Kaluza-Klein Theory

Contents

1 Kaluza-Klein Reduction on S^1 and T^n

Ten-dimensional string theory and eleven-dimensional M-theory are at present our best candidates for providing a unified description of all the fundamental forces in nature. For example, the effective low-energy limit of M-theory is an eleven-dimensional field theory whose bosonic sector comprises the metric tensor and a 4-index antisymmetric tensor field strength. The entire low-energy theory contains a fermionic field of spin $\frac{3}{2}$ as well, and together with the bosonic fields gives rise to the long-known theory of eleven-dimensional supergravity. If we concentrate just on the bosons, the equations of motion can be derived from the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\,\Big(R - \tfrac{1}{48} F_{MNPQ}\, F^{MNPQ}\Big) + \frac{1}{20736}\, \epsilon^{M_1\cdots M_{11}}\, F_{M_1\cdots M_4}\, F_{M_5\cdots M_8}\, A_{M_9\cdots M_{11}}\ , \qquad (1.1)$$

where as a 4-form, $F = dA$. In terms of indices, $F_{MNPQ} = 4\partial_{[M}\, A_{NPQ]}$.

Two things are evident. Firstly, if the eleven-dimensional theory, or string theories in ten dimensions, are truly fundamental, then we should be interested in all their predictions and consequences, including solutions in the higher dimensions. Secondly, especially if we hope that one day they may allow us to describe our four-dimensional world, we need to have a way of extracting four-dimensional physics from higher-dimensional theories. A satisfactory by-product of learning how to perform dimensional reduction is that we find that many of the lower-dimensional theories that we wish to consider are derivable from simpler theories in a higher dimension. For example, the four-dimensional $N = 8$ supergravity mentioned above can be derived by dimensional reduction from eleven-dimensional supergravity. Contrary to what one might have thought, things are immensely simpler in eleven dimensions than in four, and so this provides a very useful way of learning about the four-dimensional theory.

To begin, therefore, let us make a preliminary study of how dimensional reduction works. This will lead us on to a number of topics that will develop in various directions, including the study of complex manifolds and Kähler geometry, and a study of coset spaces and non-linear sigma models. Our first step, though, will be a relatively humble one, where we perform a dimensional reduction in which the spacetime dimension is reduced by 1. This is the original example considered by Kaluza and Klein, and although there have been many developments and advances since their days, the general procedure for dimensional reduction bears their names.

1.1 Kaluza-Klein reduction on S^1

The higher-dimensional theories that we shall consider will all be theories of gravity plus additional fields, and so a good starting point is to study how the dimensional reduction of gravity itself proceeds. In fact this is really the hardest part of the calculation, and so once this is done the rest will be comparatively simple.

Let us assume that we are starting from Einstein gravity in $(D+1)$ dimensions, described by the Einstein-Hilbert Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} \ , \tag{1.2}$$

where as usual $\hat{R}$ is the Ricci scalar and $\hat{g}$ denotes the determinant of the metric tensor. We put hats on the fields to signify that they are in $(D+1)$ dimensions. Now suppose that we wish to reduce the theory to D dimensions, by "compactifying" one of the coordinates on a circle, S^1, of radius L. Let this coordinate be called z. In principle, we could simply now expand all the components of the $(D+1)$-dimensional metric tensor as Fourier series of the form

$$\hat{g}_{MN}(x,z) = \sum_n g^{(n)}_{MN}(x)\, e^{\mathrm{i}\, n\, z/L} \ , \tag{1.3}$$

where we use x to denote collectively the D coordinates of the lower-dimensional spacetime. If one does this, one gets an infinite number of fields in D dimensions, labelled by the Fourier mode number n.

It turns out that the modes with $n \ne 0$ are associated with massive fields, while those with $n = 0$ are massless. The basic reason for this can be seen by considering a simpler toy example, of a massless scalar field $\hat{\phi}$ in flat $(D+1)$-dimensional space. It satisfies

$$\hat{\Box}\, \hat{\phi} = 0 \ , \tag{1.4}$$

where $\hat{\Box} = \partial^M\, \partial_M$. Now if we Fourier expand $\hat{\phi}$ after compactifying the coordinate z, so that

$$\hat{\phi}(x,z) = \sum_n \phi_n(x)\, e^{\mathrm{i}\, n\, z/L} \ , \tag{1.5}$$

then we immediately see that the lower-dimensional fields $\phi_n(x)$ will satisfy

$$\Box\phi_n - \frac{n^2}{L^2}\, \phi_n = 0 \ . \tag{1.6}$$

This is the wave equation for a scalar field of mass $|n|/L$.

The usual Kaluza-Klein philosophy is to assume that the radius L of the compactifying circle is very small (otherwise we would see it!), in which case the masses of the the non-zero modes will be enormous. (By small, we mean that L is roughly speaking of order the Planck length, 10^{-33} centimetres, so that the non-zero modes will have masses of order the Planck mass, 10^{-5} grammes.) Thus unless we were working with accelerators way beyond even intergalactic scales, the energies of particles that we ever see would be way below the scales of the Kaluza-Klein massive modes, and they can safely be neglected. Thus usually, when one speaks of Kaluza-Klein reduction, one has in mind a compactification together with a truncation to the massless sector. At least in a case such as our compactification on S^1, this truncation is *consistent*, in a manner that we shall elaborate on later.

Our Kaluza-Klein reduction ansatz, then, will simply be to take $\hat{g}_{MN}(x,z)$ to be independent of z. The main point now is that from the D-dimensional point of view, the index M, which runs over the $(D+1)$ values of the higher dimension, splits into a range lying in the D lower dimensions, or it takes the value associated with the compactified dimension z. Thus we may denote the components of the metric $\hat{g}_{MN}$ by $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$. From the D-dimensional viewpoint these look like a 2-index symmetric tensor (the metric), a 1-form (a Maxwell potential) and a scalar field respectively.

We could simply define $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$ to be the D-dimensional fields $g_{\mu\nu}$, $\mathcal{A}_\mu$ and ϕ respectively. There is nothing logically wrong with doing this, and it would give perfectly correct lower-dimensional equations of motion. However, as a parameterisation this simple-looking choice is actually very unnatural, and the equations of motion that result look like a dog's breakfast. The reason is that this naive parameterisation pays no attention to the underlying symmetries of the theory. A much better way to parameterise things is as follows. We write the $(D+1)$ dimensional metric in terms of D-dimensional fields $g_{\mu\nu}$, $\mathcal{A}_\mu$ and ϕ as follows:

$$d\hat{s}^2 = e^{2\alpha\phi}\, ds^2 + e^{2\beta\phi}\, (dz + \mathcal{A})^2 \ , \tag{1.7}$$

where α and β are constants that we shall choose for convenience in a moment, and $\mathcal{A} = \mathcal{A}_\mu\, dx^\mu$. All the fields on the right-hand side are independent of z. Note that this ansatz means that the components of the higher-dimensional metric $\hat{g}_{MN}$ are given in terms of the lower-dimensional fields by

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi}\, g_{\mu\nu} + e^{2\beta\phi}\, \mathcal{A}_\mu\, \mathcal{A}_\nu \ , \qquad \hat{g}_{\mu z} = e^{2\beta\phi}\, \mathcal{A}_\mu \ , \qquad \hat{g}_{zz} = e^{2\beta\phi} \ . \tag{1.8}$$

Thus as long as we choose $\beta \neq 0$, this will adequately parameterise the higher-dimensional metric.

To proceed, we make a convenient choice of vielbein basis, namely

$$\hat{e}^a = e^{\alpha\phi}\, e^a \ , \qquad \hat{e}^z = e^{\beta\phi}\, (dz + \mathcal{A}) \ . \tag{1.9}$$

(One should pause here, to take note of exactly which is a vielbein, and which is an exponential! We are using latin letters a, b, *etc.* to denote tangent-space indices in D dimensions. The use of z as the index associated with the extra dimension will not, hopefully, create too much confusion. Thus $\hat e^z$ here means the z component of the $(D+1)$-dimensional vielbein.) Notice, by the way, that if we had chosen the "naive" identification of D-dimensional fields mentioned above, we would have been hard-pressed to come up with any way of writing down a vielbein basis; it would be possible, of course, but it would have been messy.)

It is now a mechanical, if slightly tedious, exercise to compute the spin connection, and then the curvature. Our goal is to express the $(D+1)$-dimensional quantities in terms of the D-dimensional ones, so that eventually we can express the $(D+1)$-dimensional Einstein-Hilbert Lagrangian in terms of a D-dimensional Lagrangian. For the spin connection, one finds that

$$\begin{aligned} \hat\omega^{ab} &= \omega^{ab} + \alpha\, e^{-\alpha\phi}\, (\partial^b\phi\, \hat e^a - \partial^a\phi\, \hat e^b) - \tfrac12 \mathcal{F}^{ab}\, e^{(\beta-2\alpha)\phi}\, \hat e^z\ , \\ \hat\omega^{az} &= -\hat\omega^{za} = -\beta\, e^{-\alpha\phi}\, \partial^a\phi\, \hat e^z - \tfrac12 \mathcal{F}^a{}_b\, e^{(\beta-2\alpha)\phi}\, \hat e^b\ , \end{aligned} \tag{1.10}$$

where $\partial_a\phi$ means $E^\mu_a\, \partial_\mu\phi$, and E^μ_a is the inverse of the D-dimensional vielbein $e^a = e^a_\mu\, dx^\mu$. Also, $\mathcal{F}_{ab}$ denotes the vielbein components of the D-dimensional field strength $\mathcal{F} = d\mathcal{A}$.

The calculation of the curvature 2-forms proceeds uneventfully. Rather than present all the formulae here, we shall just present the key results. Firstly, we can exploit our freedom to choose the values of the constants α and β in the metric ansatz in the following way. There are two things that we would like to achieve, one of which is to ensure that the dimensionally-reduced Lagrangian is of the Einstein-Hilbert form $\mathcal{L} = \sqrt{-g}\, R + \cdots$. If the values of α and β are left unfixed, we instead end up with $\mathcal{L} = e^{(\beta+(D-2)\alpha)\phi}\, \sqrt{-g}\, R + \cdots$. Thus we immediately see that we should choose $\beta = -(D-2)\alpha$. Provided we are not reducing down to $D = 2$ dimensions, this will not present any problem. The other thing that we would like is to ensure that the scalar field ϕ acquires a kinetic term with the canonical normalisation, meaning a term of the form $-\frac12\sqrt{-g}\, (\partial\phi)^2$ in the Lagrangian. This determines the choice of overall scale, and it turns out that we should choose our constants as follows:

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}\ , \qquad \beta = -(D-2)\,\alpha\ . \tag{1.11}$$

With these choices for the constants in the metric ansatz, we can now present the results for the vielbein components of the Ricci tensor:

$$\hat R_{ab} = e^{-2\alpha\phi}\Big(R_{ab} - \tfrac12 \partial_a\phi\, \partial_b\phi - \alpha\, \eta_{ab}\, \Box\, \phi\Big) - \tfrac12 e^{-2D\alpha\phi}\, \mathcal{F}_a{}^c\, \mathcal{F}_{bc}\ ,$$

$$
\begin{aligned}
\hat{R}_{az} &= \hat{R}_{za} = \tfrac{1}{2} e^{(D-3)\alpha\phi}\, \nabla^b \Big(e^{-2(D-1)\alpha\phi}\, \mathcal{F}_{ab}\Big) \ , \\
\hat{R}_{zz} &= (D-2)\,\alpha\, e^{-2\alpha\phi}\, \Box\, \phi + \tfrac{1}{4} e^{-2D\alpha\phi}\, \mathcal{F}^2 \ ,
\end{aligned} \tag{1.12}
$$

where $\mathcal{F}^2$ means $\mathcal{F}_{ab}\, \mathcal{F}^{ab}$. From these, it follows that the Ricci scalar $\hat{R} = \eta^{AB}\, \hat{R}_{AB} = \eta^{ab}\, \hat{R}_{ab} + \hat{R}_{zz}$ is given by

$$
\hat{R} = e^{-2\alpha\phi} \Big(R - \tfrac{1}{2}(\partial\phi)^2 + (D-3)\alpha\, \Box\, \phi\Big) - \tfrac{1}{4} e^{-2D\alpha\phi}\, \mathcal{F}^2 \ . \tag{1.13}
$$

Now, finally, we calculate the determinant of the metric $\hat{g}$ in terms of the determinant of g, from the ansatz (1.7), finding

$$
\sqrt{-\hat{g}} = e^{(\beta + D\alpha)\phi}\, \sqrt{-g} = e^{2\alpha\phi}\, \sqrt{-g} \ , \tag{1.14}
$$

where the second equality follows using our relation between β and α given in (1.11). Putting all the results together, we see that the dimensional reduction of the higher-dimensional Einstein-Hilbert Lagrangian gives

$$
\mathcal{L} = \sqrt{-\hat{g}}\, \hat{R} = \sqrt{-g} \Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4} e^{-2(D-1)\alpha\phi}\, \mathcal{F}^2\Big) \ , \tag{1.15}
$$

where we have dropped the $\Box\, \phi$ term in (1.13) since it just gives a total derivative in $\mathcal{L}$, which therefore does not contribute to the field equations. In modern parlance, the scalar field ϕ is called a dilaton.

If the scalar field in (1.15) were set to zero, we would simply have the Einstein-Maxwell Lagrangian in D dimensions. This is in fact what some people thought that Kaluza and Klein originally did (which, apparently, they did not, although it is not common to encounter anyone who has ever looked at their papers). It would be a tempting thing to do, since it could then be viewed as a unification of Einstein's theory of gravity and Maxwell's electrodynamics, reformulated as pure gravity in five dimensions. However, it is not actually allowed to set the scalar field to zero; this would be in conflict with the field equation for ϕ. To see this, and for general future reference, let us pause to work out the field equations coming from (1.15). They are

$$
\begin{aligned}
R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} &= \tfrac{1}{2}\Big(\partial_\mu\phi\, \partial_\nu\phi - \tfrac{1}{2}(\partial\phi)^2\, g_{\mu\nu}\Big) + \tfrac{1}{2} e^{-2(D-1)\alpha\phi} \Big(\mathcal{F}^2_{\mu\nu} - \tfrac{1}{4}\mathcal{F}^2\, g_{\mu\nu}\Big) \ , \\
\nabla^\mu \Big(e^{-2(D-1)\alpha\phi}\, \mathcal{F}_{\mu\nu}\Big) &= 0 \ , \\
\Box\, \phi &= -\tfrac{1}{2}(D-1)\alpha\, e^{-2(D-1)\alpha\phi}\, \mathcal{F}^2 \ ,
\end{aligned} \tag{1.16}
$$

where we have defined $\mathcal{F}^2_{\mu\nu} = \mathcal{F}_{\mu\rho}\, \mathcal{F}_\nu{}^\rho$. Actually, it is usually more convenient to eliminate the $-\frac{1}{2} R\, g_{\mu\nu}$ term in the Einstein equation, by subtracting out the appropriate multiple of the trace, so that we get

$$R_{\mu\nu} = \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}e^{-2(D-1)\alpha\phi}\left(\mathcal{F}^2_{\mu\nu} - \frac{1}{2(D-2)}\,\mathcal{F}^2\,g_{\mu\nu}\right)\,. \tag{1.17}$$

We see from the last equation in (1.16) that one cannot in general set $\phi = 0$, since there is a source term on the right-hand side of the equation, involving $\mathcal{F}^2$. In other words, the details of the *interactions* between the various lower-dimensional fields prevent the truncation of the scalar ϕ. Thus it is an Einstein-Maxwell-Scalar system that comes from the consistent dimensional reduction of the higher-dimensional pure Einstein theory. One would not notice this subtlety if one simply made the ansatz (1.7) but with $\phi = 0$, and plugged the resulting Ricci scalar into the higher-dimensional Einstein-Hilbert Lagrangian. What one would be failing to notice is that such an ansatz would be inconsistent with the *higher-dimensional* equations of motion, specifically, with the $\hat{R}_{zz}$ component of the higher-dimensional Einstein equation. Neglecting some of the content of the higher-dimensional equations of motion is, from a modern viewpoint, a philosophically unattractive thing to do, since it would be denying the fundamental significance of the higher-dimensional theory. Nevertheless, the mistake of substituting an ansatz into a Lagrangian, and noticing that the resulting apparently-sensible lower-dimensional equations are masking a failure to satisfy all the components of the higher-dimensional equations, is a common one. It has been responsible for a considerable amount of confusion over the years. In these lectures we shall be careful never to believe in any ansatz until it has been verified either by substituting into the higher-dimensional equations of motion, or by constructing an argument to prove that it would satisfy all the equations if the substitution were performed.

