$, 4 and 3

Aficionados of group theory will easily recognise that if we consider the cases $n = 6$, 7 and 8, corresponding to reductions to $D = 5$, 4 and 3 dimensions, the Dynkin diagrams above will be those of the exceptional groups E_6, E_7 and E_8. One does not need to be much of an aficionado, however, to see that as things stand, there is something wrong with the counting of fields. After reduction on an n torus there will be $\frac{1}{2}n(n-1)$ axions $\mathcal{A}^i_{(0)j}$, and $\frac{1}{6}n(n-1)(n-2)$ axions $A_{(0)ijk}$. For $n=(2,3,4,5,6,7,8)$, we therefore have $(1,4,10,20,35,56,84)$ axions in total. On the other hand, the numbers of positive roots for the groups indicated by the Dynkin diagrams above are $(1,4,10,20,36,63,120)$. Thus the discrepancies set in at $n=6$ and above. We appear to be missing some axionic scalar fields.

Consider first the situation where this arises, when $n=6$, implying that we have dimensionally reduced the $D=11$ theory to $D=5$. From the counting above, we are missing one axion. The explanation for where it comes from is in fact quite simple. Recall that among the fields in the reduced theory is the 3-form potential $A_{(3)}$, with its 4-form field strength $F_{(4)}$. Now, in $D=5$, if we take the Hodge dual of a 4-form field strength, we get a 1-form, and this can be interpreted as the field strength for a 0-form potential, or axion. This is the source of our missing axion.

Before looking at this in more detail, let's just check the counting for remaining two cases. When $n=7$, we have reduced the theory to $D=4$, and in this case it is 2-form potentials that dualise into axions. The 2-form potentials are $A_{(2)i}$, and so when $n=7$ there are seven of them. This is precisley the discrepancy that we noted in the previous paragraph. Finally, when $n=8$ we have a reduction to $D=3$, and in this case it is 1-form potentials that are dual to axions. The relevant potentials are $A_{(1)ij}$ and $\mathcal{A}^i_{(1)}$, of which there are $28+8=36$ when $n=8$. Again, this exactly resolves the discrepancy noted in the previous paragraph.

Now, back to $D=5$. As usual, we shall concentrate just on the scalar sector, since this governs the global symmetry of the entire theory. Now, of course, we must include the 3-form potential too, since we are about to dualise it to obtain the "missing" axion. In fact, to start with, we may consider just those terms in the five-dimensional Lagrangian that involve the 3-form potential. From the general results in (7.61) and the associated formulae, we can see that the relevent terms are

$$\mathcal{L}(F_{(4)}) = -\tfrac{1}{2}e^{\vec{a}\cdot\vec{\phi}}\,{*F_{(4)}}\wedge F_{(4)} - \tfrac{1}{72}A_{(0)ijk}\,dA_{(0)\ell mn}\wedge F_{(4)}\,\epsilon^{ijk\ell mn}\,, \qquad (7.124)$$

where $F_{(4)}=dA_{(3)}$. In the process of dualisation, the rôle of the Bianchi identity, which is $dF_{(4)}=0$ here, is interchanged with the role of the field equation. The easiest way to achieve this is to treat $F_{(4)}$ as a fundamental field in its own right, and impose its Bianchi identity by adding the term $-\chi\,dF_{(4)}$ to the Lagrangian, where we have introduced the field

χ as a Lagrange multiplier. Thus we consider

$$\mathcal{L}(F_{(4)})' = -\tfrac{1}{2} e^{\vec{a}\cdot\vec{\phi}} *F_{(4)} \wedge F_{(4)} - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn} \wedge F_{(4)}\, \epsilon^{ijk\ell mn} - \chi\, dF_{(4)}\ . \qquad (7.125)$$

Clearly, the variation of this with respect to χ gives the required Bianchi identity. We note that $F_{(4)}$, which is now treated as a fundamental field, has a purely algebraic equation of motion. Varying $\mathcal{L}(F_{(4)})'$ with respect to $F_{(4)}$, we get the equation of motion

$$e^{\vec{a}\cdot\vec{\phi}} *F_{(4)} = d\chi - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn}\, \epsilon^{ijk\ell mn}\ . \qquad (7.126)$$

We may define this right-hand side as our new 1-form field strength,; let us call it $G_{(1)}$:

$$G_{(1)} \equiv d\chi - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn}\, \epsilon^{ijk\ell mn}\ . \qquad (7.127)$$

Thus we have $F_{(4)} = e^{-\vec{a}\cdot\vec{\phi}} *G_{(1)}$. Substituting this back into the Lagrangian (which is allowed, since it is a purely algebraic, non-differential equation), we find that $\mathcal{L}(F_{(4)})'$ has become

$$\mathcal{L}(F_{(4)})' = -\tfrac{1}{2} e^{-\vec{a}\cdot\vec{\phi}} *G_{(1)} \wedge G_{(1)}\ . \qquad (7.128)$$

In other words, we have successfully dualised the potential $A_{(3)}$, with field strength $F_{(4)} = dA_{(3)}$, and replaced it with the axion χ, whose field strength $G_{(1)}$ is given in (7.127). Note that its dilaton vector, $-\vec{a}$, is the negative of the dilaton vector $\vec{a}$ of the field prior to dualisation. This sign reversal always occurs in any dualisation. Notice that one effect of the dualisation is that the FFA term in the Lagrangian (7.124) has migrated to become a trangsression term in the definition of the new dualised field strength $G_{(1)}$ in (7.127). This interchange between FFA terms and transgression terms is a general feature in any dualisation.

Having found the missing axion, we must now consider the algebra, and the construction of the coset representative $\mathcal{V}$. We need one more generator, over and above the usual Cartan generators $\vec{H}$ and positive-root generators $E_i{}^j$ and E^{ijk}. In fact we are missing one further positive-root generator, in this $D=5$ example; let us call it J. It satisfies the following commutation relations, which extend the set given already in equations (7.118)-(7.121):

$$\begin{aligned} &[\vec{H}, J] = -\vec{a}\, J\ , \qquad [E_i{}^j, J] = 0\ , \qquad [E^{ijk}, J] = 0\ , \\ &[E^{ijk}, E^{\ell mn}] = -\epsilon^{ijk\ell mn}\, J\ . \end{aligned} \qquad (7.129)$$

The last commutator here is a reflection of the fact that in $D=5$, the sum of dilaton vectors $\vec{a}_{ijk} + \vec{a}_{\ell mn}$, when i, j, k, ℓ, m, n are all different, is equal to $-\vec{a}$, as can be seen from (7.62) and (7.65). Note that this depends crucially on a specific feature of reduction on a torus of dimension 6, since then we have that $\vec{a}_{ijk} + \vec{a}_{\ell mn} = \sum_i \vec{f}_i - 2\vec{g}$ since all of i, j, k, ℓ, m, n are different, and hence this equals $\vec{g}$.

The coset representative is now constructed as follows:

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \Big(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}\, E_i{}^j} \Big) \exp\Big(\sum_{i<j<k} A_{(0)ijk}\, E^{ijk} \Big)\, e^{\chi\, J} \ . \tag{7.130}$$

After some algebra, one can show that now we have

$$d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i{}_{(1)j}\, E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}\, F_{(1)ijk}\, E^{ijk} + e^{-\vec{a}\cdot\vec{\phi}}\, G_{(1)}\, J \ , \tag{7.131}$$

where the 1-form field strengths $\mathcal{F}^i_{(1)j}$ and $F_{(1)ijk}$ are given in (7.68), and $G_{(1)}$ is given in (7.127). As in the previous examples, the transgression terms in all the field strengths come out to be precisely correct, and arise from the various non-vanishing commutators among the positive-root generators.

From the previous general discussion, we can expect that the coset representative $\mathcal{V}$ can be used to construct an E_6-invariant scalar Lagrangian, and that this will be the Lagrangian of the scalar sector of $D = 11$ supergravity reduced on T^6. In particular, we can act on $\mathcal{V}$ from the right with a global E_6 transformation Λ , and then the Iwasawa decomposition theorem assures us that we can find a compensating field-dependent transformation $\mathcal{O}$ that acts on the left, such that $\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda$ is back in the "upper-triangular" gauge. In this case, the maximal compact subgroup of E_6 is $USp(8)$, and so $\mathcal{O}$ is a $USp(8)$ matrix. Actually, a better name for the gauge is really the *Borel gauge.* The Borel subgroup of any Lie group is the subgroup generated by the positive-root generators and the Cartan generators. Obviously this is a subgroup, since negative roots cannot be generated by commutation of non-negative ones. Sometimes, it is useful also to be able to talk of the *strict Borel subgroup*, defined to be the subgroup generated by the strictly-positive-root generators. In our cases, we obtain our coset representatives by exponentiating the entire Borel subalgebra, including the Cartan subalgebra.

Because the maximal compact subgroup in this E_6 case is no longer orthogonal, the way in which the Lagrangian is constructed from the coset representative $\mathcal{V}$ is slightly different. In general, the construction is the following. One defines the so-called *Cartan involution* τ, which has the effect of reversing the sign of every non-compact generator in the algebra $\mathcal{G}$, while leaving the sign of every compact generator unchanged. If we denote the positive-root generators, negative-root generators and Cartan generators by $(E_{\vec{\alpha}}, \{E_{-\vec{\alpha}}, \vec{H})$, where $\vec{\alpha}$ ranges over all the positive roots, then for our algebras τ effects the mapping

$$\tau: \qquad (E_{\vec{\alpha}}, E_{-\vec{\alpha}}, \vec{H}) \longrightarrow (-E_{-\vec{\alpha}}, -E_{\vec{\alpha}}, -\vec{H}) \ . \tag{7.132}$$

It should perhaps be remarked at this point that the groups that we are encountering in the toroidal compactifications of eleven-dimensional supergravity are somewhat special, in that they are always *maximally non-compact*. It is always the case, in any real group, that the generator combinations $(E_\alpha - E_{-\alpha})$ are compact while the combinations $(E_\alpha - E_{-\alpha})$ are non-compact.[4] (Thus if there are N positive roots, then there are N compact and N non-compact generators formed from the non-zero roots.) But in our case, we also have that all the Cartan generators are non-compact. Thus the group E_n that we encounter upon compactification on an n-torus is actually E_n in its maximally non-compact form, denoted by $E_{n(+n)}$. It has the n "extra" non-compact Cartan generators, in addition to the 50/50 split of compact/non-compact generators coming from the non-zero-root generators. We shall normally not bother with the extra annotation of the $(+n)$ in the subscript, but its presence will be implicit.

Getting back to the Cartan involution, we may use this to construct the required generalisation of the $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$ construction that worked when the maximal compact subgroup was orthogonal. Thus we may define a "generalised transpose" $X^\#$ of a matrix X, by

$$X^\# \equiv \tau(X^{-1}) \; . \tag{7.133}$$

From the definition of τ, and its action on the various generators, it is evident that $X^\#$ is nothing but X^T in cases where the compact generators give rise to an orthogonal group. If the compact generators form a unitary group, then $X^\#$ will be $X^\dagger$. In the case of E_6, the maximal compact subgroup is $USp(8)$, which is the intersection of $SU(8)$ and $Sp(8)$. A detailed discussion of the generalised transpose in this case would take us off into a digression about symplectic invariants, and is probably inappropriate here. Some further details can be found in hep-th/9710119.