After this little cautionary tale, one might wonder whether we ourselves might be guilty of exactly the same offence. Recall that early on, we set all the non-zero modes in the Fourier expansion (1.3) of the metric to zero. Suppose we had kept them instead, and eventually worked out the analogue of (1.15) with the entire infinite towers of massive as well as massless fields. Might we not have found that the equations of motion of the massive fields would forbid us from setting them to zero? The answer is that a little bit of (elementary) group theory saves us. The mode functions $e^{imnz/L}$ in the Fourier expansion (1.3) are representations of the $U(1)$ group of the circle S^1. The mode $n = 0$ is a singlet, while the non-zero modes are all doublets, in the sense that the modes with numbers n and $-n$ are complex conjugates of each other. When we truncated out all the non-zero modes, what we were doing was keeping all the group singlets, and throwing out all the non-singlets. This is guaranteed to be a consistent truncation, since no amount of multiplying group singlets together can ever generate non-singlets. To put it another way, the label n is like a $U(1)$

charge, and there is a charge-conservation law that must be obeyed. Each term in field equation for any particular field labelled by n will necessarily have net charge equal to n, and so at least one factor in each term in the equation must have non-zero charge whenever n is non-zero. Thus provided we truncate out *all* the non-zero modes, the consistency is guaranteed.

In more complicated Kaluza-Klein reductions, where the compactifying manifold is not simply a circle or a product of circles (a torus), the issue of the consistency of the truncation to the massless sector is a much more tricky one. It is a question that is usually ignored by those who do compactifications on K3 or Calabi-Yau manifolds, but there is always a lurking suspicion (or hope?) that one day their sins will catch up with them. We shall study this question in detail later, when we discuss Kaluza-Klein sphere reductions.

Having seen how the Kaluza-Klein S^1 reduction of the metric works, we shall now see how an antisymmetric tensor field strength is reduced from $(D+1)$ to D dimensions. Suppose we have an n-index field strength in the higher dimension, which we denote by $\hat{F}_{(n)}$. Suppose, furthermore, that this is given in terms of a potential $\hat{A}_{(n-1)}$, so that $\hat{F}_{(n)} = d\hat{A}_{(n-1)}$. In terms of indices, it is clear that after reduction on S^1 there will be two kinds of D-dimensional potentials, namely one with all $(n-1)$ indices lying in the D-dimensional spacetime, and the other with $(n-2)$ indices lying in the D-dimensional spacetime, and the remaining index being in the direction of the S^1. This is most easily expressed in terms of differential forms. Thus the ansatz for the reduction of the potential is

$$\hat{A}_{(n-1)}(x,z) = A_{(n-1)}(x) + A_{(n-2)}(x) \wedge dz \ . \tag{1.18}$$

Now, let us calculate the field strength. Clearly, we shall have

$$\hat{F}_{(n)} = dA_{(n-1)} + dA_{(n-2)} \wedge dz \ . \tag{1.19}$$

One might naively be tempted to identify $dA_{(n-1)}$ and $dA_{(n-2)}$ as the lower-dimensional field strengths $F_{(n)}$ and $F_{(n-1)}$. There is nothing logically wrong with doing so, but it is not a very convenient choice. Much better is to add and subtract a term in (1.19), so that we get

$$\begin{aligned} \hat{F}_{(n)} &= dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} + dA_{(n-2)} \wedge (dz + \mathcal{A}_{(1)}) \ , \\ &\equiv F_{(n)} + F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)}) \ , \end{aligned} \tag{1.20}$$

where $\mathcal{A}_{(1)}$ is the Kaluza-Klein potential that we encountered in the metric reduction. We have appended a subscript $_{(1)}$ to it now, in keeping with our general notation to indicate the degrees of differential forms. Thus the D-dimensional field strengths are given by

$$F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} \ , \qquad F_{(n-1)} = dA_{(n-2)} \ . \tag{1.21}$$

This is in a sense a purely notational change from the "naive" choice mentioned above; it is entirely up to us to decide what particular combination of quantities will be dignified with the name $F_{(n)}$. The point is that the specific choice in (1.21) has a particular significance, which becomes apparent when we calculate the higher-dimensional kinetic term $\hat{F}_{(n)}^2$ in terms of the lower-dimensional fields.[1] The calculation is most easily done in the vielbein basis, since then the metric is just the diagonal one η_{AB}. Consequently, in view of the definition of the vielbeins in (1.9), the vielbein components of the $(n-1)$-form field strength in D dimensions will be the ones where the n'th index is a vielbein z index, not a coordinate z index, meaning that we should read off $F_{(n-1)}$ from $F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)})$, and not from $F_{(n-1)} \wedge dz$. It is now easily seen from (1.9) and (1.21) that in terms of vielbein components we shall have

$$
\begin{aligned}
\hat{F} &= \frac{1}{n!}\, \hat{F}_{A_1\cdots A_n}\, \hat{e}^{A_1} \wedge\cdots\wedge \hat{e}^{A_n} \\
&= \frac{e^{n\alpha\phi}}{n!}\, \hat{F}_{a_1\cdots a_n}\, e^{a_1} \wedge\cdots\wedge e^{a_n} + \frac{e^{((n-1)\alpha+\beta)\phi}}{(n-1)!}\, \hat{F}_{a_1\cdots a_{n-1}z}\, e^{a_1} \wedge\cdots\wedge e^{a_{n-1}} \wedge (dz + \mathcal{A}_{(1)})\ , \\
&\equiv \frac{1}{n!}\, F_{a_1\cdots a_n}\, e^{a_1} \wedge\cdots\wedge e^{a_n} + \frac{1}{(n-1)!}\, F_{a_1\cdots a_{n-1}}\, e^{a_1} \wedge\cdots\wedge e^{a_{n-1}} \wedge (dz + \mathcal{A}_{(1)})\ ,
\end{aligned} \tag{1.22}
$$

implying that

$$
\hat{F}_{a_1\cdots a_n} = e^{-n\alpha\phi}\, F_{a_1\cdots a_n}\ , \qquad \hat{F}_{a_1\cdots a_{n-1}z} = e^{(D-n-1)\alpha\phi}\, F_{a_1\cdots a_{n-1}}\ , \tag{1.23}
$$

where we have used (1.11) to express β in terms of α. It is now easy to see, bearing in mind the relation (1.14) between the determinants of the metrics in $(D+1)$ and D dimensions, that the kinetic term for the $(D+1)$-dimensional n-form field strength $\hat{F}_{(n)}$ will give, upon Kaluza-Klein reduction to D dimensions,

$$
\mathcal{L} = -\tfrac{\sqrt{-\hat{g}}}{2\,n!}\, \hat{F}_{(n)}^2 = -\tfrac{\sqrt{-g}}{2\,n!}\, e^{-2(n-1)\alpha\phi}\, F_{(n)}^2 - \tfrac{\sqrt{-g}}{2\,(n-1)!}\, e^{2(D-n)\alpha\phi}\, F_{(n-1)}^2\ . \tag{1.24}
$$

At this point, let us pause for a moment in order to find a nicer way to present the Lagrangians that we are encountering. There are two reasons for doing so; firstly, on general aesthetic grounds, but also, and more importantly, to make the process of varying the Lagrangian to obtain the equations of motion as simple and straightforward as possible. The advantage of doing this is already evident if we consider what happens when we want to vary the reduced Lagrangian (1.24) with respect to the potential $A_{(n-2)}$. Not only does this potential appear in its "own" field strength $F_{(n-1)}$, but it also appears in the "transgression"

[1] The mathematicians have, curiously, attached the name "transgression" to the process by which these extra modifications to field strengths arise. The etymology is unclear.

term in $F_{(n)}$ (see equation (1.21)). Already in this example, therefore, it is apparent that getting the right signs, combinatoric factors, *etc.* when working out the equation of motion in index notation will be a tedious and wearisome business. It is highly preferable to be able to work with the language of differential forms.

Recall that we define the Hodge dual of the basis for p-forms in D dimensions by

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) \equiv \frac{1}{q!}\, \epsilon_{\nu_1 \cdots \nu_q}{}^{\mu_1 \cdots \mu_p}\, dx^{\nu_1} \wedge dx^{\nu_q} \ , \tag{1.25}$$

where $q = D - p$. Here, $\epsilon_{\mu_1 \cdots \mu_D}$ is the totally antisymmetric Levi-Civita tensor, whose components are $\pm\sqrt{|g|}$ or 0, given by

$$\epsilon_{\mu_1 \cdots \mu_D} = \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_D} \ , \tag{1.26}$$

where $\varepsilon_{\mu_1 \cdots \mu_D}$ is the totally antisymmetric Levi-Civita tensor *density*, with

$$\varepsilon_{\mu_1 \cdots \mu_D} \equiv (+1, -1, 0) \tag{1.27}$$

according to whether $\mu_1 \cdots \mu_D$ is an *even* permutation of the canonically-ordered set of index values, an *odd* permutation, or no permutation at all. A natural canonical ordering of indices would be $0, 1, 2, \ldots$, but it is, of course, ultimately a matter of pure convention. It is also sometimes useful to define a totally antisymmetric tensor density with upstairs indices, and components given numerically by

$$\varepsilon^{\mu_1 \cdots \mu_D} \equiv (-1)^t\, \varepsilon_{\mu_1 \cdots \mu_D} \ , \tag{1.28}$$

where t is the number of timelike coordinates. Note that this is the *one and only* time that we ever introduce a pair of objects for which we use the same symbol, but where the one with upstairs indices is not obtained by raising the indices on the one with downstairs indices using the metric. In terms of $\varepsilon^{\mu_1 \cdots \mu_D}$, the Levi-Civita *tensor* with upstairs indices is given by

$$\epsilon^{\mu_1 \cdots \mu_D} = \frac{1}{\sqrt{|g|}}\, \varepsilon^{\mu_1 \cdots \mu_D} \ . \tag{1.29}$$

This, of course, *is* obtained from $\epsilon_{\mu_1 \cdots \mu_D}$ simply by raising the indices using the metric.

It is easy to see from the definition (1.25) that if we apply the Hodge dual to a p-form A, we get a $(D-p)$-form $B = *A$ with components given by

$$B_{\mu_1 \cdots \mu_q} = \frac{1}{p!}\, \epsilon_{\mu_1 \cdots \mu_q}{}^{\nu_1 \cdots \nu_p}\, A_{\nu_1 \cdots \nu_p} \ , \tag{1.30}$$

where $q \equiv D - p$. (Note the order in which the indices appear on the epsilon tensors in (1.25) and (1.30).) As a particular case, we see that the Hodge dual of the pure number 1 (a 0-form) is the D-form whose components are the Levi-Civita tensor, and thus we may write

$$
\begin{aligned}
*1 &= \epsilon = \tfrac{1}{D!}\,\epsilon_{\mu_1\cdots\mu_D}\,dx^{\mu_1}\cdots dx^{\mu_D}\ , \\
&= \sqrt{|g|}\,dx^0\cdots dx^{D-1} = \sqrt{|g|}\,d^Dx\ .
\end{aligned} \tag{1.31}
$$

Thus $*1$ is nothing but the generally coordinate invariant volume element. Note that owing to the tiresome, but unavoidable, $(-1)^t$ factor in (1.28), we have

$$
dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} = (-1)^t\,\varepsilon^{\mu_1\cdots\mu_D}\,d^Dx = (-1)^t\,\epsilon^{\mu_1\cdots\mu_D}\,\sqrt{|g|}\,d^Dx\ . \tag{1.32}
$$

From the above definitions, the following results follow straightforwardly. If A and B are any two p-forms, then

$$
*A\wedge B = *B\wedge A = \frac{1}{p!}\,|A\cdot B|\,\epsilon = \frac{1}{p!}\,|A.B|\,*1\ , \tag{1.33}
$$

where

$$
|A\cdot B| \equiv A_{\mu_1\cdots\mu_p}\,B^{\mu_1\cdots\mu_p}\ , \tag{1.34}
$$

is the inner product of A and B. Also, applying $*$ twice, we have that if A is any p-form, then

$$
**A = (-1)^{pq+t}\,A\ , \tag{1.35}
$$

where as usual we define $q\equiv D-p$.

A Lagrangian density $\mathcal{L}$ is something which is to be multiplied by d^Dx and then integrated over the spacetime manifold to get the action. For example, the Einstein-Hilbert Lagrangian density is $\sqrt{-g}\,R$, and this is integrated to give $\int R\sqrt{-g}\,d^Dx$. From a differential-geometric point of view, it is really not 0-forms, but rather D-forms, that can be integrated over a D-dimensional manifold. Thus we can really think of the Einstein-Hilbert action as being obtained by integrating the D-form $R*1$ over the manifold. This is a convenient way to think of things, and so typically, from now on, when we speak of a Lagrangian we will mean the D-form whose integral gives the action.

It is now easily seen from the previous definitions that the D-form Lagrangian corresponding to the circle reduction of the Einstein-Hilbert Lagrangian, which we obtained in the "traditional" language in (1.15), is given by

$$
\mathcal{L} = R*1 - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 e^{-2(D-1)\alpha\phi}\,{*\mathcal{F}_{(2)}}\wedge\mathcal{F}_{(2)}\ , \tag{1.36}
$$

where we have put a $_{(2)}$ subscript on the Maxwell field strength to remind us that it is a 2-form. Similarly, we see that the Lagrangian (1.24) becomes, when written as a D-form,

$$
\mathcal{L} = -\tfrac12 e^{-2(n-1)\alpha\phi}\,{*F_{(n)}}\wedge F_{(n)} - \tfrac12 e^{2(D-n)\alpha\phi}\,{*F_{(n-1)}}\wedge F_{(n-1)}\ . \tag{1.37}
$$

Note that the previous $n!$ combinatoric denominator, associated with the kinetic term for an n-form field strength, is nicely eliminated in the Lagrangians written as differential forms.

It is now a completely straightforward matter to vary the Lagrangian for any gauge field, and to get the combinatorics and signs correct without headaches. The only rule one ever needs, apart from the usual ones for carrying differential forms over each other, is that the variation of an expression of the form $X_{(p)} \wedge dA_{(q)}$ with respect to $A_{(q)}$ gives, after integration by parts, $-(-1)^p\, dX_{(p)} \wedge \delta A_{(q)}$, when $X_{(p)}$ is a p-form. This is just the usual minus sign coming from integration by parts, accompanied by an additional $(-1)^p$ factor coming from the fact that the exterior derivative has to be taken over a p-form.

For example, if we look at the equations of motion coming from varying the Lagrangian (1.37) with respect to the potential $A_{(n-1)}$ we get

$$\delta\mathcal{L} = -e^{-2(n-1)\alpha\phi} *F_{(n)} \wedge d\delta A_{(n-1)} \longrightarrow (-1)^{D-n}\, d\Big(e^{-2(n-1)\alpha\phi} *F_{(n)}\Big) \wedge \delta A_{(n-1)}\ , \qquad (1.38)$$

where the arrow indicates that the result is obtained after integration by parts. Varying instead with respect to $A_{(n-2)}$ gives

$$\begin{aligned} \delta\mathcal{L} &= -e^{2(D-n)\alpha\phi} *F_{(n-1)} \wedge d\delta A_{(n-2)} + e^{-2(n-1)\alpha\phi} *F_{(n)} \wedge d\delta A_{(n-2)} \wedge \mathcal{A}_{(1)}\ , \quad (1.39)\\ &\longrightarrow (-1)^{D-n+1}\, d\Big(e^{2(D-n)\alpha\phi} *F_{(n-1)}\Big) \wedge \delta A_{(n-2)}\\ &\quad -(-1)^D\, d\Big(e^{-2(n-1)\alpha\phi} *F_{(n)} \wedge \mathcal{A}_{(1)}\Big)\, \delta A_{(n-2)}\ . \end{aligned}$$

The first lesson to note from this example is that when varying an expression such as $-\frac{1}{2}{*F_{(n)}} \wedge F_{(n)}$ that is quadratic in $F_{(n)}$, the terms coming from varying the potentials in each $F_{(n)}$ always simply add up, nicely removing the $\frac{1}{2}$ prefactor. The second lesson is that the chief remaining subtleties in varying Lagrangians are associated with the occurrence of the transgression terms in the various field strengths, as we have here in the definition of $F_{(n)}$ in (1.21). Having now got the variation expressed as $\delta\mathcal{L} = X \wedge \delta A$ for some X, one simply reads off the field equation as $X = 0$. In our example here, note that the field equation for $F_{(n)}$ can be used to simplify the field equation for $F_{(n-1)}$, leading simply to

$$\begin{aligned} d\Big(e^{-2(n-1)\alpha\phi} *F_{(n)}\Big) &= 0\ ,\\ d\Big(e^{2(D-n)\alpha\phi} *F_{(n-1)}\Big) + (-1)^D\, e^{-2(n-1)\alpha\phi} *F_{(n)} \wedge \mathcal{F}_{(2)} &= 0\ . \end{aligned} \qquad (1.40)$$

1.2 Lower-dimensional symmetries from the S^1 reduction

In the case where we started just from pure Einstein gravity in $(D+1)$ dimensions, we ended up with an Einstein-Maxwell-Scalar system in D dimensions. Thus the higher-dimensional theory had general coordinate covariance, while the lower-dimensional one has general coordinate covariance and the local $U(1)$ gauge invariance of the Maxwell field. In fact, as can be seen from (1.15), it also has another symmetry, namely a constant shift of the dilaton field ϕ, accompanied by an appropriate constant scaling of the Maxwell potential:

$$\phi \longrightarrow \phi + c\ , \qquad \mathcal{A}_\mu \longrightarrow e^{c(D-1)\alpha}\, \mathcal{A}_\mu\ . \tag{1.41}$$

At first sight, therefore, one might think that the lower-dimensional theory had more symmetry than the higher-dimensional one. Of course this is not really the case; the point is that the local general coordinate symmetry in the higher dimension involves coordinate reparameterisations by arbitrary functions of $(D+1)$ coordinates, while the local general coordinate and $U(1)$ gauge transformations in the lower dimension involve arbitrary functions of only D coordinates. Thus in effect the symmetries of the D-dimensional theory really constitute only an infinitesimal residue of the $(D+1)$-dimensional general coordinate symmetries. We can understand this better by looking in detail at the Kaluza-Klein reduction ansatz (1.7) for the $(D+1)$-dimensional metric.

The original $(D+1)$-dimensional Einstein theory is invariant under general coordinate transformations, which can be written (see section 5.1) in infinitesimal form as

$$\delta\hat{x}^M = -\hat{\xi}^M\ , \qquad \delta\hat{g}_{MN} = \hat{\xi}^P\, \partial_P\, \hat{g}_{MN} + \hat{g}_{PN}\, \partial_M\, \hat{\xi}^P + \hat{g}_{MP}\, \partial_N\, \hat{\xi}^P\ . \tag{1.42}$$

As yet, the parameters $\hat{\xi}^M$ are arbitrary functions of all $(D+1)$ coordinates. Now, the form of the Kaluza-Klein ansatz (1.7) will not in general be preserved by such transformations. In fact, it is rather easy to see that the most general allowed form for transformations that preserve (1.7) will be

$$\hat{\xi}^\mu = \xi^\mu(x)\ , \qquad \hat{\xi}^z = c\, z + \lambda(x)\ , \tag{1.43}$$

where the $(D+1)$-dimensional index on $\hat{\xi}^M$ is split as $\hat{\xi}^\mu$ and $\hat{\xi}^z$, with μ a D-dimensional index. The coordinates $\hat{x}^M$ are split as (x^μ, z), and the x arguments on $\xi^\mu(x)$ and $\lambda(x)$ indicate that these functions depend only on the D-dimensional coordinates x^μ. The parameter c is a constant. Note that from (1.7) we have that the components of the $(D+1)$-dimensional metric $\hat{g}_{MN}$ are given in terms of the D-dimensional metric $g_{\mu\nu}$, gauge potential $\mathcal{A}_\mu$ and dilaton ϕ by

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi}\, g_{\mu\nu} + e^{2\beta\phi}\, \mathcal{A}_\mu\, \mathcal{A}_\nu\ , \qquad \hat{g}_{\mu z} = \hat{g}_{z\mu} = e^{2\beta\phi}\, \mathcal{A}_\mu\ , \qquad \hat{g}_{zz} = e^{2\beta\phi}\ , \tag{1.44}$$

where $\beta = -(D-2)\,\alpha$.

Let us look first at the local transformations, namely those parameterised by $\xi^\mu(x)$ and $\lambda(x)$ (so we take the constant $c = 0$ for now). We shall see that these are the parameters of D-dimensional general coordinate transformations, and $U(1)$ gauge transformations, respectively. Under these transformations, we see first from (1.42) that

$$\delta \hat{g}_{zz} = \xi^\rho \, \partial_\rho \hat{g}_{zz} \ , \tag{1.45}$$

where we have dropped those terms that give zero by virtue either of the form of the metric ansatz (1.7), or by our assumption for now that c is zero. From (1.44), we thus deduce that

$$\delta\phi = \xi^\rho \, \partial_\rho \, \phi \ , \tag{1.46}$$

implying that ϕ is indeed transforming as a scalar under the D-dimensional general coordinate transformations parameterised by ξ^μ, and that is it inert (as it should be) under the $U(1)$ gauge transformations parameterised by λ.

Next, looking at the (μz) components in (1.42), we see that

$$\delta \hat{g}_{\mu z} = \xi^\rho \, \partial_\rho \, \hat{g}_{\mu z} + \hat{g}_{\rho z} \, \partial_\mu \, \xi^\rho \ . \tag{1.47}$$

Substituting from (1.44), and what we already learned about the transformations of ϕ, we deduce that $\mathcal{A}_\mu$ transforms as

$$\delta \mathcal{A}_\mu = \xi^\rho \, \partial_\rho \, \mathcal{A}_\mu + \mathcal{A}_\rho \, \partial_\mu \, \xi^\rho + \partial_\mu \, \lambda \ . \tag{1.48}$$

This shows that $\mathcal{A}_\mu$ transforms properly as a covector under general coordinate transformations ξ^ρ, and that it has the usual gauge transformation of a $U(1)$ gauge field, under the parameter λ.

Finally, looking at the $(\mu\nu)$ components in (1.42), we have

$$\delta \hat{g}_{\mu\nu} = \xi^\rho \, \partial_\rho \, \hat{g}_{\mu\nu} + \hat{g}_{\rho\nu} \, \partial_\mu \, \xi^\rho + \hat{g}_{\mu\rho} \, \partial_\nu \, \xi^\rho + \hat{g}_{z\nu} \, \partial_\mu \, \xi^z + \hat{g}_{\mu z} \, \partial_\nu \, \xi^z \ . \tag{1.49}$$

Using what we have now learned about the transformation rules for ϕ and $\mathcal{A}_\mu$, we find, after substituting from (1.44) that

$$\delta g_{\mu\nu} = \xi^\rho \, \partial_\rho \, g_{\mu\nu} + g_{\rho\nu} \, \partial_\mu \, \xi^\rho + g_{\mu\rho} \, \partial_\nu \, \xi^\rho \ , \tag{1.50}$$

showing that the D-dimensional metric indeed has the proper transformation properties under general coordinate transformations ξ^ρ, and that it is inert, as it should be, under the $U(1)$ gauge transformations λ.

We have now taken care of the local parameters in (1.43). We have seen that the subset of the original $(D+1)$-dimensional general coordinate transformations $\hat{\xi}^M$ that preserve the form of the Kaluza-Klein metric ansatz (1.7) include the D-dimensional general coordinate transformations ξ^μ, and the D-dimensional $U(1)$ local gauge transformations of the Kaluza-Klein vector potential $\mathcal{A}_\mu$. The remaining parameter to consider is the constant c in (1.43). This is associated with the constant shift symmetry of the dilaton ϕ, given in (1.41). To see how this symmetry comes out of (1.43), we have to introduce one further ingredient in the discussion.

The higher-dimensional equations of motion, namely the Einstein equations $\hat{R}_{MN} - \frac{1}{2}\hat{R}\,\hat{g}_{MN} = 0$, actually have an additional global symmetry in addition to the local general coordinate transformations. This is a symmetry under which the metric is scaled by a constant factor, $\hat{g}_{MN} \longrightarrow k^2\,\hat{g}_{MN}$. It is easily seen that the various curvature tensors transform under this constant scaling as

$$\hat{R}^M{}_{NPQ} \longrightarrow \hat{R}^M{}_{NPQ}\ , \qquad \hat{R}_{MN} \longrightarrow \hat{R}_{MN}\ , \qquad \hat{R} \longrightarrow k^{-2}\,\hat{R}\ . \tag{1.51}$$

In other words, the Riemann tensor with its coordinate indices in their "natural" positions is inert. No metric is needed in order then to construct the Ricci tensor, $\hat{R}_{MN} = \hat{R}^P{}_{MPN}$, and so it too is inert. However, the construction of the Ricci scalar then requires the use of the inverse metric, $\hat{R} = \hat{g}^{MN}\,\hat{R}_{MN}$, and so it acquires the scaling given above in (1.51). The upshot is that the Einstein equation is actually invariant under the scaling.

The reason for discussing this scaling symmetry in terms of the equations of motion is that, as is easily seen, it is not a symmetry of the Lagrangian itself. Clearly, we will have $\sqrt{-\hat{g}} \longrightarrow k^{D+1}\sqrt{-\hat{g}}$ in $(D+1)$ dimensions, and hence the Einstein-Hilbert Lagrangian will scale as $\sqrt{-\hat{g}}\,\hat{R} \longrightarrow k^{D-1}\sqrt{-\hat{g}}\,\hat{R}$. The crucial point is, however, that this is a uniform constant scaling of the Lagrangian. Now, the equations of motion that follow from two Lagrangians that are related by a constant scale factor are the same, and hence we can understand the invariance of the equations of motion from this viewpoint too. In certain less trivial examples, notably eleven-dimensional supergravity, on also finds that there is such a uniform scaling symmetry of the Lagrangian, and hence a scale-invariance of the equations of motion. It is less trivial in this example, because the various terms in the Lagrangian (1.1) must all conspire to scale the same way.

Returning now to our discussion of the symmetries of the Kaluza-Klein reduction of $(D+1)$-dimensional Einstein theory, we have learned that there is the additional symmetry $\hat{g}_{MN} \longrightarrow k^2\,\hat{g}_{MN}$ in the original $(D+1)$-dimensional theory, where k is a constant. In infinitesimal form, this translates into the statement that $\delta\hat{g}_{MN} = 2a\,\hat{g}_{MN}$, where a is an infinitesimal constant parameter. Thus if we write out the residual general-coordinate transformations (1.43), specialised to include just the constant parameter c, and include also the scaling symmetry, we will have the following infinitesimal global symmetry:

$$\delta\hat{g}_{MN} = c\,\delta_M^z\,\hat{g}_{zN} + c\,\delta_N^z\,\hat{g}_{Mz} + 2a\,\hat{g}_{MN}\ . \tag{1.52}$$

Note that the δ symbols on the right-hand side are Kronecker deltas, non-vanishing only when the m or N index takes the $(D+1)$'th value z.

Plugging in the form of the metric ansatz (1.44), and taking (MN) to be (zz), $(z\mu)$ and $\mu\nu)$ successively, we can read off the transformation rules for ϕ, $\mathcal{A}_\mu$ and $g_{\mu\nu}$, finding

$$\beta\,\delta\phi = a + c\ , \qquad \delta\mathcal{A}_\mu = -c\,\mathcal{A}_\mu\ , \qquad \delta g_{\mu\nu} = 2a\,g_{\mu\nu} - 2\alpha\,g_{\mu\nu}\,\delta\phi\ . \tag{1.53}$$

It is now evident that we can use the scaling transformation a as a compensator for the dilaton-shift transformation c, in such a way that under the appropriate combined transformation the metric $g_{\mu\nu}$ is inert, i.e. $\delta g_{\mu\nu} = 0$. Clearly to to this, we should choose

$$a = -\frac{c}{D-1}\ , \tag{1.54}$$

bearing in mind that the constants α and β in the Kaluza-Klein ansatz (1.7) were chosen so that $\beta = -(D-2)\,\alpha$. Thus we arrive at the global transformation

$$\delta\phi = -\frac{c}{\alpha\,(D-1)}\ , \qquad \delta\mathcal{A}_\mu = -c\,\mathcal{A}_\mu\ , \qquad \delta g_{\mu\nu} = 0\ . \tag{1.55}$$

After a constant scaling redefinition of the parameter c, this can be seen to be precisely the dilaton shift symmetry given in (1.41).

Of course since we have just made use of a particular linear combination of the original two global symmetries, with parameters a and c related by (1.54), it follows that the "orthogonal" combination is still also a symmetry of the D-dimensional theory. This other combination is nothing but a uniform scaling symmetry of the entire D dimensional theory. What we have done by taking combinations of the a and c transformations is to diagonalise the two symmetries, one of which, given by (1.55), is a purely *internal* symmetry that leaves the lower-dimensional metric invariant and acts only on the other fields. The other combination is a scaling symmetry that acts on all fields that carry indices; in this case, on $g_{\mu\nu}$ and $\mathcal{A}_\mu$. In fact the general rule for the scaling symmetries, if they are present in a particular theory, is that each fundamental field is scaled according to the number of indices it carries:

$$g_{\mu\nu} \longrightarrow k^2\,g_{\mu\nu}\ , \qquad A_{\mu_1\cdots\mu_n} \longrightarrow k^n\,A_{\mu_1\cdots\mu_n}\ . \tag{1.56}$$

Thus in our example of the D-dimensional Lagrangian (1.15), one can easily verify that it is invariant under

$$g_{\mu\nu} \longrightarrow k^2\, g_{\mu\nu}\ , \qquad \mathcal{A}_\mu \longrightarrow k\, \mathcal{A}_\mu\ . \tag{1.57}$$

Furthermore, it is easily established from the combined transformations (1.53) that we can indeed find a combination of the parameters, namely $a = -c$, that gives (1.57) in its infinitesimal form. This is precisely the combination that leaves ϕ invariant, which is consistent with the general rule (1.56) since ϕ has no indices. These kinds of scaling transformations have been referred to as "trombone" symmetries.

To complete the story of S^1 reductions, let us consider the dimensional reduction of $D = 11$ supergravity down to $D = 10$. In our new, improved notation, the eleven-dimensional Lagrangian can be written as the 11-form

$$\mathcal{L}_{11} = R\,{*1} - \tfrac12 {*F_{(4)}} \wedge F_{(4)} + \tfrac16 dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}\ . \tag{1.58}$$

Substituting all the previous results, we find that we can write $\mathcal{L}_{11} = \mathcal{L}_{10} \wedge dz$, with the ten-dimensional Lagrangian given by

$$\begin{aligned}\mathcal{L}_{10} &= R\,{*1} - \tfrac12 {*d\phi} \wedge d\phi - \tfrac12 e^{\frac32\phi}\, {*\mathcal{F}_{(2)}} \wedge \mathcal{F}_{(2)} \\ &\quad - \tfrac12 e^{\frac12\phi}\, {*F_{(4)}} \wedge F_{(4)} - \tfrac12 e^{-\phi}\, {*F_{(3)}} \wedge F_{(3)} + \tfrac12 dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)}\ ,\end{aligned} \tag{1.59}$$

with $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ being the Kaluza-Klein Maxwell field, and $F_{(3)} = dA_{(2)}$ and $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}$ being the two field strengths coming from the 4-form $F_{(4)}$ in $D = 11$. Note that the final term in the ten-dimensional Lagrangian comes from the cubic term $dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}$ in $D = 11$, and that this requires no metric in its construction. This ten-dimensional theory is the bosonic sector of the type IIA supergravity theory, which is the low-energy limit of the type IIA string.

Note that the eleven-dimensional theory has the "trombone" symmetry described above, namely a symmetry under the constant rescaling $g_{\mu\nu} \longrightarrow k^2\, g_{\mu\nu}$ and $A_{\mu\nu\rho} \longrightarrow k^3\, A_{\mu\nu\rho}$. Consequently, the ten-dimensional theory has the global internal symmetry $\phi \longrightarrow \phi + c$, together with

$$\mathcal{A}_{(1)} \longrightarrow e^{-\frac34 c}\, \mathcal{A}_{(1)}\ , \qquad A_{(3)} \longrightarrow e^{-\frac14 c}\, A_{(3)}\ , \qquad A_{(2)} \longrightarrow e^{\frac12 c}\, A_{(2)}\ . \tag{1.60}$$

1.3 Kaluza-Klein Reduction of $D = 11$ supergravity on T^n

It is clear that having established the procedure for performing a Kaluza-Klein reduction from $D + 1$ dimensions to D dimensions on the circle S^1, the process can be repeated for a succession of circles. Thus we may consider a reduction from $D+n$ dimensions to D dimensions on the n-torus $T^n = S^1 \times \cdots \times S^1$. At each successive step, for example the i'th reduction step, one generates a Kaluza-Klein vector potential $\mathcal{A}^i_{(1)}$, and a dilaton ϕ_i from the reduction of the metric. In addition, p-form potential already present in $D+i$ dimensions will descend to give a p-form and a $(p-1)$-form potential, by the mechanism that we have already studied. As a result, one obtains a rapidly-proliferating number of fields as one descends through the dimensions.