Suffice it to say that with the generalised transpose defined as above, the scalar Lagrangian in $D = 5$ can now be written as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\,\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M}\Big) \; , \tag{7.134}$$

where $\mathcal{M} = \mathcal{V}^\#\,\mathcal{V}$. The proof of the invariance under global E_6 transformations is then essentially identical to that in the previous examples that we discussed. Note that another

[4] By the real form of a group, we mean that the Hermitean generators are all formed by taking *real* combinations of the raising and lowering operators, not complex ones. For example, $SL(2,\mathbb{R})$ is the real form of A_1, since $E_+ \pm E_-$ and H are Hermitean, whereas $SU(2)$ is the complex form of A_1, since its Hermitean generators are the complex combinations $\tau_1 = E_+ + E_-$, $\tau_2 = i\,E_+ - i\,E_-$ and $\tau_3 = H$.

way of writing the Lagrangian, which follows directly by substitution of $\mathcal{M} = \mathcal{V}^\# \, \mathcal{V}$ into (7.134), is

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{tr}\, \Big(\partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})^\# + \partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})\Big)\ . \tag{7.135}$$

The stories for the compactifications of $D=11$ supergravity on T^7 and T^8 to $D=4$ and $D=3$ proceed in a very similar manner. Full details can be found in hep-th/9710119. As we already mentioned, in order to achieve the full E_7 or E_8 global symmetries one must dualise the seven 2-form potentials $A_{(2)i}$ to 0-forms χ^i in $D=4$, whilst in $D=3$, one must dualise the $28+8$ 1-form potentials $A_{\iota ij}$ and $\mathcal{A}^i_{(1)}$ to 0-forms χ^{ij} and χ_i in $D=3$. Thus in $D=4$ we must introduce seven extra generators J_i for the duals of the $A_{(2)i}$. They will have associated root vectors $-\vec a_i$ (remember that dualisation reverses the signs of the dialton vectors), and sure enough, these are precisely the addition positive roots that can be constructed by taking non-negative-integer linear combinations of the simple roots $\vec b_{i,i+1}$ and $\vec a_{123}$ in this case. In addition to the standard dimension-independent commutation relations (7.118)-(7.121), there will now be the further commutators involving J_i:

$$\begin{aligned}
&[\vec H, J_i] = -\vec a_i\, J_i\ , \qquad [E_i{}^j, J_j] = \delta_i^k\, J_j\ , \qquad [E^{ijk}, J_\ell] = 0\ ,\\
&[E^{ijk}, E^{\ell mn}] = \epsilon^{ijk\ell mnp}\, J_p\ .
\end{aligned} \tag{7.136}$$

We then form a coset representative by exponentiation, appending an additional factor

$$\mathcal{V}_{\rm extra} = e^{\chi^i\, J_i} \tag{7.137}$$

to the right of the standard dimension-independent expression given in (7.122). One then finds, after extensive algebra, that the scalar Lagrangian for the four-dimensional reduction from $D=11$ can be written in the form (7.134)(or (7.135), and that it has an E_7 global symmetry. The coset is $E_7/SU(8)$ in this case.

Finally, in $D=3$, one introduces extra generators J_{ij} and J^i for the axions χ^{ij} and χ_i coming from dualising $A_{(1)ij}$ and $\mathcal{A}^i_{(1)}$. In addition to the dimension-independent commutators (7.118)-(7.121), there will now in addition be

$$\begin{aligned}
&[\vec H, J_{ij}] = -\vec a_{ij}\, J_{ij}\ , \quad [\vec H, J^i] = -\vec b_i\, J^i\ , \quad [E_i{}^j, J_{k\ell}] = -2\delta^j_{[k}\, J_{\ell]i}\ , \quad [E_i{}^j, J_k] = -\delta_i^k\, J^j\ ,\\
&[E^{ijk}, J_{\ell m}] = -6\delta^{[i}_{[\ell}\,\delta^j_{m]}\, J^{k]}\ , \qquad [E^{ijk}, J_\ell] = 0\ ,\\
&[E^{ijk}, E^{\ell mn}] = -\tfrac{1}{2}\epsilon^{ijk\ell mnpq}\, J_{pq}\ .
\end{aligned} \tag{7.138}$$

In this case, the coset representative $\mathcal{V}$ is constructed by appending

$$\mathcal{V}_{\rm extra} = e^{\chi_i\, J^i}\, e^{\frac{1}{2}\chi^{ij}\, J_{ij}} \tag{7.139}$$

to the right of the usual dimension-independent terms given in (7.122). The scalar Lagrangian can then be shown to be given by (7.134) or (7.135), and its global symmetry is E_8. The coset in this case is $E_8/SO(16)$.

To summarise this discussion of the scalar cosets coming from the toroidal reductions of eleven-dimensional supergravity, we may present a table listing the coset spaces in each dimension. The numerator group G, and the maximal compact denominator subgroup K, are listed in each case.

	G	K
$D=10$	$O(1,1)$	-
$D=9$	$GL(2,\mathbb{R})$	$O(2)$
$D=8$	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$
$D=7$	$SL(5,\mathbb{R})$	$SO(5)$
$D=6$	$O(5,5)$	$O(5)\times O(5)$
$D=5$	$E_{6(+6)}$	$USp(8)$
$D=4$	$E_{7(+7)}$	$SU(8)$
$D=3$	$E_{8(+8)}$	$SO(16)$

Table 1: Scalar cosets for maximal supergravities in D dimensions

7.7 General remarks about coset Lagrangians

As we have already remarked, the scalar cosets that we encountered in the toroidal compactifications of eleven-dimensional supergravity are somewhat special, in the sense that the numerator groups (i.e. the global symmetry groups themselves) are all maximally non-compact. In addition, our way of parameterising the cosets involved making a specific "gauge choice," which in our case was achieved by choosing the coset representative $\mathcal{V}$ to be in the Borel gauge. One can perfectly well, in principle, make some other gauge choice. Alternatively, one is not obliged to make any choice of gauge at all. One could simply exponentiate the entire Lie algebra of the global symmetry group G. This would give too many fields, of course, since the dimension of the coset G/K is $\dim(G)-\dim(K)$, and so there should be this number of scalar fields, rather than the $\dim(G)$ fields that one would get if no gauge choice were made. The resolution is a simple one, and it is essentially something that we have already seen: two points $\mathcal{V}_1$ and $\mathcal{V}_2$ on the coset manifold G/K

that are related by left-multiplication by an element of K, i.e. $\mathcal{V}_1 = \mathcal{O}\,\mathcal{V}_2$, are actually the same point. Thus if one constructs $\mathcal{V}$ by exponentiating the entire algebra, then there will be local "gauge" symmetries associated with the entire group K that remove the surplus degrees of freedom. Our way of constructing the scalar cosets in the supergravity theories exploited the fact that in those cases it was actually very simple to use these local gauge symmetries explicitly, to fix a gauge in which the redundant fields were simply set to zero.

We shall not delve here into the details of how one handles the construction of coset Lagrangians in general, for example in cases where the local K invariance is left unfixed. We shall, however, make some general remarks about how to handle a wider class of cosets in the gauge-fixed formalism, namley in those cases where the numerator group G is not maximally non-compact. To illustrate the point, let us consider the family of examples of cosets

$$M_{p,q} = \frac{O(p,q)}{O(p)\times O(q)}\ , \tag{7.140}$$

where $O(p,q)$ is the group of pseudo-orthogonal matrices that leaves invariant the indefinite-signature diagonal matrix $\eta = \mathrm{diag}\,(1,1,\ldots,1,-1,-1,\ldots,-1)$, where there are p plus signs and q minus signs. Thus $O(p,q)$ matrices Λ satisfy

$$\Lambda^T\,\eta\,\Lambda = \eta\ . \tag{7.141}$$

For a given value of $n = p+q$, the algebras $O(p,q)$ are all just different forms of the same underlying algebra, which would be $D_{n/2}$ in the Dynkin classification if n were even, and $B_{(n-1)/2}$ if n were odd. However, the partition into compact and non-compact generators is different for different partitions of $n = p+q$. In fact, the denominator groups $O(p)\times O(q)$ are the maximal compact subgroups in each case, telling us that of the total of $\frac12(p+q)(p+q-1)$ generators of $O(p,q)$, there are $\frac12 p(p-1)+\frac12 q(q-1)$ compact generators, with the rest being non-compact. Evidently, then, the dimensions of the cosets are different depending on the partition of $n = p+q$; simple subtraction gives us

$$\dim\,(M_{p,q}) = \tfrac12(p+q)(p+q-1) - \tfrac12 p(p-1) - \tfrac12 q(q-1) = p\,q\ . \tag{7.142}$$

When $n = p+q$ is even, the rank of $O(p,q)$ is $\frac12 n$, and ones finds that the dimension $p\,q$ of the coset space is equal to the dimension of the Borel subalgebra, which is $\frac12 n + \frac12(\frac12 n(n-1) - n/2) = \frac14 n^2$, only if $p=q$. Thus when $n = p+q$ is even, only the cosets of the form $O(p,p)/(O(p)\times O(p))$ are maximally non-compact. (We encounted such a coset in $D=6$, where the scalar Lagrangian was $O(5,5)/(O(5)\times O(5))$.) A similar analysis for the case $n = p+q$ odd shows that only the case $O(p,p+1)/(O(p)\times O(p+1))$ (or, equivalently,

$O(p+1,p)/(O(p+1)\times O(p)))$ is maximally non-compact. These are the cases where, for a given n, the dimension of $M_{p,q}$ is largest.

Clearly, if we consider a coset of the form (7.140) that is not maximally non-compact, then if we are to construct a coset representative $\mathcal{V}$ in a gauge-fixed form, we must exponentiate only an appropriate subset of the Borel generators of $O(p,q)$. The general theory of how to do this was worked out by Alekseevski, in the 1970's. It again makes use of the Iwasawa decomposition, but this is now a little more complicated when the group G is not maximally non-compact. The decomposition asserts that there is a unique factorisation of a group element g as

$$g = g_K\, g_A\, g_N\ , \tag{7.143}$$

where g_K is in the maximal compact subgroup K of G and g_A is in the maximal non-compact Abelian subgroup of G. The factor g_N is in a nilpotent subgroup of G, which is defined as follows. It is generated by that subset of the positive-root generators that are strictly positive with respect to the maximal non-compact Abelian subalgebra (whose exponentiation gives g_A).

Now, if the group G were maximally non-compact, then *all* the Cartan subalgebra generators would be non-compact, and hence *all* the positive-root generators would be included in the nilpotent subalgebra. We would then be back to the previous statement of the Iwasawa decomposition for maximally non-compact groups, where we exponentiated the entire Borel subalgebra.

Here, however, we are by contrast considering a case where only a subset of the Cartan generators are non-compact. Accordingly, only a subset of the positive-root generators pass the test of having strictly positive weights with respect to this subset of the Cartan generators. In this more general situation, the subalgebra of the Borel algebra, comprising the non-compact Cartan generators A and the positive-root generators N that have positive weights under A, is known as the *Solvable Lie Algebra* of the group G.

We can now build a coset representative $\mathcal{V}$ by exponentiationg the non-compact Cartan generators, and the nilpotent subalgebra generators. By the modified Iwasawa decomposition (7.143), it follows that a global transformation consisting of a right-multiplication by an element of G can be compensated by a local field-dependent left-multiplication by an appropriate element of the maximal compact subgroup, thereby giving a $\mathcal{V}'$ in the same "nilpotent" gauge, corresponding to a G-transformed point in the coset G/K. Thus we again have a procedure for constructing the scalar Lagrangian for the coset, in this more general situation where G is not maximally non-compact.