Let us consider an example where we again begin with $D = 11$ supergravity, and now reduce it to D dimensions on the $n = (11-D)$ torus, with coordinates z^i. As well as the set of Kaluza-Klein vectors $\mathcal{A}^i_{(1)}$ and dilatons ϕ_i, we will have 0-form potentials or "axions" $\mathcal{A}^i_{(0)j}$ coming from the further reduction of the Kaluza-Klein vectors. Since such an axion cannot be generated until the Kaluza-Klein vector $\mathcal{A}^i_{(1)}$ has first been generated at a previous reduction step, we see that the axions $\mathcal{A}^i_{(0)j}$ will necessarily have $i < j$. In addition, the potential $A_{(3)}$ $A_{(3)}$ in $D = 11$ will give, upon reduction, the potentials $A_{(3)}$, $A_{(2)i}$, $A_{(1)ij}$ and $A_{(0)ijk}$. Here, the $i, j, \ldots$ indices are essentially internal coordinate indices corresponding to the torus directions. Thus these indices are antisymmetrised.

We will not labour too much over the details of the calculation of the torus reduction. It is clear that one just has to apply the previously-derived formulae for the single-step reduction of the Einstein-Hilbert and gauge-field actions repeatedly, until the required lower dimension $D = 11 - n$ is reached. If one does this, one obtains the following Lagrangian in D dimensions (see [1, 2])

$$\begin{aligned}
\mathcal{L} &= R{*}\mathbb{1} - \tfrac12 {*}d\vec\phi\wedge d\vec\phi - \tfrac12 e^{\vec a\cdot\vec\phi}\,{*}F_{(4)}\wedge F_{(4)} - \tfrac12\sum_i e^{\vec a_i\cdot\vec\phi}\,{*}F_{(3)i}\wedge F_{(3)i}\\
&\quad -\tfrac12\sum_{i<j} e^{\vec a_{ij}\cdot\vec\phi}\,{*}F_{(2)ij}\wedge F_{(2)ij} - \tfrac12\sum_i e^{\vec b_i\cdot\vec\phi}\,{*}\mathcal{F}^i_{(2)}\wedge\mathcal{F}^i_{(2)} - \tfrac12\sum_{i<j<k} e^{\vec a_{ijk}\cdot\vec\phi}\,{*}F_{(1)ijk}\wedge F_{(1)ijk}\\
&\quad -\tfrac12\sum_{i<j} e^{\vec b_{ij}\cdot\vec\phi}\,{*}\mathcal{F}^i_{(1)j}\wedge\mathcal{F}^i_{(1)j} + \mathcal{L}_{FFA}\,. \qquad (1.61)
\end{aligned}$$

where the "dilaton vectors" $\vec a$, $\vec a_i$, $\vec a_{ij}$, $\vec a_{ijk}$, $\vec b_i$, $\vec b_{ij}$ are constants that characterise the couplings of the dilatonic scalars $\vec\phi$ to the various gauge fields. They are given by

$$\begin{array}{lll}
 & F_{MNPQ} & \text{vielbein}\\
4-\text{form}: & \vec a = -\vec g\,, & \\
3-\text{forms}: & \vec a_i = \vec f_i - \vec g\,, & \\
2-\text{forms}: & \vec a_{ij} = \vec f_i + \vec f_j - \vec g\,, & \vec b_i = -\vec f_i\,,\\
1-\text{forms}: & \vec a_{ijk} = \vec f_i + \vec f_j + \vec f_k - \vec g\,, & \vec b_{ij} = -\vec f_i + \vec f_j\,,
\end{array} \qquad (1.62)$$

where the vectors $\vec{g}$ and $\vec{f}_i$ have $(11-D)$ components in D dimensions, and are given by

$$\begin{aligned}
\vec{g} &= 3(s_1, s_2, \ldots, s_{11-D}) \ , \\
\vec{f}_i &= \Big(\underbrace{0,0,\ldots,0}_{i-1}, (10-i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{11-D}\Big) \ ,
\end{aligned} \tag{1.63}$$

where $s_i = \sqrt{2/((10-i)(9-i))}$. It is easy to see that they satisfy

$$\vec{g}\cdot\vec{g} = \tfrac{2(11-D)}{D-2}, \qquad \vec{g}\cdot\vec{f}_i = \tfrac{6}{D-2} \ , \qquad \vec{f}_i\cdot\vec{f}_j = 2\delta_{ij} + \tfrac{2}{D-2} \ . \tag{1.64}$$

Note also that

$$\sum_i \vec{f}_i = 3\,\vec{g} \ . \tag{1.65}$$

Note that the D-dimensional metric is related to the eleven-dimensional one by

$$ds_{11}^2 = e^{\frac13 \vec{g}\cdot\vec{\phi}}\, ds_D^2 + \sum_i e^{2\vec{\gamma}_i\cdot\vec{\phi}}\, (h^i)^2 \ , \tag{1.66}$$

where $\vec{\gamma}_i = \frac16 \vec{g} - \frac12 \vec{f}_i$, and

$$h^i = dz^i + \mathcal{A}_1^i + \mathcal{A}_{0j}^i\, dz^j \ . \tag{1.67}$$

There are, of course, a number of subtleties that have been sneaked into the formulae presented above. First of all, as we already saw from the single-step reduction from $D+1$ to D dimensions, one acquires transgression terms that modify the leading-order expressions $F_{(n)} = dA_{(n-1)} + \cdots$ for the lower-dimensional field strengths. This can all be handled in a fairly mechanical, although somewhat involved, manner. After a certain amount of algebra, one can show that the various field strengths are given by

$$\begin{aligned}
F_{(4)} &= \tilde{F}_{(4)} - \gamma^i{}_j\, \tilde{F}_{(3)i} \wedge \mathcal{A}^j_{(1)} + \tfrac12 \gamma^i{}_k\, \gamma^j{}_\ell\, \tilde{F}_{(2)ij} \wedge \mathcal{A}^k_{(1)} \wedge \mathcal{A}^\ell_{(1)} \\
&\quad - \tfrac16 \gamma^i{}_\ell\, \gamma^j{}_m\, \gamma^k{}_n\, \tilde{F}_{(1)ijk} \wedge \mathcal{A}^\ell_{(1)} \wedge \mathcal{A}^m_{(1)} \wedge \mathcal{A}^n_{(1)} \ , \\
F_{(3)i} &= \gamma^j{}_i\, \tilde{F}_{(3)j} + \gamma^j{}_i\, \gamma^k{}_\ell\, \tilde{F}_{(2)jk} \wedge \mathcal{A}^\ell_{(1)} + \tfrac12 \gamma^j{}_i\, \gamma^k{}_m\, \gamma^\ell{}_n\, \tilde{F}_{(1)jk\ell} \wedge \mathcal{A}^m_{(1)} \wedge \mathcal{A}^n_{(1)} \ , \\
F_{(2)ij} &= \gamma^k{}_i\, \gamma^\ell{}_j\, \tilde{F}_{(2)k\ell} - \gamma^k{}_i\, \gamma^\ell{}_j\, \gamma^m{}_n\, \tilde{F}_{(1)k\ell m} \wedge \mathcal{A}^n_{(1)} \ , \\
F_{(1)ijk} &= \gamma^\ell{}_i\, \gamma^m{}_j\, \gamma^n{}_k\, \tilde{F}_{(1)\ell mn} \ , \\
\mathcal{F}^i_{(2)} &= \tilde{\mathcal{F}}^i_{(2)} - \gamma^j{}_k\, \tilde{\mathcal{F}}^i_{(1)j} \wedge \mathcal{A}^k_{(1)} \ , \\
\mathcal{F}^i_{(1)j} &= \gamma^k{}_j\, \tilde{\mathcal{F}}^i_{(1)k} \ ,
\end{aligned} \tag{1.68}$$

where the tilded quantities represent the unmodified pure exterior derivatives of the corresponding potentials, $\tilde{F}_{(n)} \equiv dA_{(n-1)}$, and $\gamma^i{}_j$ is defined by

$$\gamma^i{}_j = [(1+\mathcal{A}_0)^{-1}]^i{}_j = \delta^i_j - \mathcal{A}^i_{(0)j} + \mathcal{A}^i_{(0)k}\, \mathcal{A}^k_{(0)j} + \cdots \ . \tag{1.69}$$

Recalling that $\mathcal{A}^i_{(0)j}$ is defined only for $j > i$ (and vanishes if $j \le i$), we see that the series terminates after a finite number of terms. We also define here the inverse of $\gamma^i{}_j$, namely $\tilde\gamma^i{}_j$ given by

$$\tilde\gamma^i{}_j = \delta^i_j + \mathcal{A}^i_{(0)j}\ . \tag{1.70}$$

Another point still requiring explanation is the term denoted by $\mathcal{L}_{FFA}$ in (1.61). This is the D-dimensional descendant of the term $\frac16 dA_{(3)}\wedge dA_{(3)}\wedge A_{(3)}$. Again, the calculations are purely mechanical, and we can just present the results:

$$\begin{aligned}
D=10:&\quad \tfrac12 \tilde F_{(4)}\wedge\tilde F_{(4)}\wedge A_{(2)}\ ,\\
D=9:&\quad \Big(\tfrac14 \tilde F_{(4)}\wedge\tilde F_{(4)}\wedge A_{(1)ij} - \tfrac12 \tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(3)}\Big)\epsilon^{ij}\ ,\\
D=8:&\quad \Big(\tfrac{1}{12}\tilde F_{(4)}\wedge\tilde F_{(4)}A_{(0)ijk} - \tfrac16\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(2)k} - \tfrac12\tilde F_{(4)}\wedge\tilde F_{(3)i}\wedge A_{(1)jk}\Big)\epsilon^{ijk}\ ,\\
D=7:&\quad \Big(\tfrac16\tilde F_{(4)}\wedge\tilde F_{(3)i}A_{(0)jkl} - \tfrac14\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(1)kl} + \tfrac18\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(3)}\Big)\epsilon^{ijkl}\ ,\\
D=6:&\quad \Big(\tfrac{1}{12}\tilde F_{(4)}\wedge\tilde F_{(2)ij}A_{(0)klm} - \tfrac{1}{12}\tilde F_{(3)i}\wedge\tilde F_{(3)j}A_{(0)klm} + \tfrac18\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(2)m}\Big)\epsilon^{ijklm}\ ,\\
D=5:&\quad \Big(\tfrac{1}{12}\tilde F_{(3)i}\wedge\tilde F_{(2)jk}A_{(0)lmn} + \tfrac{1}{48}\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(1)mn}\\
&\qquad -\tfrac{1}{72}\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(3)}\Big)\epsilon^{ijklmn}\ ,\\
D=4:&\quad \Big(\tfrac{1}{48}\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}A_{(0)mnp} - \tfrac{1}{72}\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(2)p}\Big)\epsilon^{ijklmnp}\ ,\\
D=3:&\quad -\tfrac{1}{144}\,\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(1)pq}\,\epsilon^{ijklmnpq}\ ,\\
D=2:&\quad -\tfrac{1}{1296}\,\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}A_{(0)pqr}\,\epsilon^{ijklmnpqr}\ .
\end{aligned}\tag{1.71}$$

We may now ask the analogous question to the one we considered in the single-step S^1 reduction, namely what are the symmetries of the dimensionally-reduced theory, and how do they arise from the original higher-dimensional symmetries. Although the discussion above was aimed at the specific example of the T^n reduction of $D=11$ supergravity, it is obvious that much of the general structure, for example in the reduction of the Einstein-Hilbert term, is applicable to any starting dimension.

Let us consider the higher-dimensional general coordinate transformations, which, in infinitesimal form, are paramameterised in terms of the vector $\hat\xi^M$ as before: $\delta\hat x^M = -\hat\xi^M(\hat x)$. The difference now is that we have n reduction coordinates z^i, and so the higher-dimensional coordinates $\hat x^M$ are split as $\hat x^M = (x^\mu, z^i)$. As in the S^1 reduction, we must first identify the subset of these higher-dimensional general coordinate transformations that leaves the *structure* of the dimensional-reduction ansatz (1.66) invariant. (In other words, we need to find the transformations which allow the metric still to be written in the same form (1.66), but with, in general, transformed lower-dimensional fields $g_{\mu\nu}$, $\mathcal{A}^i_{(1)}$, $\mathcal{A}^i_{(0)j}$ and $\vec\phi$.

The crucial point is that only those higher-dimensional general coordinate transformations that preserve the z^i-independence of the lower-dimensional fields are allowed.)

It is not hard to see, using the expression (1.42) for the infinitesimal general coordinate transformations of $\hat{g}_{MN}$, that the subset that preserves the structure of (1.66) is

$$\hat{\xi}^\mu(x,z) = \xi^\mu(x) \ , \qquad \hat{\xi}^i(x,z) = \Lambda^i{}_j \, z^j + \lambda^i(x) \ , \tag{1.72}$$

where the quantities $\Lambda^i{}_j$ are constants. This generalises the expression (1.43) that we obtained in the case of the S^1 reduction. Clearly, we can expect that $\xi^\mu(x)$ will again describe the general coordinate transformations of the lower-dimensional theory. The n local parameters $\lambda^i(x)$, which generalise the single local parameter $\lambda(x)$ of the S^1-reduction case, will now describe the local $U(1)$ gauge invariances of the n Kaluza-Klein vector fields $\mathcal{A}^i_\mu$.

This leaves only the global transformations, parameterised by the constants $\Lambda^i{}_j$ to interpret. These generalise the single constant c of the S^1 reduction example. In that case, we saw that after taking into account the additional scaling symmetry of the higher-dimensional equations of motion, which could be used as a compensating transformation, we could extract a symmetry in the lower dimension that left the metric invariant, and described a constant shift of the dilaton, combined with appropriate constant rescalings of the gauge fields. In group-theoretic terms, that was an $\mathbb{R}$ transformation; the group parameter c took values anywhere on the real line.

In our present case with a reduction on the torus T^n, we have n^2 constant parameters $\Lambda^i{}_j$ appearing in (1.72). They act by matrix multiplication on the "column vector" composed of the internal coordinates z^i on the torus,

$$\delta z^i = -\Lambda^i{}_j \, z^j \ . \tag{1.73}$$

The matrix $\Lambda^i{}_j$ is unrestricted; it just has n^2 real components. This is the general linear group of real $n \times n$ matrices, denoted by $GL(n,\mathbb{R})$. There is, of course, again also the uniform scaling symmetry of the higher-dimensional equations of motion. One can use this as a "compensator," to allow all of the $\Lambda^i{}_j$ transformations to become purely internal symmetries, which act on the various lower-dimensional potentials and dilatons, but which leave the lower-dimensional metric invariant. This can be seen by calculations that are precisely analogous to the ones for the S^1 reduction in the previous section.

The conclusion, therefore, from the above discussion is that when the Einstein-Hilbert action is dimensionally reduced on the n-dimensional torus T^n, it gives rise to a theory in the lower dimension that has a $GL(n,\mathbb{R})$ global symmetry, in addition to the local general coordinate and gauge symmetries generated by $\xi^\mu(x)$ and $\lambda^i(x)$. In fact, the $GL(n,\mathbb{R})$ transformations are also symmetries of the theory that we get when we include the other terms in the eleven-dimensional supergravity Lagrangian. This is a rather general feature; any theory with gravity coupled to other matter fields will, upon dimensional reduction on T^n, give rise to a theory with a $GL(n,\mathbb{R})$ global symmetry. (Strictly speaking, one can only be sure of $SL(n,\mathbb{R})$ as an *internal* symmetry that leaves the metric invariant; getting the full $GL(n,\mathbb{R})$ depends on having the extra homogeneous scaling symmetry of the higher-dimensional equations of motion; note that $GL(n,\mathbb{R}) \sim SL(n,\mathbb{R}) \times \mathbb{R}$.)

Actually, as we shall see later, the reduction of eleven-dimensional supergravity on T^n actually typically gives a *bigger* global symmetry than $GL(n,\mathbb{R})$. The reason for this is that there is actually a "conspiracy" between the metric and the 3-form potential of $D=11$, and between them they create a lower-dimensional system that has an enlarged global symmetry. The phenomenon first sets in when one descends down to eight dimensions on the 3-torus, for which the global symmetry is $SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$, rather than the naively-expected $GL(3,\mathbb{R})$. By the time one considers a reduction from $D=11$ to $D=3$ on the 8-torus, the naively-expected $GL(8,\mathbb{R})$ is enlarged to an impressive E_8. We won't study all the details of how these enlargements occur, but we will look at some of the elements in the mechanism. First, let us consider the simplest non-trivial example of a global symmetry, which arises in a reduction of pure gravity on a 2-torus.