Let us close this discussion with an illustrative example. There is string theory in $D = 10$, known as the heterotic string, whose low-energy effective Lagrangian is different from the ten-dimensional theory that comes by S^1 reduction from eleven-dimensional supergravity. For our present purposes, it suffices to say that the Lagrangian in $D = 10$ can be taken to have the general form

$$\mathcal{L}_{10} = R\,{*1} - \tfrac{1}{2}{*d\phi_1} \wedge d\phi_1 - \tfrac{1}{2}e^{\phi_1}\,{*F_{(3)}} \wedge F_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\phi_1} \sum_{I=1}^{N} {*G^I_{(2)}} \wedge G^I_{(2)}\ , \tag{7.144}$$

where $G^i_{(2)} = dB^i_{(1)}$ are a set of N 2-form field strengths, and

$$F_{(3)} = dA_{(2)} + \tfrac{1}{2}B^I_{(1)} \wedge dB^I_{(1)}\ . \tag{7.145}$$

(Actually, in the heterotic string itself $N = 16$, and the 16 gauge fields $B^I_{(1)}$ are just in the $U(1)^{16}$ Cartan subgroup of a 496-dimensional Yang-Mills group, which can be $E_8 \times E_8$ or $SO(32)$. But for our purposes it suffices to consider the Abelian subgroup fields, and also we can generalise the discussion by allowing N to be arbitrary.)

Clearly there is a global $O(N)$ symmetry in $D = 10$, under which the N gauge fields are rotated amongst each other. If one performs a Kaluza-Klein dimensional reduction of the theory on T^n, then it turns out that the resulting theory in $D = 10 - n$ has an $O(n, n+N)$ global symmetry, and that the scalar manifold is the coset

$$\frac{O(n, n+N)}{O(n) \times O(n+N)}\ . \tag{7.146}$$

These cosets are of precisely the type that we discussed above, which can be parameterised by means of an exponentiation of their solvable Lie algebras. To keep things simple, let us consider the case $n = 1$. Thus we shall reduce (7.144) on a circle, and show that the scalar sector in $D = 9$ has an $O(1, N+1)/O(N+1)$ coset structure. (Actually, there will be another $\mathbb{R}$ factor too, associated with an extra scalar that decouples from the rest.)

Let us denote the dilaton of the $d = 10$ to $D = 9$ reduction by ϕ_2. After performing the reduction, using the standard rules that we established previously, we find, after making a convenient rotation of the dilatons, that the nine-dimensional Lagrangian is

$$\begin{aligned}\mathcal{L}_9 \;=\;& R\,{*1} - \tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2}{*d\varphi} \wedge d\varphi - \tfrac{1}{2}e^{\sqrt{2}\varphi} \sum_I {*dB^I_{(0)}} \wedge dB^I_{(0)} - \tfrac{1}{2}e^{-\sqrt{\frac{8}{7}}\phi}\,{*F_{(3)}} \wedge F_{(3)} \\ & -\tfrac{1}{2}e^{-\sqrt{\frac{2}{7}}\phi}\Big(e^{\sqrt{2}\varphi}\,{*F_{(2)}} \wedge F_{(2)} + e^{-\sqrt{2}\varphi}\,{*\mathcal{F}_{(2)}} \wedge \mathcal{F}_{(2)} + \sum_I {*G^I_{(2)}} \wedge G^I_{(2)}\Big)\ ,\end{aligned} \tag{7.147}$$

where $\mathcal{F}_{(2)}$ is the Kaluza-Klein gauge field, and $F_{(2)}$ and $G^i_{(1)} = dB^I_{(0)}$ are the dimensional reductions of $F_{(3)}$ and $G^i_{(2)}$ respectively. The various field strengths are given in terms of potentials by

$$
\begin{aligned}
&F_{(3)} = dA_{(2)} + \tfrac{1}{2}B^I_{(1)}\, dB^I_{(1)} - \tfrac{1}{2}\mathcal{A}_{(1)}\, dA_{(1)} - \tfrac{1}{2}A_{(1)}\, d\mathcal{A}_{(1)} \ , \\
&\mathcal{F}_{(2)} = d\mathcal{A}_{(1)} \ , \qquad G^I_{(2)} = m dB^I_{(1)} + dB^I_{(0)}\, \mathcal{A}_{(1)} \ , \\
&F_{(2)} = dA_{(1)} + B^I_{(0)}\, dB^I_{(1)} + \tfrac{1}{2}B^I_{(0)}\, B^I_{(0)}\, d\mathcal{A}_{(1)} \ .
\end{aligned}
\tag{7.148}
$$

(A field redefinition has been made here, to move the derivative off the axionic scalars $B^I_{(0)}$; thi is analogous to the one we did in the nine-dimensional theory coming from the T^2 reduction of eleven-dimensional supergravity.) Note that we are omitting the wedge symbols here, to avoid some clumsiness in the appearance of the equations.

Let us just focus on the scalar part of the Lagrangian, namely

$$
\mathcal{L} = -\tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2}{*d\varphi} \wedge d\varphi - \tfrac{1}{2}e^{\sqrt{2}\varphi} \sum_I {*dB^I_{(0)}} \wedge dB^I_{(0)} \ . \tag{7.149}
$$

We may first observe that the dilaton ϕ is decoupled from the rest of the scalar Lagrangian; it just contributes a global $\mathbb{R}$ symmetry of constant shift transformations $\phi \longrightarrow \phi + c$. We shall ignore ϕ from now on. The rest of the scalar manifold can be described as follows. First, introduce a Cartan generator H, and positive-root generators E_I, with the commutation relations

$$
[H, H] = 0 \ , \qquad [H, E_I] = \sqrt{2}\, E_I \ , \qquad [E_I, E_J] = 0 \ . \tag{7.150}
$$

We define the coset representative $\mathcal{V}$ as

$$
\mathcal{V} = e^{\frac{1}{2}\varphi\, H}\, e^{B^I_{(0)}\, E_I} \ . \tag{7.151}
$$

It is easily seen that

$$
d\mathcal{V}\, \mathcal{V}^{-1} = \tfrac{1}{2}d\varphi\, H + dB^I_{(0)}\, E_I \ . \tag{7.152}
$$

Now, we wish to argue that H and E_I generate a subalgabra of $O(1, N+1)$. In, fact, we want to argue that they generate the solvable Lie algebra of $O(1, N+1)$. The orthogonal algebras $O(p, q)$ divide into two cases, namely the D_n series when $p + q = 2n$, and the B_n series when $p + q = 2n + 1$. The positive roots are given in terms of an orthonormal basis e_i as follows:

$$
\begin{aligned}
&D_n : \qquad e_i \pm e_j \ , \qquad i < j \le n \ , \\
&B_n : \qquad e_i \pm e_j \ , \qquad i < j \le n \ , \qquad \text{and} \qquad e_i \ ,
\end{aligned}
\tag{7.153}
$$

where $e_i \cdot e_j = \delta_{ij}$. It is sometimes convenient to take e_i to be the n-component vector $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the "1" component occurs at the i'th position. The

Cartan subalgebra generators, specified in a basis-independent fashion, are h_{e_i}, which satisfy $[h_{e_i}, E_{e_j \pm e_k}] = (\delta_{ij} \pm \delta_{ik})\, E_{e_j \pm e_k}$, *etc.* Of these, $\min(p,q)$ are non-compact, with the remainder being compact. It is convenient to take the non-compact ones to be h_{e_i} with $1 \le i \le \min(p,q)$.

Returning now to our algebra (7.150), we find that the generators H and E_I can be expressed in terms of the $O(1,N+1)$ basis as follows:

$$\begin{aligned} H &= \sqrt{2}\, h_{e_1}\ , \\ E_{2k-1} &= E_{e_1 - e_{2k}}\ , \qquad E_{2k} = E_{e_1 + e_{2k}} \qquad 1 \le k \le [\tfrac{1}{2} + \tfrac{1}{4}N]\ , \\ E_{1+\frac{1}{2}N} &= E_{e_1}\ , \qquad \text{if } N \text{ is even}\ . \end{aligned} \tag{7.154}$$

It is easily seen that h_{e_1} and $E_{e_1 \pm e_i}$, together with E_{e_1} in the case of N even, are precisely the generators of the solvable Lie algebra of $O(1,N+1)$. In other words, h_{e_1} is the non-compact Cartan generator of $O(1,N+1)$, while the other generators in (7.154) are precisely the subset of positive-root $O(1,N+1)$ generators that have strictly positive weights under h_{e_1}. Thus it follows from the general discussion at the beginning of this section that the scalar Lagrangian for the $D=9$ theory is described by the coset[5] $(O(1,N{+}1)/O(N{+}1)) \times \mathbb{R}$. (Recall that there is the additional decoupled scalar field ϕ with an $\mathbb{R}$ shift symmetry.)

[5]It should be emphasised that the mere fact that one can embed the algebra (7.150) into the Lie algebra of a larger Lie group G does not, of itself, mean that the group G acts effectively on the scalar manifold. Only when (7.150) is the solvable Lie algebra of the group G can one deduce that G has an effective group action on the scalar manifold.

8 Complex Manifolds and Calabi-Yau Spaces

8.1 Introduction

In the previous chapter, we studied in some detail the process of Kaluza-Klein dimensional reduction on an n-dimensional torus. Other possibilities for dimensional reduction exist also, in which the compactifying manifold is taken to be a certain kind of complex manifold, known as a Calabi-Yau manifold. Complex manifolds have made appearances in other branches of theoretical physics too, but probably the most important is in the framework of string compactifications. There are various reasons why they can be regareded as preferable to simple toroidal compactifying manifolds, principally because they offer some prospect of yielding lower-dimensional spacetime theories with more phenomenologically promising characteristics.

A n-dimensional manifold M is a topological space together with an *atlas*, i.e. a collection of *charts* (U_i, x_j) where U_j are open subsets of M and the x_j are one-to-one maps of the corresponding U_j to open subsets of $\mathbb{R}^n$. In other words, x_j represents a set of coordinates x_j^μ, $1 \le \mu \le n$, which covers the open region U_j in M. The compete atlas of charts covers the entire manifold M, but in general, no single chart can cover all of M. If two of the regions J_j and U_k have an overlap, then the map obtained by composing $x_j \cdot x_k^{-1}$ takes us from one copy of $\mathbb{R}^n$ to the another. Put another way, this means that in the overlap region, one can view the coordinates x_j^μ as functions of the x_k^ν, i.e. $x_j^\mu = f_{jk}^\mu(x_k^\nu)$. The manifold is said to be C^r if the transition functions are r-times differentiable. Normally, one considers manifolds that are C^∞.