1.4 $SL(2,\mathbb{R})$ and the 2-torus

Let us consider pure gravity in $D+2$ dimensions, reduced to D dimensions on T^2. From the earlier discussions it is clear that we will get the following fields in the dimensionally-reduced theory: $(g_{\mu\nu}, \mathcal{A}^i_{(1)}, \mathcal{A}^1_{(0)2}, \vec{\phi})$. The notation is a little ugly-looking here, so let us just review what we have. There are two Kaluza-Klein gauge potentials $\mathcal{A}^i_{(1)}$, and then there is the 0-form potential, or axion, $A^1_{(0)2}$. This is what comes from the dimensional reduction of the first of the two Kaluza-Klein vectors, $\mathcal{A}^1_{(1)}$, which, at the second reduction step gives not only a vector, but also the axion. We can make things look nicer by using the symbol χ to represent $A^1_{(0)2}$. From the previous results, it is not hard to see that the dimensionally-reduced Lagrangian is

$$\mathcal{L} = R\,{*1} - \tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}\sum_i e^{\vec{c}_i\cdot\vec{\phi}}\,{*\mathcal{F}_{(2)i}} \wedge \mathcal{F}_{(2)i} - \tfrac{1}{2} e^{\vec{c}\cdot\vec{\phi}}\,{*d\chi} \wedge d\chi\ , \qquad (1.74)$$

where the dilaton vectors are given by

$$\vec{c}_1 = \Big(-\sqrt{\tfrac{2D}{D-1}}, -\sqrt{\tfrac{2}{(D-1)(D-2)}}\Big) \ , \qquad \vec{c}_2 = \Big(0, -\sqrt{\tfrac{2(D-1)}{D-2}}\Big) \ ,$$
$$\vec{c} = \Big(-\sqrt{\tfrac{2D}{D-1}}, \sqrt{\tfrac{2(D-2)}{D-1}}\Big) \ . \tag{1.75}$$

The field strengths are given by

$$\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} - d\chi\wedge \mathcal{A}^2_{(1)} \ , \qquad \mathcal{F}^2_{(2)} = d\mathcal{A}^2_{(1)} \ . \tag{1.76}$$

Things simplify a lot if we rotate the basis for the two dilatons $\vec{\phi} = (\phi_1, \phi_2)$. Make the orthogonal transformation to two new dilaton combinations, which we may call ϕ and φ:

$$\phi = -\tfrac12\sqrt{\tfrac{2D}{D-1}}\, \phi_1 + \tfrac12\sqrt{\tfrac{2(D-2)}{D-1}}\, \phi_2 \ , \qquad \varphi = -\tfrac12\sqrt{\tfrac{2(D-2)}{D-1}}\, \phi_1 - \tfrac12\sqrt{\tfrac{2D}{D-1}}\, \phi_2 \ . \tag{1.77}$$

After a little algebra, the Lagrangian (1.74) can be seen to become

$$\mathcal{L} = R\, {*1} - \tfrac12 {*d\varphi}\wedge d\varphi - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 e^{\phi+q\varphi}\, {*\mathcal{F}^1_{(2)}}\wedge \mathcal{F}^1_{(2)} - \tfrac12 e^{-\phi+q\varphi}\, {*\mathcal{F}^2_{(2)}}\wedge \mathcal{F}^2_{(2)} - \tfrac12 e^{2\phi}\, {*d\chi}\wedge d\chi \ , \tag{1.78}$$

where $q = \sqrt{D/(D-2)}$.

Note also that from the expression (1.66) for the dimensionally-reduced metric, we have

$$ds^2_{D+2} = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}\, ds^2_D + e^{\sqrt{(D-2)/D}\,\varphi}\Big(e^{\phi}\,(dz_1 + \mathcal{A}^1_{(1)} + \chi\, dz_2)^2 + e^{-\phi}\,(dz_2 + \mathcal{A}^2_{(2)})^2\Big) \ . \tag{1.79}$$

This shows that the scalar φ has the interpretation of parameterising the volume of the 2-torus, since it occurs in an overall multiplicative factor of the internal compactifying metric, while ϕ parameterises a shape-changing mode of the torus, since it scales the lengths of the two circles of the torus in opposite directions. In fact ϕ and χ completely characterise the *moduli* of the torus. The moduli are parameters that change the shape of the torus, at fixed volume, while keeping it flat. One can see that as ϕ varies, the relative radii of the two circles change, while as χ varies, the angle between the two circles changes.

Let us now look at the scalars in the Lagrangian (1.78), namely ϕ, φ and χ, described by the scalar Lagrangian

$$\mathcal{L}_{\rm scal} = -\tfrac12 (\partial\varphi)^2 - \tfrac12(\partial\phi)^2 - \tfrac12 e^{2\phi}\,(\partial\chi)^2 \ . \tag{1.80}$$

It is evident that φ is decoupled from the others. It has a global shift symmetry, $\varphi \longrightarrow \varphi + k$. This gives an $\mathbb{R}$ factor in the global symmetry group. Now look at the dilaton-axion system (ϕ,χ). This is best analysed by defining a complex field $\tau = \chi + {\rm i}\, e^{-\phi}$. The Lagrangian for ϕ and χ can then be written as

$$\mathcal{L}_{(\phi,\chi)} \equiv -\tfrac12(\partial\phi)^2 - \tfrac12 e^{2\phi}\,(\partial\chi)^2 = -\frac{\partial\tau\cdot\partial\bar\tau}{2\,\tau_2^2} \ , \tag{1.81}$$

where τ_2 means the imaginary part of τ; one commonly writes $\tau = \tau_1 + \mathrm{i}\,\tau_2$. Now, it is not hard to see that if τ is subjected to the following *fractional linear transformation*,

$$\tau \longrightarrow \frac{a\,\tau + b}{c\,\tau + d}\ , \tag{1.82}$$

where a, b, c and d are constants that satisfy

$$a\,d - b\,c = 1\ , \tag{1.83}$$

then the Lagrangian (1.81) is left invariant. But we can write the constants in a 2×2 matrix,

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ , \tag{1.84}$$

with the condition (1.83) now restated as $\det\Lambda = 1$. What we have here is real 2×2 matrices of unit determinant. They therefore form the group $SL(2,\mathbb{R})$. This $SL(2,\mathbb{R})$ is a symmetry that acts non-linearly on the complex scalar field τ, as in (1.82).

Thus we have seen that the scalar Lagrangian (1.80) has in total an $\mathbb{R} \times SL(2,\mathbb{R})$ global symmetry. This makes the $GL(2,\mathbb{R})$ symmetry that was promised in the previous section. Note that the $SL(2,\mathbb{R})$ transformation (1.82) can be expressed directly on the dilaton and axion, where it becomes

$$\begin{aligned} e^{\phi} &\longrightarrow e^{\phi'} = (c\,\chi + d)^2\, e^{\phi} + c^2\, e^{-\phi}\ , \\ \chi\, e^{\phi} &\longrightarrow \chi'\, e^{\phi'} = (a\,\chi + b)(c\,\chi + d)\, e^{\phi} + a\,c\,e^{-\phi}\ . \end{aligned} \tag{1.85}$$

To complete the story, we should go back to analyse the full Lagrangian (1.78) that includes the gauge fields $\mathcal{F}^i_{(2)}$. First of all, it is helpful to make a field redefinition $\mathcal{A}^1_{(1)} \longrightarrow \mathcal{A}^1_{(1)} + \chi\,\mathcal{A}^2_{(1)}$, which has the effect of changing the expression for the field strength $\mathcal{F}^1_{(2)}$, so that instead of (1.76) we have

$$\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} + \chi\, d\mathcal{A}^2_{(1)}\ , \qquad \mathcal{F}_{(2)} = d\mathcal{A}^2_{(1)}\ . \tag{1.86}$$

In other words, the derivative has bee shifted off χ, and onto $\mathcal{A}^2_{(1)}$ instead. The statement of how the $SL(2,\mathbb{R})$ transformations act on the gauge fields now becomes very simple; it is

$$\begin{pmatrix} \mathcal{A}^2_{(1)} \\ \mathcal{A}^1_{(1)} \end{pmatrix} \longrightarrow (\Lambda^T)^{-1} \begin{pmatrix} \mathcal{A}^2_{(1)} \\ \mathcal{A}^1_{(1)} \end{pmatrix}\ , \tag{1.87}$$

where Λ was defined in (1.84). This transformation on the potentials is to be performed at the same time as the transformation (1.85) is performed on the scalars. (If one spots the right way to do this calculation, the proof is not too difficult.) Note that while the scalars transform non-linearly under $SL(2,\mathbb{R})$, the two gauge potentials transform linearly, as a doublet. In other words, they just transform by matrix multiplication of $(\Lambda^T)^{-1}$ on the column vector formed from the two potentials.

1.5 Scalar coset Lagrangians

Many of the features of the 2-torus reduction that we saw in the previous section are rather general in all the toroidal dimensional reductions. In particular, one thing that we encountered was that the global symmetry of the lower-dimensional Lagrangian was already established by looking just at the scalar fields, and their symmetry transformations. Showing that the full Lagrangian had the symmetry was then a matter of showing that the terms in the full lower-dimensional Lagrangian that involve the higher-rank potentials (the two 1-form gauge potentials, in our 2-torus reduction example) also share the same symmetry. It is in fact essentially true in general that the extension of the global symmetry to the entire Lagrangian is "guaranteed," once it is established as a symmetry of the scalar sector. Furthermore, the higher-rank potentials always transform in linear representations of the global symmetry group, while the scalars transform non-linearly. One can, for example, show without too much further trouble that if one reduces $D = 11$ supergravity on the 2-torus, so that now the 3-form gauge potential is included also, the resulting additional gauge potentials in $D = 9$ will again transform linearly under the $GL(2,\mathbb{R})$ global symmetry. These additional gauge potentials will comprise $A_{(3)}$, transforming as a singlet under the $SL(2,\mathbb{R})$ subgroup, two 2-forms $A_{(2)i}$, transforming as a doublet, and one 1-form, $A_{(1)12}$, transforming as a singlet. Under the $\mathbb{R}$ factor of $GL(2,\mathbb{R})$, which corresponds to the constant shift symmetry of the other dilaton φ, all the potentials will transform by appropriate constant scaling factors.

To understand the structure of the global symmetries better, we need to study the nature of the scalar Lagrangians that arise from the dimensional reduction. This is instructive not only in its own right, but also because it leads us into the subject of non-linear sigma models, and coset spaces, which are of importance in many other areas of physics too. Let us begin by considering the $SL(2,\mathbb{R})$ example from the previous section. It exhibits many of the general features that one encounters in non-linear sigma models, while having the merit of being rather simple and easy to calculate explicitly.

The group $SL(2,\mathbb{R})$ is the non-compact version of $SU(2)$, and consequently, its associated Lie algebra (the elements infinitesimally close to the identity) is essentially the same as that of $SU(2)$. Thus we have the generators (H, E_+, E_-), satisfying the Lie algebra

$$[H, E_\pm] = \pm 2\, E_\pm \ , \qquad [E_+, E_-] = H \ . \tag{1.88}$$

H is the Cartan subalgebra generator, while $E_\pm$ are the raising and lowering operators. A convenient representation for the generators is in terms of 2×2 matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{1.89}$$

(So $H = \tau_3$, $E_\pm = 1/2(\tau_1 \pm \mathrm{i}\, \tau_2)$, where τ_i are the Pauli matrices.)

Consider now the exponentiation of the H and E_+, and define

$$\mathcal{V} = e^{\frac{1}{2}\phi\, H}\, e^{\chi\, E_+} , \tag{1.90}$$

where ϕ and χ are thought of as fields depending on the coordinates of a D-dimensional spacetime. A simple calculation shows that

$$\mathcal{V} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi\, e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} . \tag{1.91}$$

We now compute the exterior derivative, to find

$$d\mathcal{V}\, \mathcal{V}^{-1} = \begin{pmatrix} \frac{1}{2}d\phi & e^{\phi}\, d\chi \\ 0 & -\frac{1}{2}d\phi \end{pmatrix} = \tfrac{1}{2}d\phi\, H + e^{\phi}\, d\chi\, E_+ . \tag{1.92}$$

Let us define also the matrix $\mathcal{M} = \mathcal{V}^T\, \mathcal{V}$. It is easy to see from (1.91) that we have

$$\mathcal{M} = \begin{pmatrix} e^{\phi} & \chi\, e^{\phi} \\ \chi\, e^{\phi} & e^{-\phi} + e^{\phi}\, \chi^2 \end{pmatrix} , \qquad \mathcal{M}^{-1} = \begin{pmatrix} e^{-\phi} + e^{\phi}\, \chi^2 & -\chi\, e^{\phi} \\ -\chi\, e^{\phi} & e^{\phi} \end{pmatrix} . \tag{1.93}$$

Thus we see that we may write a scalar Lagrangian as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\,\Big(\partial\mathcal{M}^{-1}\, \partial\mathcal{M}\Big) = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}\, (\partial\chi)^2 . \tag{1.94}$$

This is nothing but the $SL(2,\mathbb{R})$-invariant scalar Lagrangian that we encountered in the previous section. The advantage now is that we have a very nice way to see why it is $SL(2,\mathbb{R})$ invariant.

To do this, observe that if we introduce an arbitrary constant $SL(2,\mathbb{R})$ matrix Λ, given by

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad a\, d - b\, c = 1 , \tag{1.95}$$

then if we send $\mathcal{V} \longrightarrow \mathcal{V}'' = \mathcal{V}\, \Lambda$, we get $\mathcal{M} \longrightarrow (\mathcal{V}'')^T\, \mathcal{V}'' = \Lambda^T\, \mathcal{V}^T\, \mathcal{V}\, \Lambda = \Lambda^T\, \mathcal{M}\, \Lambda$, which manifestly leaves $\mathcal{L}$ invariant:

$$\mathcal{L} \longrightarrow \tfrac{1}{4}\mathrm{tr}\,\Big(\Lambda^{-1}\, \partial\mathcal{M}^{-1}\, (\Lambda^T)^{-1}\, \Lambda^T\, \partial\mathcal{M}\, \Lambda\Big) = \tfrac{1}{4}\mathrm{tr}\,\Big(\partial\mathcal{M}^{-1}\, \partial\mathcal{M}\Big) . \tag{1.96}$$

The only trouble with this transformation is that when we sent $\mathcal{V} \longrightarrow \mathcal{V}'' = \mathcal{V}\, \Lambda$ we actually did something improper, because in general the transformed matrix $\mathcal{V}''$ is *not* of the upper-triangular form that the original matrix $\mathcal{V}$ given in (1.91 is. Thus by acting with Λ, we have done something that cannot, as it stands, be expressed as a transformation on the fields ϕ and χ. Happily, there is a simple remedy for this. What we must do is make a compensating *local* transformation $\mathcal{O}$ that acts on $\mathcal{V}$ from the left, at the same time as we multiply by the constant $SL(2,\mathbb{R})$ matrix from the right. Thus we define a transformed matrix $\mathcal{V}'$ by

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda\ , \tag{1.97}$$

where, by definition, $\mathcal{O}$ is the matrix that does the job of restoring $\mathcal{V}'$ to the upper-triangular gauge. There is a unique orthogonal matrix that does the job, and after a little algebra, one finds that it is

$$\mathcal{O} = \big(c^2 + e^{2\phi}\,(c\,\chi + a)^2\big)^{-1/2}\,\begin{pmatrix} e^{\phi}\,(c\,\chi+a) & c \\ -c & e^{\phi}\,(c\,\chi+a)\end{pmatrix}\ . \tag{1.98}$$

The matrix $\mathcal{O}$ that we have just constructed does the job of restoring the $SL(2,\mathbb{R})$-transformed matrix $\mathcal{V}$ to the upper-triangular gauge of (1.91), which means that we can now interpret the action of $SL(2,\mathbb{R})$ in terms of transformations on ϕ and χ. But does it give us an invariance of the Lagrangian (1.94)? The answer is yes, and this is easily seen. The matrix $\mathcal{O}$ is the specific one that does the job of compensating for the $SL(2,\mathbb{R})$ transformation with constant parameters a, b, c and d. It is itself local, since it depends not only on the constant $SL(2,\mathbb{R})$ parameters but also on the fields ϕ and χ themselves. This does not cause trouble, however, because, crucially, $\mathcal{O}$ is an *orthogonal* matrix. This means that when we calculate how $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$ transforms, we find

$$\mathcal{M} \longrightarrow \mathcal{M}' = (\mathcal{V}')^T\,\mathcal{V}' = \Lambda^T\,\mathcal{V}^T\,\mathcal{O}^T\,\mathcal{O}\,\mathcal{V}\,\Lambda = \Lambda^T\,\mathcal{V}^T\,\mathcal{V}\,\Lambda = \Lambda^T\,\mathcal{M}\,\Lambda\ . \tag{1.99}$$

Thus the local compensating transformation cancels out when the transformed $\mathcal{M}$ matrix is calculated, and hence the previous calculation (1.96) demonstrating the invariance of the Lagrangian goes through without modification.