A *complex* n-manifold is a topological space M of complex dimension n with a holomorphic, or complex-analytic, atlas. Thus one now has a collection of charts (U_j, z_j), where in every non-empty intersection the maps $z_j \cdot z_k^{-1}$ are holomorphic. Of course the z_j are now maps into C^n. Thus the transition functions are now required to be holomorphic functions of the complex coordinates in the two overlapping charts: $z_j^\mu = f_{jk}^\mu(z_k^\nu)$, rather than being C^∞. Thus the z_j^μ are functions of z_k^ν, but not of $\bar z_k^\nu$. Since C^n can be thought of as $\mathbb{R}^{2n}$, it follows that every complex n-manifold is also a real $(2n)$-manifold.[6]

Not every real $(2n)$-manifold is a complex n-manifold, however. A simple non-trivial example that *is* a complex manifold is the 2-sphere. Imagine sandwiching a 2-sphere between

[6]Do not confuse the use of C^r to mean an r-times differentiable function with C^n to mean complex n-dimensional space! Amost all the time, we mean the complex n space when the symbol C^n appears. It shopuld be clear from the context, and so there should be no confusion.

two infinite parallel plates T_1 and T_2, which are tangent to the sphere at the south and the north poles S and N respectively. We may parameterise a point P on the sphere in terms of the (x, y) coordinates in the planes T_1 or T_2 of the points obtained by passing a straight line from N through P to T_1, or S through P to T_2, respectively. Call these coordinates (x_1, y_1) and (x_2, y_2) respectively. Obviously, for a generic point P on the sphere, there are corresponding well-defined points (x_1, y_1) and (x_2, y_2) in the planes T_1 and T_2, and there is a one-to-one map between the two descriptions. This will break down only if $P = N$ or $P = S$, since then (x_1, y_1) or (x_2, y_2) respectively will go to infinity. For generic points P, simple geometry shows that the relation between the coordinates in the two charts is

$$x_1 = \frac{x_2}{x_2^2 + y_2^2}\,, \qquad y_1 = -\frac{y_2}{x_2^2 + y_2^2}\,. \tag{8.1}$$

Clearly these functions are C^∞ for generic points where the two charts overlap, i.e. provided the north and south poles are excluded. To see that the 2-sphere is a complex manifold, we now introduce the complex coordinate $z_1 = x_1 + \mathrm{i}\, y_1$ on T_1, and likewise $z_2 = x_2 + \mathrm{i}\, y_2$ on T_2. It is easy to see that the real C^∞ transition functions defined by (8.1) can be re-expressed in terms of z as

$$z_1 = \frac{1}{z_2}\,. \tag{8.2}$$

This is holomorphic, or complex analytic, in the overlap region (i.e. for $z_2 \ne 0,\ \infty$), thus demonstrating that S^2 is a complex manifold.

8.2 Almost Complex Structures and Complex Structures

Suppose that M is a complex n-manifold, with coordinates z^μ in some neighbourhood U. We define the 2-index mixed tensor J, by

$$J = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\, dz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\,, \tag{8.3}$$

where we use the notation $z^{\bar\mu}$ to stand for $\bar z^\mu$. In terms of components, we see that

$$J_\mu{}^\nu = \mathrm{i}\,\delta_\mu{}^\nu\,, \qquad J_{\bar\mu}{}^{\bar\nu} = -\mathrm{i}\,\delta_{\bar\mu}{}^{\bar\nu}\,, \qquad J_{\bar\mu}{}^\nu = 0\,, \qquad J_\mu{}^{\bar\nu} = 0\,. \tag{8.4}$$

J is called an *Almost Complex Structure.*

Note that J as defined is indeed a tensor, since it is independent of the choice of complex coordinates. The crucial point here is that we allow only *holomorphic* coordinate transformations, and so the first and the second terms in (8.3) are separately unchanged under such transformations. Thus if $z^\mu = z^\mu(w^\nu)$, then

$$dz^\mu \otimes \frac{\partial}{\partial z^\mu} = \frac{\partial z^\mu}{\partial w^\nu}\,\frac{\partial w^\rho}{\partial z^\mu}\, dw^\nu \otimes \frac{\partial}{\partial w^\rho} = dw^\mu \otimes \frac{\partial}{\partial w^\mu}\,, \tag{8.5}$$

with an analogous result for the complex conjugate. It is also evident that J is itself a real tensor.

As we remarked previously, we may think of the complex n-manifold as being also a real $(2n)$-manifold. Suppose that in the local neighbourhood U we have real coordinates x^i, with $1 \le i \le 2n$. It can also be useful to think of the real coordinates as occurring in pairs, so that $z^\mu = x^\mu + \mathrm{i}\, y^\mu$, for $1 \le \mu \le n$. We then see that the definition (8.3) for J becomes

$$J = dx^\mu \otimes \frac{\partial}{\partial y^\mu} - dy^\mu \otimes \frac{\partial}{\partial x^\mu} \ . \tag{8.6}$$

We may therefore represent the tensor J as a real $(2n)\times(2n)$ matrix in $n \times n$ blocks as

$$J = \begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix} \ , \tag{8.7}$$

where O_n denotes the zero $n\times n$ matrix, and $\mathbb{1}_n$ dentotes the unit $n\times n$ matrix. Evidently, therefore, the tensor J squares to minus 1:

$$J_i{}^j\, J_j{}^k = -\delta_i{}^k \ . \tag{8.8}$$

It is a theorem that every complex manifold admits a *globally defined* almost complex structure. The emphasis here is on the fact that it is globally defined, since obviously *any* real $(2n)$-manifold, since it looks locally like $\mathbb{R}^{2n}$, must look locally like C^n. The converse, however, is not true: not every manifold that admits an almost complex structure is a complex manifold. Rather, such a manifold is, by definition an *Almost Complex Manifold.* In the special case where an almost complex manifold is actually a complex manifold, the almost complex structure is called a complex structure. To see why it is the case that not every almost complex manifold is a complex manifold, we need to delve a little deeper into the properties of the almost complex structure tensor.

From $J_i{}^j$, in a complex manifold we can define projection operators $P_i{}^j$ and $Q_i{}^j$:

$$P_i{}^j = \tfrac12(\delta_i{}^j - \mathrm{i}\, J_i{}^j) \ , \qquad Q_i{}^j = \tfrac12(\delta_i{}^j + \mathrm{i}\, J_i{}^j) \ , \tag{8.9}$$

which clearly satisfy the relations

$$P^2 = P \ , \qquad Q^2 = Q \ , \qquad P\,Q = Q\,P = 0 \ , \qquad P + Q = \mathbb{1} \ , \tag{8.10}$$

in the obvious matrix notation. These operators project into the holomorphic and anti-holomorphic components of tensors. Thus, for example,

$$V_i\, dx^i = (P_i{}^j + Q_i{}^j)\, V_j\, dx^i = V_\mu\, dz^\mu + V_{\bar\mu}\, dz^{\bar\mu} \ , \tag{8.11}$$

where

$$P_i{}^j\, V_j\, dx^i = V_\mu\, dz^\mu\ , \qquad Q_i{}^j\, V_j\, dx^i = V_{\bar\mu}\, dz^{\bar\mu}\ . \tag{8.12}$$

These 1-forms are called $(1,0)$-forms and $(0,1)$-forms respectively. Generally, it is clear that the existence of an almost complex structure allows the refinement of the notion of n-forms, into subsets of (p,q)-forms, where $p+q=n$.

The question now is the following. What further conditions are necessary in order for an almost complex structure J to be a complex structure? In other words, what are the conditions for the almost complex manifold, with almost complex structure J, to be a complex manifold?

First of all, note that we can still define the projection operators $P_i{}^j$ and $Q_i{}^j$ as in (8.9) whenever we have an almost complex structure, although we should not yet think of them as projections into holomorphic and anti-holomorphic subspaces of forms. To have a complex manifold, we must be able to introduce complex coordinates z^μ. Thus, consider a neighbourhood U in the almost complex manifold M, with real coordinates x^i. We wish to see if we can find complex coordinates $z^\mu(x^i)$. Thus we can write

$$dz^\mu = \frac{\partial z^\mu}{\partial x^i}\, dx^i\ , \tag{8.13}$$

which can be expressed, by inserting $\delta_i{}^j = P_i{}^j + Q_i{}^j$, as

$$dz^\mu = \partial_j z^\mu\, P_i{}^j\, dx^i + \partial_j z^\mu\, Q_i{}^j\, dx^i\ . \tag{8.14}$$

Now, we saw previously that the two terms on the right-hand side are respectively $(1,0)$ and $(0,1)$ forms, while the left-hand side is manifestly what we should call a $(1,0)$ form if the complex coordinates do indeed exist. Consequently, is must be that

$$\partial_j z^\mu\, Q_i{}^j = 0\ . \tag{8.15}$$

This can be viewed as a system of differential equations for the complex coordinates z^μ. If the equations are satisfied, then we can act with $Q_k{}^\ell\, \partial_\ell$ to get

$$\partial_\ell \partial_j z^\mu\, Q_i{}^j\, Q_k{}^\ell + \partial_j z^\mu\, \partial_\ell Q_i{}^j\, Q_k{}^\ell = 0\ . \tag{8.16}$$

Taking the projection of this equation that is skew-symmetric in i and k, we therefore obtain the integrability condition

$$\partial_j z^\mu\, \partial_\ell Q_{[i}{}^j\, Q_{k]}{}^\ell = 0\ . \tag{8.17}$$

We can now insert $P_m{}^j + Q_m{}^j = \delta_m{}^j$, and make use of (8.15), to re-express the integrability condition as

$$\partial_j z^\mu\, P_m{}^j\, \partial_\ell Q_{[i}{}^m\, Q_{k]}{}^\ell = 0\ . \tag{8.18}$$

In order for the derivatives $\partial z^\mu/\partial x_j$, which must already satisfy (8.15), not to be overconstrained, it must be that

$$P_m{}^j\, \partial_\ell Q_{[i}{}^m\, Q_{k]}{}^\ell = 0\ . \tag{8.19}$$

By taking the real and imaginary parts of this equation, one can easily show that each is equivalent to the statement that the following real tensor must vanish:

$$N_{ij}{}^k \equiv \partial_{[j} J_{i]}{}^k - J_{[i}{}^\ell\, J_{j]}{}^m\, \partial_m J_\ell{}^k\ . \tag{8.20}$$

This is known as the Nijenhuis tensor. (Note that it is indeed a tensor, even though it is defined using partial derivatives. This can be verified by direct calculation of its behaviour under general coordinate transformations. Alternatively, one can consider the manifestly tensorial object defined by replacing the partial derivatives by covariant derivatives, and then verify that all the connection terms cancel out by virtue of the antisymmetrisations.)

To summarise, then, we have seen that the vanishing of the Nijenhuis tensor is an integrability condition for the existence of a complex coordinate system in an almost complex manifold. As usual, establishing a completely watertight "if and only if" theorem is something best left to the hard-core mathemeticians. The bottom line, in any case, is that an almost complex manifold can be shown to be a complex manifold if and only if the Nijenhuis tensor vanishes.

Also, for future reference, let us establish some further notation and terminology for differential forms on almost complex and complex manifolds. We have seen that the tensors $P_i{}^j$ and $Q_i{}^j$ project 1-forms into two subspaces, which we are denoting by $(1,0)$ and $(0,1)$ forms. More generally, given any n form ω , we can make projections into $(n+1)$ subspaces, of (p,q)-forms where $p+q=n$, as follows:

$$\omega^{(p,q)}_{i_1\cdots i_p j_1\cdots j_q} = P_{i_1}{}^{k_1} \cdots P_{i_p}{}^{k_p}\, Q_{j_1}{}^{\ell_1} \cdots Q_{j_q}{}^{\ell_q}\, \omega_{k_1\cdots k_p \ell_1\cdots\ell_q}\ . \tag{8.21}$$

It is evident from the properties of P and Q as projection operators that the sum over these various (p,q)-forms gives back the original n-form:

$$\omega = \sum_{p+q=n} \omega^{(p,q)}\ . \tag{8.22}$$

Now consider the action of the exterior derivative d. It is easy to see from the definitions that if we apply d to a (p,q)-form in an almost complex manifold, we will obtain a $(p+q+1)$-form that is expressible in general as the sum of four distinct terms, namely

$$d\omega^{(p,q)} = (d\omega)^{(p,q+1)} + (d\omega)^{(p+1,q)} + (d\omega)^{(p+2,q-1)} + (d\omega)^{(p-1,q+2)}\ . \tag{8.23}$$

If J is in fact a *complex structure*, then the last two terms in this decomposition are absent. To see how this works, consider, for simplicity, a $(1,0)$ form A, which we may construct from a generic 1-form ω by defining $A = P_i{}^j\,\omega_j\,dx^i$. We now calculate dA, and then insert $1 = P + Q$ judicously in all necessary places so as to project out the various structures:

$$\begin{aligned} dA &= (\partial_k P_i^j\,\omega_j + P_i^j\,\partial_k\omega_j)dx^k\wedge dx^i \\ &= (\partial_\ell P_m{}^j\,\omega_j\,P_k{}^\ell\,P_i{}^m + \partial_\ell P_m{}^j\,\omega_j\,P_k{}^\ell\,Q_i{}^m + \partial_\ell P_m{}^j\,\omega_j\,Q_k{}^\ell\,P_i{}^m + \partial_\ell P_m{}^j\,\omega_j\,Q_k{}^\ell\,Q_i{}^m \\ &\quad + P_i^j\,P_k{}^\ell\,\partial_\ell\omega_j + P_i^j\,Q_k{}^\ell\,\partial_\ell\omega_j)dx^k\wedge dx^i\ . \end{aligned} \tag{8.24}$$

It is manifest that these six terms are of types $(2,0)$, $(1,1)$, $(1,1)$, $(0,2)$, $(2,0)$ and $(1,1)$ respectively. If the almost complex structure is in fact a complex structure, we expect that dA should have only $(2,0)$ and $(1,1)$ components, and so it would then have to be that the $(0,2)$ component were zero. This would imply that we need

$$\partial_\ell P_m{}^j\,Q_{[k}{}^\ell\,Q_{i]}{}^m = 0\ . \tag{8.25}$$

Now, since the projection operators satisfy $P_m{}^j\,Q_i{}^m = 0$, it follows that $\partial_\ell P_m{}^j\,Q_i{}^m + P_m{}^j\,\partial_\ell Q_i{}^m = 0$, and using this, we see that (8.25) reduces to (8.19). Thus we see that indeed the exterior derivative of a $(1,0)$-form gives only a $(2,0)$ and a $(1,1)$ form, but no $(0,2)$ form, provided that the Nijenhuis tensor vanishes, implying that the almost complex structure is a complex structure.

If we do have a complex structure, so that $d\omega^{(p,q)} = (d\omega)^{(p+1,q)} + (d\omega)^{(p,q+1)}$, we may then define holomorphic and antiholomorphic exterior derivative operators ∂ and $\bar\partial$, where

$$\begin{aligned} &d = \partial + \bar\partial\ , \\ &\partial\omega^{(p,q)} = (d\omega)^{(p+1,q)}\ , \qquad \bar\partial\omega^{(p,q)} = (d\omega)^{(p,q+1)}\ . \end{aligned} \tag{8.26}$$

Thus $\partial f(z,\bar z) = \partial f/\partial z^\mu\,dz^\mu$, and $\bar\partial f(z,\bar z) = \partial f/\partial z^{\bar\mu}\,dz^{\bar\mu}$, *etc.* Note that we have $d^2 = (\partial+\bar\partial)^2 = \partial^2 + \bar\partial^2 + \partial\bar\partial + \bar\partial\partial$, and hence in a complex manifold we have

$$\partial^2 = 0\ , \qquad \bar\partial^2 = 0\ , \qquad \partial\bar\partial + \bar\partial\partial = 0\ , \tag{8.27}$$

since these three parts of d^2 have different holomorphic degrees.

A further consequence of the vanishing of the Nijenuis tensor is that there exists a holomorphic atlas with respect to which the components of the complex structure are given by

$$J_\mu{}^\nu = \mathrm{i}\,\delta_\mu{}^\nu\ , \qquad J_{\bar\mu}{}^{\bar\nu} = -\mathrm{i}\,\delta_{\bar\mu}{}^{\bar\nu}\ , \qquad J_{\bar\mu}{}^\nu = 0\ , \qquad J_\mu{}^{\bar\nu} = 0\ . \tag{8.28}$$

To see this, note that we can write (8.15) as

$$\partial_j\, z^\mu + \mathrm{i}\, J_j{}^k \partial_k z^\mu = 0\ . \tag{8.29}$$

In fact this is precisely the n-dimensional generalisation of the Cauchy-Riemann equations. Contracting with the basis $dx^j \otimes \partial/\partial z^\mu$, we have

$$J_j{}^k\, dx^j\, \partial_k z^\mu \otimes \frac{\partial}{\partial z^\mu} = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu}\ . \tag{8.30}$$

If we add the complex conjugate to this equation, we get

$$J_j{}^k \Big(dx^j\, \partial_k z^\mu \otimes \frac{\partial}{\partial z^\mu} + dx^j\, \partial_k z^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\Big) = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\, dz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\ , \tag{8.31}$$

which, by the chain rule, is nothing but

$$J_j{}^k\, dx^j \otimes \partial_k = \mathrm{i}\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\, dz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}}\ . \tag{8.32}$$

Thus the complex structure tensor $J \equiv J_j{}^k\, dx^j \otimes \partial_k$ does indeed have components, with respect to the complex coordinate basis, given by (8.28).

8.3 Metrics on Almost Complex and Complex Manifolds

So far in the discussion, our considerations have been entirely independent of any metric on the manifold. Suppose that an almost complex manifold M has a metric h_{ij}, i.e. a real, symmetric 2-index tensor with positive-definite eigenvalues. We can always construct from this a so-called *almost Hermitean metric* g_{ij}, defined as

$$g_{ij} = \tfrac12 (h_{ij} + J_i{}^k\, J_j{}^\ell\, h_{k\ell})\ , \tag{8.33}$$

which is also real, symmetric and positive-definite. It clearly satisfies the almost-hermiticity condition

$$g_{ij} = J_i{}^k\, J_j{}^\ell\, g_{k\ell}\ . \tag{8.34}$$

Another way of expressing this, by multiplying by $J_m{}^i$, and defining $J_m{}^i\, g_{ij} = J_{mj}$, is

$$J_{mj} = -J_{jm}\ . \tag{8.35}$$

Thus with respect to an almost Hermitean metric, the almost complex structure defines a natural 2-form.

If M is actually a complex manifold, then it is evident that an Hermitean metric has the property that

$$ds^2 = 2\, g_{\mu\bar\nu}\, dz^\mu\, dz^{\bar\nu}\ . \tag{8.36}$$

This should be contrasted with a generic real symmetric metric, for which we would have

$$ds^2 = 2\, g_{\mu\bar\nu} dz^\mu\, dz^{\bar\nu} + g_{\mu\nu}\, dz^\mu\, dz^\nu + g_{\bar\mu\bar\nu}\, dz^{\bar\mu}\, dz^{\bar\nu}\ . \tag{8.37}$$

A consequence of the structure (8.36) of the Hermitean metric is that when complex indices are raised or lowered, barred become unbarred, and *vice versa.*

Suppose now that M is an Hermitean manifold, meaning that it is a complex manifold endowed with an Hermitean metric. We can now introduce the notion of a covariant derivative. Normally, in real geometry, we define the unique Christofel connection $\Gamma^i{}_{jk}$ by two conditions, namely that the covariant derivative (defined using $\Gamma^i{}_{jk}$) of the metric be zero, and that $\Gamma^i{}_{jk}$ be symmetric in its two lower indices. More generally, we could consider a connection for which the metric is still covariantly constant, but where there is an antisymmetric part $\Gamma^i{}_{[jk]}$ also. This extra term is known as a torsion tensor.

For the Hermitean manifold M, we may define a unique connection as follows. We require that the covariant derivative both of the metric, and of the complex structure tensor, vanish. In addition, we require that the torsion $\Gamma^i{}_{[jk]}$ be *pure* in its lower indices. In other words, if we use complex coordinates, we require that $\Gamma^i{}_{[\mu\bar\nu]}$ be zero, where i represents either ρ or $\bar\rho$, while no requirement is placed on $\Gamma^i{}_{[\mu\nu]}$ or $\Gamma^i{}_{[\bar\mu\bar\nu]}$. To see what this leads to, we may consider taking the covariant derivative of $P_i{}^j$, which, by our requirements for the connection, must vanish.

First, let us write down the general expression for the covariant derivative:

$$\nabla_i\, P_j{}^k \equiv \partial_i\, P_j{}^k + \Gamma^k{}_{i\ell}\, P_j{}^\ell - \Gamma^\ell{}_{ij}\, P_\ell{}^k\ . \tag{8.38}$$

Now, noting that we may choose complex coordinates such that the complex structure has components given by (8.28), it follows, from (8.9), that the only non-vanishing components of $P_i{}^j$ are given by

$$P_\mu{}^\nu = \delta_\mu{}^\nu\ . \tag{8.39}$$

Thus if we consider the covariant-constancy condition $\nabla_i\, P_j{}^k = 0$, then the content of this equation is encompassed by taking (j,k) to be either (μ,ν), or else $(\mu,\bar\nu)$. From (8.38), the first case tells us nothing, since we get

$$0 = \nabla_i\, P_\mu{}^\nu = \Gamma^\nu{}_{i\rho}\, \delta_\mu{}^\rho - \Gamma^\rho{}_{i\mu}\, \delta_\rho{}^\nu\ . \tag{8.40}$$

On the other hand, we do learn something from taking $(i,j) = (\mu,\bar\nu)$, since then we get

$$0 = \nabla_i\, P_\mu{}^{\bar\nu} = \Gamma^{\bar\nu}{}_{k\rho}\, \delta_\mu{}^\rho\ , \tag{8.41}$$

and hence $\Gamma^{\bar\nu}{}_{k\mu} = 0$. Thus we have

$$\begin{aligned} \Gamma^{\bar\nu}{}_{\rho\mu} &= 0 \ , \qquad \Gamma^{\bar\nu}{}_{\bar\rho\mu} = 0 \ , \\ \Gamma^{\nu}{}_{\bar\rho\bar\mu} &= 0 \ , \qquad \Gamma^{\nu}{}_{\rho\bar\mu} = 0 \ , \end{aligned} \tag{8.42}$$

where the second line follows by complex conjugation of the first. Now, we also have the condition that the mixed components $\Gamma^i{}_{[\rho\bar\mu]}$ of the torsion vanish. Together with what we already have, this therefore implies that *all* mixed components of $\Gamma^i{}_{jk}$ vanish. In other words, the only non-vanishing components of the Hermitean connection are the pure ones,

$$\Gamma^{\mu}{}_{\nu\rho} \qquad \text{and} \qquad \Gamma^{\bar\mu}{}_{\bar\nu\bar\rho} \ . \tag{8.43}$$