After a little algebra, it is not hard to see that the transformed fields ϕ' and χ', defined by (1.97), are precisely the ones that we obtained in the previous section, given in (1.85). It is not hard to see that at a given spacetime point (i.e. for fixed values of ϕ and χ), we can use the $SL(2,\mathbb{R})$ transformation to get from *any* pair of values for ϕ and χ to any other pair of values. This means that $SL(2,\mathbb{R})$ acts *transitively* on the *scalar manifold*, which is the manifold where the fields ϕ and χ take their values.

Let us take stock of what we have found. We have parameterised points in the scalar manifold in terms of the matrix $\mathcal{V}$ in (1.91). We have seen that acting from the right with an $SL(2,\mathbb{R})$ matrix Λ, we can get to any other point in the scalar manifold. But we must, in general, make a compensating $O(2)$ transformation as we do so, to make sure that we stay within our original parameterisation scheme in terms of the upper-triangular matrices $\mathcal{V}$. Thus we may specify points in the scalar by the *coset* $SL(2,\mathbb{R})/O(2)$, consisting of $SL(2,\mathbb{R})$ motions modulo the appropriate $O(2)$ compensators. Thus we may say that the scalar manifold for the (ϕ,χ) dilaton/axion system is the coset space $SL(2,\mathbb{R})/O(2)$, and that it has $SL(2,\mathbb{R})$ as its global symmetry group.

In this example, the points in the $SL(2,\mathbb{R})/O(2)$ coset were parameterised by the *coset representative* $\mathcal{V}$, given in (1.91). We obtained this by exponentiating just two of the $SL(2,\mathbb{R})$ generators, namely the Cartan generator H and the raising operator E_+. Things don't always go quite so smoothly and easily as this, but in the case of the various scalar coset manifolds that arise in the toroidal compactifications of eleven-dimensional supergravity they do. Let us, therefore, pursue these examples a bit further.

Our discussion above was for the reduction of the Einstein-Hilbert action on T^2, starting in any dimension $D+2$ and ending up in D dimensions. We could generalise this to include some additional antisymmetric tensors in $D+2$ dimensions, and we would find in general that they give rise to sets of fields in D dimensions that transform linearly under $SL(2,\mathbb{R})$. In the case where we start with supergravity in $D=11$, we would have an additional 3-form potential, therefore. After reduction to $D=9$ on T^2, we would get the fields discussed above in from the gravity sector, together with fields $A_{(3)}$, $A_{(2)i}$ and $A_{(1)12}$ that descend from $A_{(3)}$. One finds that $A_{(3)}$ is a singlet under $SL(2,\mathbb{R})$, the two $A_{(2)i}$ form a doublet, and $A_{(1)12}$ is again a singlet.

The situation changes if we descend from $D=11$ on a higher-dimensional torus. The reason is that we now start to get additional axionic scalar fields from the descendants of $A_{(3)}$, over and above the scalars that come from the eleven-dimensional metric. For example, if we descend on T^3 to $D=8$, we now have not only the three dilatons $\vec{\phi}$, and three axions $\mathcal{A}^i_{(0)j}$, but also one additional axion $A_{(0)123}$. Now the scalars $\vec{\phi}$ and $\mathcal{A}^i_{(0)j}$ have a Lagrangian with the "expected" $GL(3,\mathbb{R})$ global symmetry. In fact, they parameterise points in the six-dimensional coset manifold $GL(3,\mathbb{R})/O(3)$. But what happens with the symmetry is the following. We saw in $D=9$, in the T^2 reduction, that the $\mathbb{R}$ factor in the $GL(2,\mathbb{R})$ symmetry "factored off" from the rest of the $SL(2,\mathbb{R})$. The same thing happens here, and there is one dilaton which contributes the $\mathbb{R}$ factor in $GL(3,\mathbb{R})$, and which is decoupled from the remaining five scalars that form the $SL(3,\mathbb{R})/O(3)$ coset. It does, however, couple to the the additional axion, $A_{(0)123}$, coming from the reduction of $A_{(3)}$. In fact they form a dilaton/axion system with an $SL(2,\mathbb{R})$ global symmetry, working just like the $SL(2,\mathbb{R})$

that we saw in the T^2 reduction. Thus the final conclusion is that the reduction of $D = 11$ supergravity on T^3 to $D = 8$ gives a theory whose scalars parameterise the coset

$$\frac{SL(3,\mathbb{R})}{O(3)} \times \frac{SL(2,\mathbb{R})}{O(2)}\ , \tag{1.100}$$

and so there is an $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ global symmetry.

To see the details in this eight-dimensional example, let us consider just the scalar sector of the dimensionally-reduced theory. From (1.61), we will have

$$\mathcal{L}_8 = -\tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\, {*\mathcal{F}^i_{(1)j}} \wedge \mathcal{F}^i_{(1)j} - \tfrac{1}{2}e^{\vec{a}_{123}\cdot\vec{\phi}}\, {*F_{(1)123}} \wedge F_{(1)123}\ , \tag{1.101}$$

where

$$\mathcal{F}^1_{(1)2} = d\mathcal{A}^1_{(0)2}\ , \qquad \mathcal{F}^2_{(1)3} = d\mathcal{A}^2_{(0)3}\ , \qquad \mathcal{F}^1_{(1)3} = d\mathcal{A}^1_{(0)3} - \mathcal{A}^2_{(0)3}\, d\mathcal{A}^1_{(0)2}\ , \qquad F_{(1)123} = dA_{(0)123}\ . \tag{1.102}$$

From the general results for the dilaton vectors, it is not hard to see that after performing an orthogonal transformation to make things look nicer, we can make the dilaton vectors become

$$\begin{aligned} &\vec{b}_{12} = (0,1,\sqrt{3})\ , \qquad \vec{b}_{23} = (0,1,-\sqrt{3})\ , \qquad \vec{b}_{13} = (0,2,0)\ , \\ &\vec{a}_{123} = (2,0,0)\ . \end{aligned} \tag{1.103}$$

We see that indeed the axion $A_{(0)123}$ and the dilaton ϕ_1 form an independent $SL(2,\mathbb{R})/O(2)$ scalar coset, which is decoupled from the rest of the scalar sector.

This leaves the $SL(3,\mathbb{R})$ part of the scalar coset still to understand. Perhaps the easiest way to see what's happening here is to recall a couple of facts about group theory. The generators of a Lie algebra $\mathcal{G}$ can be organised into Cartan generators, $\vec{H}$, which mutually commute with each other, and raising and lowering operators $E_{\vec{\alpha}}$. If the rank of the algebra is n, then there are n Cartan generators, $\vec{H} = (H_1, \ldots, H_n)$. The raising and lowering operators have the commutation relations

$$[\vec{H}, E_{\vec{\alpha}}] = \vec{\alpha}\, E_{\vec{\alpha}} \tag{1.104}$$

with the Cartan generators, where $\vec{\alpha}$ are called the root vectors associated with the generators $E_{\vec{\alpha}}$. One sets up a scheme for defining root vectors to be positive or negative. The standard way to do this is to look at the components of the root vector $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, working from the left to the right. The *sign* of the root vector is defined to be the sign of the first non-zero component that is encountered. Generators with positive root vectors are called raising operators, and those with negative roots are called lowering operators. It is easily seen from (1.104) that if the commutator of two non-zero-root generators $E_{\vec\alpha}$ and $E_{\vec\beta}$ is non-vanishing, then it will be a generator with root vector $\vec\alpha+\vec\beta$. Thus in general we have

$$[E_{\vec\alpha}, E_{\vec\beta}] = N(\alpha,\beta)\, E_{\vec\alpha+\vec\beta}\ , \tag{1.105}$$

for some constant (possibly zero) $N(\alpha,\beta)$.

The classification of all the possible Lie algebras is quite straightforward, but it is a lengthy business, and we shall not stray into it here. Suffice it to say that it turns out that the Lie algebras can be classified by classifying all the possible root systems, which means determining all the possible sets of roots that satisfy certain consistency requirements. In turn, these root systems can be characterised in terms of the *simple roots*. These are defined to be the subset of the positive roots that allow one to express *any* positive root in the system as a linear combination of the simple roots with non-negative integer coefficients. One can show that the number of simple roots is equal to the rank of the algebra. In other words, there are as many simple roots as there are components to the root vectors.

In the example of $SL(2,\mathbb{R})$, which has rank 1, we had the single Cartan generator H, and the single positive-root generator E_+, with the single-component "root vector" 2, as in (1.88). In general, $SL(n+1,\mathbb{R})$ has rank n, and so for $SL(3,\mathbb{R})$ we have rank 2. Thus we expect two Cartan generators $\vec H$, and 2-component root vectors. In fact this is just what we are seeing in our eight-dimensional scalar Lagrangian. Forgetting now about the $SL(2,\mathbb{R})$ part, which, as we have seen, factors off from the rest, we have two dilatons $\vec\phi=(\phi_2,\phi_3)$, and 2-component dilaton vectors

$$\vec b_{12} = (1,\sqrt3)\ ,\qquad \vec b_{23} = (1,-\sqrt3)\ ,\qquad \vec b_{13} = (2,0)\ . \tag{1.106}$$

(These follow from (1.103) by dropping the first component of each dilaton vector; i.e. the component associated with the decoupled $SL(2,\mathbb{R})$ part.) We can recognise the $\vec b_{ij}$ dilaton vectors as the positive roots of $SL(3,\mathbb{R})$, with $\vec b_{12}$ and $\vec b_{23}$ as the two simple roots, and $\vec b_{13}=\vec b_{12}+\vec b_{23}$. We may introduce positive-root generators ${E_i}^j$, defined for $i<j$, associated with the root-vectors $\vec b_{ij}$, and Cartan generators $\vec H$, with the commutation relations

$$[\vec H, {E_i}^j] = \vec b_{ij}\, {E_i}^j\ ,\qquad [{E_i}^j, {E_k}^\ell] = \delta^j_k\, {E_i}^\ell - \delta^\ell_i\, {E_k}^j\ . \tag{1.107}$$

Observe that the only non-zero commutator among the positive-root generators here is $[{E_1}^2, {E_2}^3] = {E_1}^3$.

One can represent the various generators here in terms of 3×3 matrices. For $E_i{}^j$, we define it to be the matrix with zeroes everywhere except for a 1 at the position of row i and column j, and so

$$E_1{}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E_2{}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E_1{}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{1.108}$$

The two Cartan generators $\vec{H} = (H_1, H_2)$ are then diagonal, with

$$H_1 = \mathrm{diag}\,(1, 0, -1) \ , \qquad H_2 = \frac{1}{\sqrt{3}}\, \mathrm{diag}\,(1, -2, 1) \ . \tag{1.109}$$

The strategy for constructing the $SL(3, \mathbb{R})/O(3)$ coset Lagrangian is now to follow the same path that we used for $SL(2, \mathbb{R})$. We write down a coset representative $\mathcal{V}$, by exponentiating the Cartan and positive-root generators of $SL(3, \mathbb{R})$, with the dilatons and axions as coefficients. We do this in the following way:

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}\, e^{\mathcal{A}^2_{(0)3}\, E_2{}^3}\, e^{\mathcal{A}^1_{(0)3}\, E_1{}^3}\, e^{\mathcal{A}^1_{(0)2}\, E_1{}^2} \ . \tag{1.110}$$

Note that there are obviously many different ways that one could organise this exponentiation; here, we exponentiate each generator separately, and then multiply the results together. An alternative would be to exponentiate the sum of generators times fields. This would, in general, give a slightly different expression for $\mathcal{V}$, since if A and B are two matrices that do not commute, then $e^A\, e^B \ne e^{A+B}$. (One can use the Baker-Campbell-Hausdorf formula to relate them.) The different possibilities correspond to making different choices for exactly how to parameterise points in the coset space, and eventually one choice is related to any other by making redefinitions of the fields. Thus any choice is equally as "good" as any other. The choice we are making here happens to be convenient, because it happens to correspond exactly to the choice of field variables in our eight-dimensional Lagrangian.

It is not hard to establish that with the coset representative $\mathcal{V}$ defined as in (1.110) above, one has

$$d\mathcal{V}\, \mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i_{(1)j}\, E_i{}^j \ , \tag{1.111}$$

where the 1-form field strengths $\mathcal{F}^i_{(1)j}$ are given in (1.102). In particular, the transgression term in $\mathcal{F}^1_{(1)3}$ comes from the fact that the commutator of $E_1{}^2$ and $E_2{}^3$ is non-zero, as given in (1.107). (One needs to use the following matrix relations in order to derive the result:

$$\begin{aligned} de^X\, e^{-X} &= dX + \tfrac{1}{2}[X, dX] + \tfrac{1}{6}[X,[X, dX]] + \cdots \ , \\ e^X\, Y\, e^{-X} &= Y + [X, Y] + \tfrac{1}{2}[X,[X, Y]] + \cdots \ . \end{aligned} \tag{1.112}$$

Only the first couple of terms in these expansions are ever needed, since the multiple commutators of positive-root generators rapidly expire.)

It is also straightforward to calculate $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$, and hence the Lagrangian

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\,\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M}\Big)\,. \tag{1.113}$$

(In practice, Mathematica is handy for this sort of calculation.) After a little algebra, one finds that it is given by

$$\mathcal{L} = -\tfrac{1}{2}{*d\vec{\phi}}\wedge d\vec{\phi} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(1)j}}\wedge\mathcal{F}^i_{(1)j}\,. \tag{1.114}$$

In other words, we have succeeded in writing the part of the eight-dimensional scalar Lagrangian (1.101) in a manifestly $SL(3,\mathrm{I\!R})$-invariant fashion.

To make the $SL(3,\mathrm{I\!R})$ symmetry fully explicit, we should really repeat the steps that we followed in the case of the $SL(2,\mathrm{I\!R})$ example. Namely, we should consider a general global $SL(3,\mathrm{I\!R})$ transformation Λ acting *via* right-multiplication on the coset representative $\mathcal{V}$. This will in general take us out of the upper-triangular gauge of (1.110), and so we should then show that there exists a local, field-dependent, compensating $O(3)$ transformation $\mathcal{O}$, such that

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda \tag{1.115}$$

is back in the upper-triangular gauge. This means that one can then interpret $\mathcal{V}'$, *via* the definition (1.110), as the coset representative for a different point in the coset manifold, corresponding to the transformed fields with primes on them. The matrix $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$ that is used to construct the scalar Lagrangian (1.113) then transforms nicely as $\mathcal{M} \longrightarrow \mathcal{M}' = \Lambda^T\,\mathcal{M}\,\Lambda$, hence implying the invariance of the Lagrangian under global $SL(3,\mathrm{I\!R})$ transformations.

In this particular case, it is perfectly possible to do this calculation explicitly, and to exhibit the required $O(3)$ compensator (again, Mathematica can be handy here). However, it is clear that in more complicated examples it would become increasingly burdensome to construct the compensator $\mathcal{O}$. Furthermore, we don't actually really need to know what it is; all we really need is to know that it exists. Luckily, there is a general theorem in the theory of Lie algebras, which does the job for us. It is known as the *Iwasawa Decomposition*, and it goes as follows. The claim is that every element g in the Lie group G obtained by exponentiating the Lie algebra $\mathcal{G}$ can be uniquely expressed as the following product:

$$g = g_K\,g_H\,g_N\,. \tag{1.116}$$

Here g_K is in the maximal compact subgroup K of G, g_H is in the Cartan subalgebra of G, and g_N is in the exponentiation of the positive-root part of the algebra $\mathcal{G}$.[2]

This is precisely what is needed for the discussion of the cosets that arise in these supergravity reductions. Our coset representative $\mathcal{V}$ is constructed by exponentiating the Cartan generators, and the full set of positive-root generators (see (1.90) for $SL(2,\mathrm{I\!R})$, and (1.110) for $SL(3,\mathrm{I\!R})$). Thus our coset representative is written as $\mathcal{V} = g_H\, g_N$. Now, we act by right-multiplication with a general group element Λ in G. This means that $\mathcal{V}\,\Lambda$ is *some* element of the group G. Now, we invoke the Iwasawa decomposition (1.116), which tells us that we must be able to write the group element $\mathcal{V}\,\Lambda$ in the form $g_K\,\mathcal{V}'$, where $\mathcal{V}'$ itself is of the form $g'_H\, g'_N$. This does what we wanted; it assures us that there exists a way of pulling out an element $\mathcal{O}$ of the maximal compact subgroup K of G on the left-hand side, such that we can write $\mathcal{V}\,\Lambda$ as $\mathcal{O}\,\mathcal{V}'$.