Since the claim is that the Hermitean connection just defined is unique, we expect to be able to solve for it in terms of the Hermitean metric. This is indeed possible. Since the metric is covariantly constant, we have

$$\nabla_i\, g_{jk} \equiv \partial_i\, g_{jk} - \Gamma^{\ell}{}_{ij}\, g_{\ell k} - \Gamma^{\ell}{}_{ik}\, g_{j\ell} = 0 \ . \tag{8.44}$$

If we take $(i,j,k) = (\mu,\nu,\bar\rho)$, we get, in view of the previous results for the purity of the connection,

$$\partial_\mu\, g_{\nu\bar\rho} - \Gamma^{\lambda}{}_{\mu\nu}\, g_{\lambda\bar\rho} = 0 \ , \tag{8.45}$$

which can therefore be immediately solved to give

$$\Gamma^{\lambda}{}_{\mu\nu} = g^{\lambda\bar\rho}\, \partial_\mu\, g_{\nu\bar\rho} \ . \tag{8.46}$$

As a consequence of the purity of the Hermitean connection, it follows that the Riemann tensor has a simple structure also. To see this, let us first write down the general expression for the Riemann tensor, namely

$$R^i{}_{jk\ell} = \partial_k\, \Gamma^i{}_{\ell j} - \partial_\ell\, \Gamma^i{}_{kj} + \Gamma^i{}_{km}\, \Gamma^m{}_{\ell j} - \Gamma^i{}_{\ell m}\, \Gamma^m{}_{kj} \ . \tag{8.47}$$

Taking first $(i,j) = (\bar\mu,\nu)$, we see that

$$R^{\bar\mu}{}_{\nu k\ell} = \partial_k\, \Gamma^{\bar\mu}{}_{\ell\nu} - \partial_\ell\, \Gamma^{\bar\mu}{}_{k\nu} + \Gamma^{\bar\mu}{}_{km}\, \Gamma^m{}_{\ell\nu} - \Gamma^{\bar\mu}{}_{\ell m}\, \Gamma^m{}_{k\nu} \ , \tag{8.48}$$

and that all the terms here vanish by virtue of the purity of the connection coefficients. Thus lowering the $\bar\mu$ index, and recalling that the only non-vanishing metric components are of the form $g_{\mu\bar\nu}$, we see that

$$R_{\mu\nu k\ell} = 0 \ . \tag{8.49}$$

Similarly, one can see that the purity of Γ implies that the components $R_{\mu\bar\nu\rho\sigma}$ must vanish. The components $R_{\bar\mu\nu\rho\sigma}$ vanish for a different reason, namely because of the expression (8.46) for the connection coefficients in terms of the metric. The upshot is that the only non-vanishing components of the Riemann tensor are those given by

$$R^\mu{}_{\nu\bar\rho\sigma} = -R^\mu{}_{\nu\sigma\bar\rho} = \partial_{\bar\rho}\,\Gamma^\mu{}_{\sigma\nu}\ , \tag{8.50}$$

together with those following by complex conjugation. In other words, the only non-vanishing components are those which, if we lower the upper index, are mixed on both their first and second index pairs:

$$R_{\mu\bar\nu\rho\bar\sigma}\ , \qquad R_{\bar\nu\mu\rho\bar\sigma}\ , \qquad R_{\mu\bar\nu\bar\sigma\rho}\ , \qquad R_{\bar\nu\mu\bar\sigma\rho}\ . \tag{8.51}$$

Owing to the existence of the complex structure tensor J, it is possible to define from the Riemann tensor a 2-form $\mathcal{R}$, known as the *Ricci form*, as follows:

$$\mathcal{R} = \tfrac14 R^i{}_{jk\ell}\, J_i{}^j\, dx^k\wedge dx^\ell\ . \tag{8.52}$$

In terms of complex coordinates, it follows from (8.28), and the structure that we have learnt for the Riemann tensor, that we have

$$\mathcal{R} = \mathrm{i}\, R^\mu{}_{\mu\rho\bar\sigma}\, dz^\rho\wedge dz^{\bar\sigma}\ . \tag{8.53}$$

From (8.46) and (8.50), it now follows that we can express the Ricci form as

$$\mathcal{R} = \mathrm{i}\,\partial\bar\partial\log\sqrt{g}\ , \tag{8.54}$$

where g is the determinant of the metric. From the properties of ∂ and $\bar\partial$ given in (8.27), it follows that $\partial\bar\partial = -\frac12 d(\partial-\bar\partial)$, and hence we have that

$$d\mathcal{R} = 0\ . \tag{8.55}$$

Note, however, although the Ricci form is closed, it is not, in general, exact, since the determinant of the metric is not a coordinate scalar. In fact, the Ricci form defines a cohomology class, namely the first Chern class, of the complex manifold. This is a topological class, which is invariant under smooth deformations of the complex structure J, and the metric. In other words, under any such deformation, the Ricci form changes by an exact form, and thus its integral over any closed 2-cycle is unchanged. The first Chern class c_1 is defined as the equivalence class of all 2-forms related to a certain multiple of the Ricci-form by the addition of an exact form, and is written as

$$c_1 = \Big[\frac{1}{2\pi}\,\mathcal{R}\Big]\ . \tag{8.56}$$

It is easy to see that $\mathcal{R}$ changes by an exact form under infinitesimal deformations of the metric, since under $g_{ij} \longrightarrow g_{ij} + \delta g_{ij}$ we have $\delta\sqrt{g} = \frac{1}{2} g^{ij}\, \delta g_{ij}\, \sqrt{g}$, and hence

$$\delta\mathcal{R} = \mathrm{i}\, \partial\bar\partial (g^{\mu\bar\nu}\, \delta g_{\mu\bar\nu}) = -\frac{\mathrm{i}}{2}\, d\Big((\partial - \bar\partial) g^{\mu\bar\nu}\, \delta g_{\mu\bar\nu}\Big) \ . \tag{8.57}$$

Since $g^{\mu\bar\nu}\, \delta g_{\mu\bar\nu}$ *is* a genuine general-coordinate scalar, even though $\det(g_{\mu\bar\nu})$ is not, it follows that $\mathcal{R}$ changes by an exact form, and thus c_1 is unaltered.

8.4 Kähler Manifolds

An Hermitean manifold M has, as we have seen, a natural 2-form $J = \frac{1}{2} J_{ij}\, dx^i \wedge dx^j$ that is obtained by lowering the upper index on the complex structure tensor $J_i{}^j$. We can now impose one further level of structure on the Hermitean manifold, by requiring that the 2-form be closed,

$$dJ = 0 \ . \tag{8.58}$$

An Hermitean manifold that satisfies this condition is called a Kähler manifold.[7] The 2-form J is then called the Kähler form. Note that all manifolds of complex dimension 1 are necessarily Kähler, since the exterior derivative of the 2-form J is a 3-form, which exceeds the real dimension of the manifold.

Note that from the pattern of the non-vanishing components of $J_i{}^j$ given in (8.28), it follows that the Kähler form can be written as

$$J = \mathrm{i}\, g_{\mu\bar\nu}\, dz^\mu \wedge dz^{\bar\nu} \ . \tag{8.59}$$

It is therefore a $(1,1)$-form.

Writing dJ as $\partial J + \bar\partial J = 0$, we may note that these two pieces must vanish separately, since they are forms of different types, namely $(2,1)$ and $(1,2)$. Thus we have

$$dJ = \mathrm{i}\, \partial_\rho\, g_{\mu\bar\nu}\, dz^\rho \wedge dz^\mu \wedge dz^{\bar\nu} + \mathrm{i}\, \partial_{\bar\rho}\, g_{\mu\bar\nu}\, dz^{\bar\rho} \wedge dz^\mu \wedge dz^{\bar\nu} = 0 \ , \tag{8.60}$$

and hence

$$\partial_\rho\, g_{\mu\bar\nu} - \partial_\mu\, g_{\rho\bar\nu} = 0 \ , \qquad \partial_{\bar\rho}\, g_{\mu\bar\nu} - \partial_{\bar\nu}\, g_{\mu\bar\rho} = 0 \ . \tag{8.61}$$

These equations imply that locally we must be able to express the Kähler metric in the form

$$g_{\mu\bar\nu} = \partial_\mu\, \partial_{\bar\nu}\, K \ , \tag{8.62}$$

[7] By now, one might almost suspect that there would exist also the notion of an "almost Kähler manifold," for which the 2-form J in an almost Hermitean manifold would be closed. In fact, it can be shown that an almost Kähler manifold is actually Kähler. It was some while before this was appreciated, and so in some older literature one can find a distinction between the two concepts.

where $K = K(z, \bar{z})$ is a real function of the complex coordinates and their complex conjugates. This implies that we have

$$J = \mathrm{i}\,\partial\,\bar{\partial}\,K\ . \tag{8.63}$$

The function K is called the Kähler function. However, it should be emphasised that it is not, in general, a general-coordinate scalar. To see this, consider the n-fold wedge product $J^n = J \wedge J \wedge \cdots \wedge J$ on the complex n-manifold. From (8.59), it is evident that

$$\begin{aligned} J^n &= \mathrm{i}^n\, g_{\mu_1\bar{\nu}_1} \cdots g_{\mu_n\bar{\nu}_n}\, dz^{\mu_1}\, dz^{\bar{\nu}_1} \cdots dz^{\mu_n}\, dz^{\bar{z}_n}\ , \\ &= \mathrm{i}^n\, \epsilon^{\mu^1\cdots\mu_n}\, \epsilon^{\bar{\nu}_1\cdots\bar{\nu}_n}\, g_{\mu_1\bar{\nu}_1} \cdots g_{\mu_n\bar{\nu}_n}\, dz^1\, dz^{\bar{1}} \cdots dz^n\, dz^{\bar{n}}\ , \\ &= \mathrm{i}^n\, n!\, \det(g_{\mu\bar{\nu}})\, dz^1\, dz^{\bar{1}} \cdots dz^n\, dz^{\bar{n}}\ . \end{aligned} \tag{8.64}$$

Now clearly $\det(g_{\mu\bar{\nu}}) = \sqrt{\det(g_{ij})}$, in view of the off-diagonal Hermitean structure of g_{ij}, and so we have $J^n = c\,{*1}$, for some specific n-dependent constant c, where $*1$ is the volume $(2n)$-form on the manifold M. Thus on a compact manifold it must be that $\int_M J^n$ is a non-vanishing constant. But (8.63) can be rewritten as

$$J = -\tfrac{1}{2} d(\partial - \bar{\partial})\, K\ . \tag{8.65}$$

Thus if K were a coordinate scalar then it would follow that $J = dA$ for some globally-defined 1-form A. However, we would then be able to write $\int_M J^n$ as $\int_M d(A\, J^{n-1}) = \int_{\partial M} A\, J^{n-1}$, and so if M had no boundary, we would have $\int_M J^n = 0$, in contradiction to the previous result. Therefore A is not globally defined, and so K is not a general-coordinate scalar.

In fact, if we consider Kähler functions K_1 and K_2 defined in open neighbourhoods U_1 and U_2 in M, with a non-trivial intersection, then they are related by

$$K_1 = K_2 + f(z) + \overline{f(z)}\ , \tag{8.66}$$

where $f(z)$ is an arbitrary holomorphic function of the coordinates. Clearly, these functions are by the $\partial\,\bar{\partial}$ derivatives that act on K, and so the Kähler form itself is well defined and transforms properly across the open neighbourhoods.