We are now in a position to proceed to the lower-dimensional theories obtained by compactifying eleven-dimensional supergravity on torii of higher dimensions. We can benefit from the lessons of the previous examples, and home in directly on the key points. Let us first, for reasons that will become clear later, consider the cases where the n-torus has $n \le 5$, meaning that we end up in dimensions $= 11 - n \ge 6$. The full set of axionic scalars will be $\mathcal{A}^i_{(0)j}$ and $A_{(0)ijk}$ in each dimension. From our T^2 and T^3 examples, we have seen that the dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ for these axions form the positive roots of a Lie algebra, and that by exponentiating the associated positive-root generators, with the axions as coefficients, and exponentiating the Cartan generators, with the dilatons as coefficients, we constructed a coset representative $\mathcal{V}$ for G/K, where G is the Lie group associated with the Lie algebra, and K is its maximal compact subgroup.

How do we identify what the group G is in each dimension? If we can identify the subset of the dilaton vectors that corresponds to the simple roots of the Lie algebra then we will have solved the problem. But this is easy; we just need to find what subset of the dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ allows us to express *all* of the dilaton vectors as linear combinations of the simple roots, with non-negative integer coefficients. The answer is very straightforward; the simple roots are given by

$$\vec{b}_{i,i+1}\ , \quad \text{for} \quad 1 \le i \le n-1\ , \qquad \text{and} \quad \vec{a}_{123}\ . \tag{1.117}$$

To check that this is correct, it is only necessary to look at the results in (1.63)-(1.65). It

[2] Actually, as we shall see later, this statement of the Iwasawa decomposition is appropriate only in the rather special circumstance we have here, where G is maximally non-compact. We shall give a more general statement later.

is manifest from the fact that $\vec{b}_{ij} = -\vec{f}_i + \vec{f}_j$ that any $\vec{b}_{ij}$ can be expressed as multiples of the $\vec{b}_{i,i+1}$, with non-negative integer coefficients. It is also clear that by adding appropriate integer multiples of the $\vec{b}_{i,i+1}$ to $\vec{a}_{123}$, all of the $\vec{a}_{ijk}$ can be constructed.

Having found the simple roots, it is easy to determine what the Lie algebra is. All the Lie algebras are classified in terms of their Dynkin diagrams, which encode the information about the lengths of the simple roots, and the angles between them. The notation is as follows. The angle between any two simple roots can be only one out of four possibilities, namely 90, 120, 135 or 150 degrees. The simple roots are denoted by dots in the Dynkin diagram, and the angle between two roots is indicated by the number of lines joining the corresponding dots. The rule is no line, 1 line, 2 lines or 3 lines, corresponding to 90, 120, 135 or 150 degrees. The lengths of the simple roots are either all equal (such groups are called *simply laced*), or they have exactly two different lengths, in groups that are called, unimaginatively, *non-simply-laced*. In this latter case, the dots in the Dynkin diagram are filled-in to denote the shorter roots, and unfilled for the longer roots. In our case, it turns out that the roots are all of the same length. From the expressions in (1.64), it is easily seen that our simple roots are characterised by the Dynkin diagram

$$
\begin{array}{ccccccccccccc}
\vec{b}_{12} & & \vec{b}_{23} & & \vec{b}_{34} & & \vec{b}_{45} & & \vec{b}_{56} & & \vec{b}_{67} & & \vec{b}_{78} \\
\mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} \\
 & & & & | & & & & & & & & \\
 & & & & \mathrm{o} & & & & & & & & \\
 & & & & \vec{a}_{123} & & & & & & & &
\end{array}
$$

This diagram is telling us that all the angles that are not 90 degrees are 120 degrees, and that all the simple roots have equal lengths. The understanding is that in a given dimension $D = 11 - n$, only those dilaton vectors which are defined for $i \le n$ arise. Those familiar with group theory and Dynkin diagrams will be able to recognise the diagrams for the various n values as follows. For $n = 2$, we have just $\vec{b}_{12}$, and the algebra is $SL(2, \mathrm{I\!R})$. For $n = 3$, we have $(\vec{b}_{12}, \vec{b}_{23}, \vec{a}_{123})$, and the algebra is $SL(3, \mathrm{I\!R}) \times SL(2, \mathrm{I\!R})$. These are the two cases that we have already studied in detail. For $n = 4$, we have $(\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}, \vec{a}_{123})$, and the Dynkin diagram is that of $SL(5, \mathrm{I\!R})$. For $n = 5$, we have $(\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}, \vec{b}_{45}, \vec{a}_{123})$, and the Dynkin diagram is that of D_5, or $O(5,5)$. We shall postpone the discussion of $n \ge 6$ for a while.

From our previous discussion of the T^2 and T^3 reductions, we expect now that we should introduce the appropriate positive-root generators associated with each of the dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$. For the $\vec{b}_{ij}$, we just use the same notation as before, with generators $E_i{}^j$, except that now the range of the i and j indices is extended to $1 \le i < j \le n$. For the $\vec{a}_{ijk}$, we introduce generators E^{ijk}. The commutation relations for these, and the Cartan generators $\vec{H}$, will be

$$[\vec{H}, E_i{}^j] = \vec{b}_{ij}\, E_i{}^j \ , \qquad [\vec{H}, E^{ijk}] = \vec{a}_{ijk}\, E^{ijk} \qquad \text{no sum} \tag{1.118}$$

$$[E_i{}^j, E_k{}^\ell] = \delta_k^j\, E_i{}^\ell - \delta_i^\ell\, E_k{}^j \ , \tag{1.119}$$

$$[E_\ell{}^m, E^{ijk},] = -3\delta_\ell^{[i}\, E^{|m|jk]} \ , \tag{1.120}$$

$$[E^{ijk}, E^{\ell mn}] = 0 \ , \tag{1.121}$$

We can recognise the commutation relations for the $\vec{H}$ and the $E_i{}^j$ as being precisely those of the Lie algebra $SL(n,\mathbb{R})$. This is reasonable on two counts. Firstly, since these are the generators associated with the fields coming from the reduction of pure gravity, namely $\vec{\phi}$ and $\mathcal{A}^i_{(0)j}$, we already expected to find a $GL(n,\mathbb{R})$ symmetry after reduction on the n-torus. (One never really sees the extra $\mathbb{R}$ factor of $GL(n,\mathbb{R}) \sim \mathbb{R} \times SL(n,\mathbb{R})$ in the Dynkin diagrams; it is associated with the fact that there is one extra Cartan generator over and above the $(n-1)$ that are needed for $SL(n,\mathbb{R})$.) Another way of seeing why this $SL(n,\mathbb{R})$ subgroup is reasonable is by looking at the Dynkin diagram above; if we delete the simple root $\vec{a}_{123}$, then the remaining simple roots $\vec{b}_{i,i+1}$ do indeed precisely give us the Dynkin diagram of $SL(n,\mathbb{R})$.

The extra commutation relations involving E^{ijk} extend the algebras from $SL(n,\mathbb{R})$ to the larger ones discussed above. Thus in addition to the $D = 9$ and $D = 8$ cases discussed previously, in $D = 7$ we will have the scalar coset $SL(5,\mathbb{R})/O(5)$, and in $D = 6$ we will have $O(5,5)/(O(5) \times O(5))$. In each case, in accordance with our discussion of the Iwasawa decomposition, the denominator group in the coset is the maximal compact subgroup of the numerator. The coset representatives in all cases $n \le 5$ are constructed as follows:

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \Big(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}\, E_i{}^j} \Big) \exp\Big(\sum_{i<j<k} A_{(0)ijk}\, E^{ijk} \Big) \ , \tag{1.122}$$

where the ordering of terms is *anti-lexigraphical*, i.e. $\cdots (24)(23) \cdots (14)(13)(12)$, in the product. With this specific way of organising the exponentiation, it turns out that, with the commutation relations given above, one has

$$d\mathcal{V}\, \mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\phi} \cdot \vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i{}_{(1)j}\, E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}\, F_{(1)ijk}\, E^{ijk} \ , \tag{1.123}$$

where the various 1-form field strengths, with all their transgression terms, are precisely as given in equation (1.68). (It is quite an involved calculation to show this!) In all the cases with $n \le 5$, one can define the matrix $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$, and it will follow that the scalar Lagrangian can be written as $\mathcal{L} = \frac{1}{4}\mathrm{tr}\,(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})$.

1.6 Scalar cosets in $D = 5$, 4 and 3

Aficionados of group theory will easily recognise that if we consider the cases $n = 6$, 7 and 8, corresponding to reductions to $D = 5$, 4 and 3 dimensions, the Dynkin diagrams above will be those of the exceptional groups E_6, E_7 and E_8. One does not need to be much of an aficionado, however, to see that as things stand, there is something wrong with the counting of fields. After reduction on an n torus there will be $\frac{1}{2}n(n-1)$ axions $\mathcal{A}^i_{(0)j}$, and $\frac{1}{6}n(n-1)(n-2)$ axions $A_{(0)ijk}$. For $n = (2, 3, 4, 5, 6, 7, 8)$, we therefore have $(1, 4, 10, 20, 35, 56, 84)$ axions in total. On the other hand, the numbers of positive roots for the groups indicated by the Dynkin diagrams above are $(1, 4, 10, 20, 36, 63, 120)$. Thus the discrepancies set in at $n = 6$ and above. We appear to be missing some axionic scalar fields.

Consider first the situation where this arises, when $n = 6$, implying that we have dimensionally reduced the $D = 11$ theory to $D = 5$. From the counting above, we are missing one axion. The explanation for where it comes from is in fact quite simple. Recall that among the fields in the reduced theory is the 3-form potential $A_{(3)}$, with its 4-form field strength $F_{(4)}$. Now, in $D = 5$, if we take the Hodge dual of a 4-form field strength, we get a 1-form, and this can be interpreted as the field strength for a 0-form potential, or axion. This is the source of our missing axion.

Before looking at this in more detail, let's just check the counting for remaining two cases. When $n = 7$, we have reduced the theory to $D = 4$, and in this case it is 2-form potentials that dualise into axions. The 2-form potentials are $A_{(2)i}$, and so when $n = 7$ there are seven of them. This is precisely the discrepancy that we noted in the previous paragraph. Finally, when $n = 8$ we have a reduction to $D = 3$, and in this case it is 1-form potentials that are dual to axions. The relevant potentials are $A_{(1)ij}$ and $\mathcal{A}^i_{(1)}$, of which there are $28 + 8 = 36$ when $n = 8$. Again, this exactly resolves the discrepancy noted in the previous paragraph.

Now, back to $D = 5$. As usual, we shall concentrate just on the scalar sector, since this governs the global symmetry of the entire theory. Now, of course, we must include the 3-form potential too, since we are about to dualise it to obtain the "missing" axion. In fact, to start with, we may consider just those terms in the five-dimensional Lagrangian that involve the 3-form potential. From the general results in (1.61) and the associated formulae, we can see that the relevant terms are

$$\mathcal{L}(F_{(4)}) = -\tfrac{1}{2} e^{\vec{a}\cdot\vec{\phi}} {*F_{(4)}} \wedge F_{(4)} - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn} \wedge F_{(4)}\, \epsilon^{ijk\ell mn} \ , \tag{1.124}$$

where $F_{(4)} = dA_{(3)}$. In the process of dualisation, the rôle of the Bianchi identity, which is $dF_{(4)} = 0$ here, is interchanged with the role of the field equation. The easiest way to achieve this is to treat $F_{(4)}$ as a fundamental field in its own right, and impose its Bianchi identity by adding the term $-\chi\, dF_{(4)}$ to the Lagrangian, where we have introduced the field χ as a Lagrange multiplier. Thus we consider

$$\mathcal{L}(F_{(4)})' = -\tfrac{1}{2} e^{\vec{a}\cdot\vec{\phi}} {*F_{(4)}} \wedge F_{(4)} - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn} \wedge F_{(4)}\, \epsilon^{ijk\ell mn} - \chi\, dF_{(4)} \ . \tag{1.125}$$

Clearly, the variation of this with respect to χ gives the required Bianchi identity. We note that $F_{(4)}$, which is now treated as a fundamental field, has a purely algebraic equation of motion. Varying $\mathcal{L}(F_{(4)})'$ with respect to $F_{(4)}$, we get the equation of motion

$$e^{\vec{a}\cdot\vec{\phi}} {*F_{(4)}} = d\chi - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn}\, \epsilon^{ijk\ell mn} \ . \tag{1.126}$$

We may define this right-hand side as our new 1-form field strength,; let us call it $G_{(1)}$:

$$G_{(1)} \equiv d\chi - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn}\, \epsilon^{ijk\ell mn} \ . \tag{1.127}$$

Thus we have $F_{(4)} = e^{-\vec{a}\cdot\vec{\phi}} {*G_{(1)}}$. Substituting this back into the Lagrangian (which is allowed, since it is a purely algebraic, non-differential equation), we find that $\mathcal{L}(F_{(4)})'$ has become

$$\mathcal{L}(F_{(4)})' = -\tfrac{1}{2} e^{-\vec{a}\cdot\vec{\phi}} {*G_{(1)}} \wedge G_{(1)} \ . \tag{1.128}$$

In other words, we have successfully dualised the potential $A_{(3)}$, with field strength $F_{(4)} = dA_{(3)}$, and replaced it with the axion χ, whose field strength $G_{(1)}$ is given in (1.127). Note that its dilaton vector, $-\vec{a}$, is the negative of the dilaton vector $\vec{a}$ of the field prior to dualisation. This sign reversal always occurs in any dualisation. Notice that one effect of the dualisation is that the FFA term in the Lagrangian (1.124) has migrated to become a transgression term in the definition of the new dualised field strength $G_{(1)}$ in (1.127). This interchange between FFA terms and transgression terms is a general feature in any dualisation.

Having found the missing axion, we must now consider the algebra, and the construction of the coset representative $\mathcal{V}$. We need one more generator, over and above the usual Cartan generators $\vec{H}$ and positive-root generators $E_i{}^j$ and E^{ijk}. In fact we are missing one further positive-root generator, in this $D=5$ example; let us call it J. It satisfies the following commutation relations, which extend the set given already in equations (1.118)-(1.121):

$$
\begin{aligned}
&[\vec{H}, J] = -\vec{a}\, J \ , \qquad [E_i{}^j, J] = 0 \ , \qquad [E^{ijk}, J] = 0 \ , \\
&[E^{ijk}, E^{\ell mn}] = -\epsilon^{ijk\ell mn}\, J \ .
\end{aligned}
\tag{1.129}
$$

The last commutator here is a reflection of the fact that in $D=5$, the sum of dilaton vectors $\vec{a}_{ijk} + \vec{a}_{\ell mn}$, when i,j,k,ℓ,m,n are all different, is equal to $-\vec{a}$, as can be seen from (1.62) and (1.65). Note that this depends crucially on a specific feature of reduction on a torus of dimension 6, since then we have that $\vec{a}_{ijk} + \vec{a}_{\ell mn} = \sum_i \vec{f}_i - 2\vec{g}$ since all of i,j,k,ℓ,m,n are different, and hence this equals $\vec{g}$.

The coset representative is now constructed as follows:

$$
\mathcal{V} = e^{\frac12 \vec{\phi}\cdot\vec{H}} \Big(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}\, E_i{}^j} \Big) \exp\Big(\sum_{i<j<k} A_{(0)ijk}\, E^{ijk} \Big)\, e^{\chi\, J} \ . \tag{1.130}
$$

After some algebra, one can show that now we have

$$
d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac12 d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac12 \vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i{}_{(1)j}\, E_i{}^j + \sum_{i<j<k} e^{\frac12 \vec{a}_{ijk}\cdot\vec{\phi}}\, F_{(1)ijk}\, E^{ijk} + e^{-\vec{a}\cdot\vec{\phi}}\, G_{(1)}\, J \ , \tag{1.131}
$$

where the 1-form field strengths $\mathcal{F}^i_{(1)j}$ and $F_{(1)ijk}$ are given in (1.68), and $G_{(1)}$ is given in (1.127). As in the previous examples, the transgression terms in all the field strengths come out to be precisely correct, and arise from the various non-vanishing commutators among the positive-root generators.