A couple of examples will be instructive at this point. First, let us consider the natural flat metric on C^n, namely $ds^2 = dz^\mu\, dz^{\bar{\nu}}\, \delta_{\mu\bar{\nu}} = |dz^1|^2 + |dz^2|^2 + \cdots + |dz^n|^2$. It is easy to see that if we define the Kähler function

$$K = z^\mu\, z^{\bar{\nu}}\, \delta_{\mu\bar{\nu}}\ , \tag{8.67}$$

then substituting into (8.62) we indeed find the desired metric

$$g_{\mu\bar\nu} = \partial_\mu\, \partial_{\bar\nu}\, (z^\rho\, z^{\bar\sigma}\, \delta_{\rho\bar\sigma}) = \delta_{\mu\bar\nu}\ . \tag{8.68}$$

Similarly, the Kähler form, given by (8.63), is

$$J = \mathrm{i}\, \partial\, \bar\partial\, (z^\mu\, z^{\bar\nu}\, \delta_{\mu\bar\nu}) = \mathrm{i}\, dz^\mu \wedge dz^{\bar\nu}\, \delta_{\mu\bar\nu}\ , \tag{8.69}$$

as expected. There is no issue of looking at overlaps between coordinate patches in this case, since this is the one example where a single coordinate patch covers the entire manifold.

For a less trivial example, consider the complex projective spaces CP^n. These are defined as follows. Begin by taking the flat metric on the complex manifold C^{n+1}, with coordinates Z^M for $1 \le M \le n+1$:

$$ds^2_{n+1} = \sum_{M=1}^{n+1} |dZ^M|^2\ . \tag{8.70}$$

Now impose the quadratic condition

$$\sum_{M=1}^{n+1} |Z^M|^2 = 1\ . \tag{8.71}$$

It is evident that both (8.70) and (8.71) are invariant under $SU(n+1)$ transformations, acting by matrix multiplication on Z^M viewed as a column vector. Imposing the constraint (8.71) clearly places the Z^M coordinates on the surface of a unit-radius sphere S^{2n+1}.

Now introduce the so-called inhomogeneous coordinates ζ^i, defined by

$$\zeta^i = Z^i/Z^{n+1}\ , \qquad 1 \le i \le n\ . \tag{8.72}$$

Actually, this is just one choice for the definition, where Z^{n+1} among the original homogeneous coordinates Z^M is singled out for special treatment. We could, and indeed later will, consider a different choice where one of the other Z^M is singled out as the special one.

Proceeding with the choice (8.72) for now, we may now express the Z^i in terms of ζ^i and Z^{n+1} using (8.72), and express $|Z^{n+1}|^2$ in terms of $|\zeta^i|^2$ using (8.71). Substituting into the metric (8.70), we therefore find

$$ds^2 = F^{-1}\, d\zeta^i\, d\bar\zeta^i + \frac{|dZ^{n+1}|^2}{|Z^{n+1}|^2} + (\bar\zeta^i\, d\zeta^i\, Z^{n+1}\, d\bar Z^{n+1} + \zeta^i\, d\bar\zeta^i\, \bar Z^{n+1}\, dZ^{n+1})\ , \tag{8.73}$$

where

$$F \equiv 1 + \sum_i |\zeta^i|^2 = |Z^{n+1}|^{-2}\ . \tag{8.74}$$

The metric can be re-expressed in the following form, by completing the square in the terms involving dZ^{n+1} and $d\bar{Z}^{n+1}$:

$$ds^2 = \Big|\frac{dZ^{n+1}}{Z^{n+1}} + F^{-1}\,\zeta^i\,d\bar{\zeta}^i\Big|^2 + F^{-1}\,d\zeta^i\,d\bar{\zeta}^i - F^{-2}\,\bar{\zeta}^i\,\zeta^j\,d\zeta^i\,d\bar{\zeta}^j\ . \tag{8.75}$$

If we now parameterise the coordinate Z^{n+1} as $Z^{n+1} = e^{\mathrm{i}\,\psi}\,F^{-1/2}$, we see that the metric becomes

$$ds^2 = (d\psi + A)^2 + F^{-1}\,d\zeta^i\,d\bar{\zeta}^i - F^{-2}\,\bar{\zeta}^i\,\zeta^j\,d\zeta^i\,d\bar{\zeta}^j\ , \tag{8.76}$$

where

$$A = \tfrac{\mathrm{i}}{2}\,F^{-1}\,(\bar{\zeta}^i\,d\zeta^i - \zeta^i\,d\bar{\zeta}^i)\ . \tag{8.77}$$

It will be recalled that (8.76) is still a metric on the unit sphere S^{2n+1}, since we have really done nothing more than reparameterise the metric we had at the beginning of the construction. Now, let us project the metric orthogonally to the orbits of the Killing vector $\partial/\partial\psi$. This is achieved by simply dropping the first term in (8.76), leading to the $(2n)$-dimensional metric

$$ds^2 = F^{-1}\,d\zeta^i\,d\bar{\zeta}^i - F^{-2}\,\bar{\zeta}^i\,\zeta^j\,d\zeta^i\,d\bar{\zeta}^j\ . \tag{8.78}$$

It will be recognised that what we are doing here is really a Kaluza-Klein dimensional reduction from $D = 2n+1$ to $D = 2n$, with ψ being the coordinate on the internal circle, and A the Kaluza-Klein vector. The metric that we have thus obtained in (8.78) is a metric on CP^n, or complex projective n-space. It is know as the Fubini-Study[8] metric on CP^n.

The CP^n manifold is a complex n-manifold. This can be seen from the fact that the complex coordinates ζ^i, defined in (8.72), are valid in the open neighbourhood where $Z^{n+1} \ne 0$. A different open neighbourhood can be covered by sigling out a different one of the $(n+1)$ homogeneous coordinates Z^M, say Z^A, for some specific value of A chosen from the range $1 \le A \le n+1$. Then we can define inhomogeneous coordinates ζ^i_A, valid in the open neighbourhood U_A defined by $Z^A \ne 0$, by

$$\zeta^i_A = Z^i/Z^A\ , \qquad i \ne A\ . \tag{8.79}$$

The construction of CP^n proceeds analogously in the neighbourhood U_A. To see that CP^n is a complex manifold we just have to look at the transition functions relating the coordinates ζ^i_A in region U_A to the coordinates ζ^i_B in region U_B, in their intersection, which

[8]Following in the tradition of mathematicians with misleading names, we may now add Study to the list that includes also Killing and Lie.

comprises all points for which $Z^A \neq 0$ and $Z^B \neq 0$. Then we have:

$$\begin{aligned} \zeta^i_A &= Z^i/Z^A = (Z^i/Z^B)(Z^B/Z^A)\,, \\ &= \zeta^i_B/\zeta^A_B\,. \end{aligned} \tag{8.80}$$

This shows that the complex coordinates of different open neighbourhoods are related holomorphically in their overlap regions, thus establishing that CP^n is a complex manifold.

Having seen that CP^n is a complex manifold, let us now show that it is a Kähler manifold. To do this, let us go back to the specific choice of the open neighbourhood U_{n+1}, for which the inhomogeneous coordinates are given by (8.72). Let K be the function

$$K = \log F\,. \tag{8.81}$$

To adjust our notation to fit better with the previous general discussion of Kähler manfolds, let us change the labelling for the homogeneous coordinates ζ^i to z^μ, so that $F = 1 + z^\mu\, z^{\bar\nu}\, \delta_{\mu\bar\nu}$. If we take K as the Kähler function, then from (8.62) we will have that

$$g_{\mu\bar\nu} = \partial_\mu\, \partial_{\bar\nu}\, K = F^{-1}\, \delta_{\mu\bar\nu} - F^{-2}\, z^{\bar\mu}\, z^\nu\,, \tag{8.82}$$

which is easily seen to be equivalent to the Fubini-Study metric (8.78) on CP^n that we derived previously. The Kähler form $J = \mathrm{i}\, \partial\, \bar\partial\, K$ is likewise easily calculated, and comes out to be

$$J = \mathrm{i}\, F^{-1}\, dz^\mu \wedge dz^{\bar\mu} - \mathrm{i}\, F^{-2}\, z^{\bar\mu}\, z^\nu\, dz^\mu \wedge dz^{\bar\nu}\,. \tag{8.83}$$

Now we are in a position to check how the Kähler form transforms under the change between coordinate systems in overlapping patches. Using (8.80), we see that the Kähler function K_A in region U_A is related in the overlap to the Kähler function K_B in region U_B by

$$\begin{aligned} K_A &= \log\Big(1 + \sum_{i\neq A} |\zeta^i_A|^2\Big) = \log\Big(1 + |\zeta^A_B|^{-2} \sum_{i\neq A} |\zeta^i_B|^2\Big) \\ &= -\log|\zeta^A_B|^2 + \log\Big(1 + \sum_{i\neq B} |\zeta^i_B|^2\Big) \\ &= K_B - \log\zeta^A_B - \log\bar\zeta^A_B\,. \end{aligned} \tag{8.84}$$

Thus, as we saw in general in (8.66), the Kähler function transforms by the addition of purely holomorphic and anti-holomorphic functions under a change of coordinates.

Let us now return to a general Kähler manifold. Recall that we found in the previous subsection that on any Hermitean manifold the uniquely-defined Hermitean connection is given by (8.46) together with its complex conjugate. Thus the Hermitean connection is always pure in its indices. However, in general it has torsion, reflected in the fact that $\Gamma^\mu{}_{\nu\rho}$ can have a part that is antisymmetric in ν and ρ. However, in a Kähler manifold we saw that the Kähler metric can be written in terms of the Kähler function K, as given in (8.62). It is therefore immediately evident, on account of the commutativity of partial derivatives, that the Hermitean connection $\Gamma^\mu{}_{\nu\rho}$ for a Kähler metric is in fact symmetric in ν and ρ. Thus the torsion vanishes, and so in fact the Hermitean connection for the Kähler metric coincides with the usual Christoffel connection. In particular, this means that all the usual additional symmetries of the Riemann tensor for a torsion-free connection hold, namely

$$R_{ijk\ell} = R_{k\ell ij} \ , \qquad R_{i[jk\ell]} = 0 \ . \tag{8.85}$$

In terms of holomorphic and anti-holomorphic indices, this means that the Riemann tensor has the symmetries

$$R_{\mu\bar\nu\rho\bar\sigma} = R_{\rho\bar\nu\mu\bar\sigma} = R_{\mu\bar\sigma\rho\bar\nu} \ . \tag{8.86}$$

In other words, it is symmetric in its holomorphic indices, and asymmetric in its anti-holomorphic indices.

It then follows that the Ricci tensor is symmetric, and has only mixed components:

$$R_{\mu\bar\nu} = g^{\rho\bar\sigma}\, R_{\bar\sigma\mu\rho\bar\nu} = -g^{\rho\bar\sigma}\, R_{\mu\bar\nu\rho\bar\sigma} \ . \tag{8.87}$$

Comparing with the expression (8.53) for the Ricci form of an Hermitean manifold, we see that for a Kähler metric, the components $\mathcal{R}_{\mu\bar\nu}$ of the Ricci form are precisely given by the components $-R_{\mu\bar\nu}$ of the Ricci tensor. Of course since the former is antisymmetric, while the latter is symmetric, we have also that $\mathcal{R}_{\bar\nu\mu} = R_{\bar\nu\mu}$.

8.5 The Monge-Ampère Equation

As we saw in (8.53), the Ricci form can be expressed very simply in terms of holomorphic and antiholomorphic derivatives of the metric. Furthermore, in a Kähler manifold we have the metric written very simply in terms of holomorphic and antiholomorphic derivatives of the Kähler function. Suppose now that we wish to find a Kähler solution of the vacuum Einstein equations (in Euclidean signature), i.e. we wish to find a Ricci-flat Kähler metric. Since in a Kähler manifold the Ricci form really is just the Ricci tensor, in that $\mathcal{R}_{\mu\bar\nu} = -R_{\mu\bar\nu}$, it follows from (8.53) that Ricci-flatness means that locally we have

$$\log g = f(z) + \overline{f(z)} \ , \tag{8.88}$$

where g is the determinant of the metric, and f is an arbitrary holomorphic function. Equivalently, we may say that $g = |h(z)|^2$, where $h(z)$ is an arbitrary holomorphic function. Now, under a holomorphic general coordinate transformation, the determinant g will change by a multiplicative Jacobian factor, which itself is the modulus-squared of the holomorphic Jacobian. Thus we may use this coordinate transformation freedom to choose a coordinate frame where we simply have $g = 1$. Now, from (8.62) we therefore find that the condition of Ricci-flatness on a Kähler manifold can be expressed simply as

$$\det\Big(\partial_\mu\,\partial_{\bar\nu}\,K\Big) = 1\ , \tag{8.89}$$

where the determinant is taken over the μ and $\bar\nu$ indices. This very simple re-expression of the vacuum Einstein equations is a special case of the *Monge-Ampère equation*.

More generally, we can look for Kähler metrics that are not Ricci flat, but whose Ricci tensor is proprtional to the metric; this condition on a metric defines what is known as an *Einstein metric*:

$$R_{ij} = \Lambda\, g_{ij}\ . \tag{8.90}$$

The factor Λ is necessarily constant, as can be seen from the Bianchi identity for the curvature. In physical terms, when the metric signature is Lorentzian, these are solutions of the vacuum Einstein equations with a cosmological constant Λ. For this more general case, the condition $R_{\mu\bar\nu} = \Lambda\, g_{\mu\bar\nu}$ for an Einstein-Kähler metric can be expressed as

$$\partial_\mu\,\partial_{\bar\nu}\,\log g^{1/2} = -\Lambda\,\partial_\mu\,\partial_{\bar\nu}\,K\ , \tag{8.91}$$

and, exploiting the various reparameterisation freedoms as before, we can without loss of generality reduce this to the condition

$$\det\Big(\partial_\mu\,\partial_{\bar\nu}\,K\Big) = e^{-\Lambda\,K}\ . \tag{8.92}$$

This is the general case of the Monge-Ampère equation. It can provide a useful way of solving for Einstein-Kähler metrics.

For example, suppose we make the ansatz that the Kähler function K on a complex n-manifold will depend on the complex coordinates z^μ only through the isotropic quantity $x \equiv \sum_\mu |z^\mu|^2$. This is, of course, a great specialisation, but it does allow one to obtain a rather simple result. Since $\partial_\mu\, x = z^{\bar\mu}$ and $\partial_{\bar\mu}\, x = z^\mu$, we see that

$$\partial_\mu\,\partial_{\bar\nu}\,K(x) = K'\,\delta_{\mu\nu} + K''\,z^{\bar\mu}\,z^\nu\ , \tag{8.93}$$

where $K' = \partial K/\partial x$, *etc.* After a little matrix algebra, it is easy to see that this implies that

$$\det\Big(\partial_\mu\,\partial_{\bar\nu}\,K\Big) = K'^{n-1}\,(K' + x\,K'')\ , \tag{8.94}$$

and consequently the Monge-Ampère equation becomes

$$(K')^{n-1}\,(K' + x\,K'') = e^{\Lambda\,K}\ . \tag{8.95}$$

Thus for this particular isotropic ansatz, the Einstein equation is reduced to an ordinary differential equation for K.

A particular solution to (8.95) can be obtained by taking $K = \log(1+x)$. Substituting into (8.95), we see that it is satisfied if $\Lambda = n+1$. Comparing with (8.74) and (8.81), we see that the solution $K = \log(1+x)$ is nothing but the Kähler function for CP^n. Our calculation has therefore shown that the Fubini-Study metric on CP^n is an Einstein-Kähler metric. An equivalent way to express this is that the Ricci-form is proportional to the Kähler form; in fact, in this CP^n case we have

$$\mathcal{R} = -(n+1)\,J\ . \tag{8.96}$$

Recall from previously that we saw that the equivalence class (8.56) of all 2-forms related to $\mathcal{R}/(2\pi)$ by the additional of an arbitrary exact 2-form defines the topological class c_1 known as the first Chern class. We have also seen that in a compact manifold M the Kähler form J is topologically non-trivial, since J^n integrates over M to give a non-zero constant. Thus J is closed, but not exact; it is *harmonic.* The expression (8.96) therefore shows that the first Chern class of CP^n is non-trivial. A consequence of this is that it is not possible to find a Ricci-flat metric on CP^n. Of course we have already seen that the Fubini-Study metric is not Ricci flat, but this, in itself, would not rule out the logical possibility that one might find a different metric that was Ricci flat. But since we know that c_1 is non-trivial, that means that we are guaranteed that no metric deformation could take us to a new metric for which the Ricci form vanished, since if it could, this would mean that c_1 would then be zero, contradicting the fact that it is a topological invariant.

Thus we have the result that a *necessary* condition for having a Ricci-flat Kähler metric is that the first Chern class c_1 must vanish. In the 1950's it was conjectured by Calabi that this is the *only* obstruction to the existence of a Ricci-flat Kähler metric on a Kähler manifold. It took until the 1970's before the Calabi conjecture was proved by Yau. The precise statement of Yau's result is the following:

Given a complex manifold M with $c_1 = 0$, and any Kähler metric g_{ij} on M with Kähler form J, then there exists a unique Ricci-flat metric g'_{ij} whose Kähler form J' is in the same cohomology class as J.

Put more plainly, the claim is that one can find a Ricci-flat Kähler metric on any Kähler manifold with vanishing first Chern class. The metric is known as a Calabi-Yau metric. The proof is highly intricate and involved, and essentially consists of an "epsilon and delta" analysis of the Monge-Ampère equation.

8.6 Holonomy and Calabi-Yau Manifolds

An important concept in any manifold with curvature is the notion of *holonomy.* This is the characterisation of the way in which a vector is rotated after being parallely transported around a closed curve, and it is a way in which inhabitants of a curved world can "detect" the curvature. A classic example is the explorer on the earth who, like superman, starts at the north pole and then walks south. At the equator he turns through 90 degrees, walks along it for a while, and then turns a further 90 degrees and returns to the north pole. All the while, he carefully follows the rules of parallel transport for his vector that he carries with him. He finds that it is pointing in a different direction from that of the original vector before he started the trip. In fact, it is rotated through an angle ϕ, where ϕ is the azimuthal angle that he has traversed while marching along the line of latitude. This $SO(2)$ rotation is an element of the *holonomy group* of the manifold S^2. Any rotation angle ϕ can be achieved, by walking the appropriate distance along the equator. Since the manifold in this example is two-dimensional, this in fact means that the most general possible rotation of a vector can be achieved by parallel transport around an appropriate closed curve. More generally, an explorer on an m-sphere would find that he could achieve any desired $SO(m)$ rotation of a vector, by parallely transporting it appropriately. Again, this would be the most general possible rotation that a vector in m dimensions could undergo.

It is not necessary to take such long walks in order to see the holonomy of the manifold. Parallel transprt around a small closed path will also reveal the presence of curvature, although now the rotation will correspondingly be only a small one. But still, on a sphere, for example, one would be able to achieve *any* desired small rotation, by choosing the path appropriately. An infinitesimal closed path can be characterised by an infinitesimal 2-form $d\Sigma_{ij}$, which defines the 2-surface spanning the closed curve. It is a straightforward result from differential geometry that a vector V^i parallely-propagated around this curve will suffer a rotation

$$\delta V^i = V^j\, R^i{}_{jk\ell}\, d\Sigma^{k\ell}\ . \tag{8.97}$$

The fact that it is a pure rotation, with no change in length, is assured by the fact that the Riemann tensor is antisymmetric in its first two indices; $\delta(V^i\, V_i) = 2V_i\, \delta V^i = 2V_i\, V^j\, R^i{}_{jk\ell}\, d\Sigma^{k\ell} = 0$. In fact, we can think of the infinitesimal rotation as being

$$\delta V^i = \Lambda^i{}_j\, V^j\ , \tag{8.98}$$

where $\Lambda_{ij} = -\Lambda_{ji}$ is an infinitesimal generaor of the holonomy group, given by $\Lambda_{ij} = R_{ijk\ell}\, d\Sigma^{k\ell}$.

In a generic manifold, and for these purposes the n-sphere is an example of such, the generators $\Lambda^i{}_j$ fill out the entire set of $SO(m)$ Lie algebra generators, in m dimensions. In fact, for the sphere with its standard unit-radius metric we have

$$R_{ijk\ell} = g_{ik}\, g_{j\ell} - g_{i\ell}\, g_{jk}\ , \tag{8.99}$$

and so we have $\Lambda_{ij} = 2d\Sigma_{ij}$. Thus we indeed see that we can achieve *any* desired infinitesimal Λ_{ij}, by choosing our closed curve appropriately.

A Kähler manifold, however, is *not* a generic manifold. It has, as we have seen, a very special kind of curvature where, in terms of complex components, only the mixed-index components $R_{\mu\bar\nu\rho\bar\sigma}$, and those related by the usual Riemann-tensor symmetries, are non-zero. If we raise the first index, we have that $R^\mu{}_{\nu\rho\bar\sigma} = -R^\mu{}_{\nu\bar\sigma\rho}$ and $R^{\bar\mu}{}_{\bar\nu\rho\bar\sigma} = -R^{\bar\mu}{}_{\bar\nu\bar\sigma\rho}$ can be non-zero, while the components with mixed indices on the first pair must vanish. From the general expression (8.97) for infinitesimal parallel transport, we see that a holomorphic vector V^μ can suffer only holomorphic rotations, while an antiholomorphic vector $V^{\bar\mu}$ can suffer only antiholomorphic ones. In other words, instead of being infinitesimal rotations of the generic $SO(2n)$ holonomy group that one would expect in a generic real $(2n)$-manifold, the rotations here are elements of $U(n)$. Thus the holonomy group of a Kähler metric on a complex n-manifold is $U(n)$. This is, of course, a subgroup of $SO(2n)$.

There is a slight further specialisation of the holonomy group that occurs if the Kähler metric is Ricci flat. It is clear from the form of the rotation of a holomorphic vector,

$$\delta V^\mu = V^\nu\, R^\mu{}_{\nu k\ell}\, d\Sigma^{k\ell} \equiv \Lambda^\mu{}_\nu\, V^\nu\ , \tag{8.100}$$

that the $U(n)$ rotation-group element will have unit determinant if the generator $\Lambda^\mu{}_\nu$ is traceless. But from (8.87), and the symmetries of the Riemann tensor, this is exactly what happens if the Kähler metric is Ricci-flat. Thus we arrive at the conclusion that a Ricci-flat Kähler metric on a complex n-manifold has $SU(n)$ holonomy.