From the previous general discussion, we can expect that the coset representative $\mathcal{V}$ can be used to construct an E_6-invariant scalar Lagrangian, and that this will be the Lagrangian of the scalar sector of $D=11$ supergravity reduced on T^6. In particular, we can act on $\mathcal{V}$ from the right with a global E_6 transformation Λ , and then the Iwasawa decomposition theorem assures us that we can find a compensating field-dependent transformation $\mathcal{O}$ that acts on the left, such that $\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda$ is back in the "upper-triangular" gauge. In this case, the maximal compact subgroup of E_6 is $USp(8)$, and so $\mathcal{O}$ is a $USp(8)$ matrix. Actually, a better name for the gauge is really the *Borel gauge.* The Borel subgroup of any Lie group is the subgroup generated by the positive-root generators and the Cartan generators. Obviously this is a subgroup, since negative roots cannot be generated by commutation of non-negative ones. Sometimes, it is useful also to be able to talk of the *strict Borel subgroup*, defined to be the subgroup generated by the strictly-positive-root generators. In our cases, we obtain our coset representatives by exponentiating the entire Borel subalgebra, including the Cartan subalgebra.

Because the maximal compact subgroup in this E_6 case is no longer orthogonal, the way in which the Lagrangian is constructed from the coset representative $\mathcal{V}$ is slightly different. In general, the construction is the following. One defines the so-called *Cartan involution* τ, which has the effect of reversing the sign of every non-compact generator in the algebra $\mathcal{G}$, while leaving the sign of every compact generator unchanged. If we denote the positive-root generators, negative-root generators and Cartan generators by $(E_{\vec{\alpha}}, \{E_{-\vec{\alpha}}, \vec{H})$, where $\vec{\alpha}$ ranges over all the positive roots, then for our algebras τ effects the mapping

$$\tau : \qquad (E_{\vec{\alpha}}, E_{-\vec{\alpha}}, \vec{H}) \longrightarrow (-E_{-\vec{\alpha}}, -E_{\vec{\alpha}}, -\vec{H}) \ . \tag{1.132}$$

It should perhaps be remarked at this point that the groups that we are encountering in the toroidal compactifications of eleven-dimensional supergravity are somewhat special, in that they are always *maximally non-compact*. It is always the case, in any real group, that the generator combinations $(E_\alpha - E_{-\alpha})$ are compact while the combinations $(E_\alpha + E_{-\alpha})$ are non-compact.[3] (Thus if there are N positive roots, then there are N compact and N non-compact generators formed from the non-zero roots.) But in our case, we also have that all the Cartan generators are non-compact. Thus the group E_n that we encounter upon compactification on an n-torus is actually E_n in its maximally non-compact form, denoted by $E_{n(+n)}$. It has the n "extra" non-compact Cartan generators, in addition to the 50/50 split of compact/non-compact generators coming from the non-zero-root generators. We shall normally not bother with the extra annotation of the $(+n)$ in the subscript, but its presence will be implicit.

Getting back to the Cartan involution, we may use this to construct the required generalisation of the $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$ construction that worked when the maximal compact subgroup was orthogonal. Thus we may define a "generalised transpose" $X^\#$ of a matrix X, by

$$X^\# \equiv \tau(X^{-1}) \ . \tag{1.133}$$

From the definition of τ, and its action on the various generators, it is evident that $X^\#$ is nothing but X^T in cases where the compact generators give rise to an orthogonal group. If the compact generators form a unitary group, then $X^\#$ will be $X^\dagger$. In the case of E_6, the maximal compact subgroup is $USp(8)$, which is the intersection of $SU(8)$ and $Sp(8)$. A detailed discussion of the generalised transpose in this case would take us off into a

[3]By the real form of a group, we mean that the Hermitean generators are all formed by taking *real* combinations of the raising and lowering operators, not complex ones. For example, $SL(2,\mathbb{R})$ is the real form of A_1, since $E_+ \pm E_-$ and H are Hermitean, whereas $SU(2)$ is the complex form of A_1, since its Hermitean generators are the complex combinations $\tau_1 = E_+ + E_-$, $\tau_2 = i\,E_+ - i\,E_-$ and $\tau_3 = H$.

digression about symplectic invariants, and is probably inappropriate here. Some further details can be found in [2].

Suffice it to say that with the generalised transpose defined as above, the scalar Lagrangian in $D=5$ can now be written as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\left(\partial\mathcal{M}^{-1}\,\partial\mathcal{M}\right)\,, \tag{1.134}$$

where $\mathcal{M} = \mathcal{V}^{\#}\,\mathcal{V}$. The proof of the invariance under global E_6 transformations is then essentially identical to that in the previous examples that we discussed. Note that another way of writing the Lagrangian, which follows directly by substitution of $\mathcal{M} = \mathcal{V}^{\#}\,\mathcal{V}$ into (1.134), is

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{tr}\left(\partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})^{\#} + \partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})\right)\,. \tag{1.135}$$

The stories for the compactifications of $D=11$ supergravity on T^7 and T^8 to $D=4$ and $D=3$ proceed in a very similar manner. Full details can be found in [2]. As we already mentioned, in order to achieve the full E_7 or E_8 global symmetries one must dualise the seven 2-form potentials $A_{(2)i}$ to 0-forms χ^i in $D=4$, whilst in $D=3$, one must dualise the 28+8 1-form potentials $A_{\iota ij}$ and $\mathcal{A}^i_{(1)}$ to 0-forms χ^{ij} and χ_i in $D=3$. Thus in $D=4$ we must introduce seven extra generators J_i for the duals of the $A_{(2)i}$. They will have associated root vectors $-\vec{a}_i$ (remember that dualisation reverses the signs of the dilaton vectors), and sure enough, these are precisely the addition positive roots that can be constructed by taking non-negative-integer linear combinations of the simple roots $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$ in this case. In addition to the standard dimension-independent commutation relations (1.118)-(1.121), there will now be the further commutators involving J_i:

$$\begin{aligned}
&[\vec{H}, J_i] = -\vec{a}_i\, J_i\,, \qquad [E_i{}^j, J_j] = \delta_i^k\, J_j\,, \qquad [E^{ijk}, J_\ell] = 0\,,\\
&[E^{ijk}, E^{\ell mn}] = \epsilon^{ijk\ell mnp}\, J_p\,.
\end{aligned} \tag{1.136}$$

We then form a coset representative by exponentiation, appending an additional factor

$$\mathcal{V}_{\rm extra} = e^{\chi^i\, J_i} \tag{1.137}$$

to the right of the standard dimension-independent expression given in (1.122). One then finds, after extensive algebra, that the scalar Lagrangian for the four-dimensional reduction from $D=11$ can be written in the form (1.134) or (1.135), and that it has an E_7 global symmetry. The coset is $E_7/SU(8)$ in this case.

Finally, in $D=3$, one introduces extra generators J_{ij} and J^i for the axions χ^{ij} and χ_i coming from dualising $A_{(1)ij}$ and $\mathcal{A}^i_{(1)}$. In addition to the dimension-independent commutators (1.118)-(1.121), there will now in addition be

$$[\vec{H}, J_{ij}] = -\vec{a}_{ij}\, J_{ij}\ , \quad [\vec{H}, J^i] = -\vec{b}_i\, J^i\ , \quad [E_i{}^j, J_{k\ell}] = -2\delta^j_{[k}\, J_{\ell]i}\ , \quad [E_i{}^j, J_k] = -\delta^k_i\, J^j\ ,$$
$$[E^{ijk}, J_{\ell m}] = -6\delta^{[i}_{[\ell}\, \delta^{j}_{m]}\, J^{k]}\ , \qquad [E^{ijk}, J_\ell] = 0\ , \tag{1.138}$$
$$[E^{ijk}, E^{\ell mn}] = -\tfrac{1}{2}\epsilon^{ijk\ell mnpq}\, J_{pq}\ .$$

In this case, the coset representative $\mathcal{V}$ is constructed by appending

$$\mathcal{V}_{\rm extra} = e^{\chi_i\, J^i}\, e^{\frac{1}{2}\chi^{ij}\, J_{ij}} \tag{1.139}$$

to the right of the usual dimension-independent terms given in (1.122). The scalar Lagrangian can then be shown to be given by (1.134) or (1.135), and its global symmetry is E_8. The coset in this case is $E_8/SO(16)$.

To summarise this discussion of the scalar cosets coming from the toroidal reductions of eleven-dimensional supergravity, we may present a table listing the coset spaces in each dimension. The numerator group G, and the maximal compact denominator subgroup K, are listed in each case.

	G	K
$D=10$	$O(1,1)$	-
$D=9$	$GL(2,\mathbb{R})$	$O(2)$
$D=8$	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$
$D=7$	$SL(5,\mathbb{R})$	$SO(5)$
$D=6$	$O(5,5)$	$O(5)\times O(5)$
$D=5$	$E_{6(+6)}$	$USp(8)$
$D=4$	$E_{7(+7)}$	$SU(8)$
$D=3$	$E_{8(+8)}$	$SO(16)$

Table 1: Scalar cosets for maximal supergravities in D dimensions

1.7 Fermions

So far in our discussion of the maximal supergravities, we have said almost nothing about fermions. We shall not go into great detail about them, because it would become a major topic and there would not be time to cover it properly. Eleven-dimensional supergravity has a single Majorana spin-$\frac{3}{2}$ gravitino field $\hat{\psi}_M$, which appears both quadratically and quartically in the eleven-dimensional Lagrangian. For details, see [3].

1.7.1 Dimensional reduction of fermions

We have seen that for vectors and tensors, the essential idea in dimensional reduction is to split the higher-dimensional index M into two ranges, $M = (\mu, m)$, where μ denotes index values in the lower-dimensional theory and m denotes the remainder, namely the index values ranging over the internal circle or torus directions. For spinors, the decomposition goes a little differently. Suppose, for example, we want to reduce the gravitino from $D = 11$ to $D = 4$ on the 7-torus. For a moment, let's forget about the vector index on $\hat{\psi}_M$, and just think about a spin-$\frac{1}{2}$ fermion $\hat{\psi}$. In eleven dimensions spinors have 32 components, whereas in four dimensions they have 4 components. Thus to get the counting right in the reduction, we can expect that a single spin-$\frac{1}{2}$ fermion in $D = 11$ should give $32/4 = 8$ spin-$\frac{1}{2}$ fermions in $D = 4$. How does this work? The answer is that in the internal space, being seven dimensional, has fermions that are 8-component objects, and this supplies us with the factor of 8 we were looking for. Thus the way to decompose a spin-$\frac{1}{2}$ fermion is by means of a tensor product:

$$\hat{\psi}(x, y) = \sum_i \psi^{(i)}(x) \otimes \eta^{(i)}(y)\,, \tag{1.140}$$

where x denotes the coordinates of the lower-dimensional spacetime, and y denotes the coordinates on the internal space. In practice, in the truncation to the zero-mode (massless) sector that we will make, the only 8-component spinors $\eta^{(i)}(y)$ on T^7 that we would keep would be the constant ones (assuming we use the obvious Cartesian coordinate system on the torus). There are 8 of these (for example, the i'th could be taken to be the 8-component coulmn vector with zeros everywhere excpet for a "1" in the i'th row.) Thus in the zero-mode sector, we would end up with a sum over 8 terms in (1.140), giving 8 spin-$\frac{1}{2}$ fields $\psi^{(i)}$ in four dimensions.

To make a dimensional reduction of the gavitino $\hat{\psi}_M$ we just have to combine the method for spin-$\frac{1}{2}$ reduction described above with the familiar way we previously handled the reduction of vectors. Thus we shall have

$$\begin{aligned}
\hat{\psi}_\mu(x, y) &= \sum_i \psi_\mu^{(i)} \otimes \eta^{(i)}(y)\,, \\
\hat{\psi}_m(x, y) &= \sum_\alpha \chi^{(\alpha)}(x) \otimes \eta_m^{(\alpha)}(y)\,.
\end{aligned} \tag{1.141}$$

Notice how the vector index resides either on the lower-dimensional spinor, or the internal spinor, as appropriate. The quantities $\eta_m^{(\alpha)}(y)$ denote a set of spin-$\frac{3}{2}$ fermions in the internal space. Again, of course, in the truncation to the zero-mode sector we would end up keeping only those with constant components. There would be $8 \times 7 = 56$ of these. Thus the reduction of the gravitino from $D = 11$ to $D = 4$ gives 8 massless gravitini $\psi_\mu^{(i)}(x)$ and 56 massless spin-$\frac{1}{2}$ fermions $\chi^{(\alpha)}(x)$. This is exactly correct for $N = 8$ supergravity in four dimensions. The discussion in all other dimensions proceeds in an entirely analogous fashion.

Another thing one needs to know is how to decompose the Dirac matrices in the dimensional reduction. Clearly the dimensions of these matrices are the same as the dimensions of the spin-$\frac{1}{2}$ fermions in the various dimensions. This means that the Dirac matrices will also be decomposed as tensor-products of the lower-dimensional and internal ones. There are slightly different rules here depending on whether the higher, lower and internal dimensions are even or odd. Let us first state the rule for the case of $D = 11$ reduced to $D = 4$. The higher-dimensional Dirac matrices $\hat\Gamma_M$ will then be decomposed as

$$\hat\Gamma_\mu = \gamma_\mu \otimes 1\!\!1\,, \qquad \hat\Gamma_m = \gamma_5 \otimes \Gamma_m\,. \tag{1.142}$$

Here γ_μ are the 4×4 Dirac matrices of the four-dimensional spacetime, and Γ_m are the 8×8 Dirac matrices of the internal 7-space. The symbol $1\!\!1$ denotes the 8×8 unit matrix, and γ_5 is the usual chirality matrix of four dimensions, $\gamma_5 = \mathrm{i}\,\gamma_{0123}$. (We are thinking of M, μ and m here as being local-Lorentz, or tangent-space, indices. There are not really enough alphabets to go round, so we use the same labels as we sometimes use for coordinate indices.) Notice that the use of the γ_5 in the definition of $\hat\Gamma_m$ in (1.142) is crucial here. If we tried replacing it by $1\!\!1$, meaning the 4×4 unit matrix, we wouldn't get the correct Clifford algebra relations. In $D = 11$, $D = 4$ and the internal space, these are

$$\{\hat\Gamma_M, \hat\Gamma_N\} = 2\eta_{MN}\,, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,, \qquad \{\Gamma_m, \Gamma_n\} = 2\delta_{mn}\,. \tag{1.143}$$

We would fail to get $\{\hat\Gamma_\mu, \hat\Gamma_m\} = 0$ if we didn't use γ_5 in (1.142).

It should be clear that if we were reducing from $D = 11$ to an even dimension instead, we would play the same kind of game, but now the chirality operator would be on the internal side. Suppose we were reducing to $D = 5$, for example. We would then have

$$\hat\Gamma_\mu = \gamma_\mu \otimes \Gamma_7\,, \qquad \hat\Gamma_m = 1\!\!1 \otimes \Gamma_m\,, \tag{1.144}$$

where $\Gamma_7 = \mathrm{i}\,\Gamma_{123456}$ is the chirality matrix in the six-dimensional internal space.

For completeness, there are two further cases for reductions that might arise. Suppose we are starting from an *even* higher dimension. Then either the lower dimension and the internal dimension will both be even, or they will both be odd. In the former case, we are spoilt for choice, and we can use either of the schemes given in (1.142) and (1.144). In the end, the two will give equivalent results. In the latter case, when both the lower dimension and the internal dimension are odd, we seem at first sight to be stuck. The answer now is that we must introduce a third factor into the tensor product, which is a factor involving 2×2 matrices. This is perfectly understandable, if one looks at the counting. Suppose, for example, we were reducing from $D = 10$ to $D = 5$. In $D = 10$ spinors have 32 components, while in $D = 5$ and in the corresponding interal 5-space, the spinors all have 4 components. Since $4 \times 4 = 16$ we see there is a shortfall of a factor of 2, and so that is why we need the extra 2×2 matrix factors.

We can summarise the Dirac-matrix decompositions as follows:

$$
\begin{aligned}
\text{(even,odd)}: &\quad \hat\Gamma_\mu = \gamma_\mu \otimes \mathbb{1}\,, &&\hat\Gamma_m = \gamma \otimes \Gamma_m\,,\\
\text{(odd,even)}: &\quad \hat\Gamma_\mu = \gamma_\mu \otimes \Gamma\,, &&\hat\Gamma_m = \mathbb{1} \otimes \Gamma_m\,,\\
\text{(even,even)}: &\quad \hat\Gamma_\mu = \gamma_\mu \otimes \mathbb{1}\,, &&\hat\Gamma_m = \gamma \otimes \Gamma_m\,,\\
\text{or} &\quad \hat\Gamma_\mu = \gamma_\mu \otimes \Gamma\,, &&\hat\Gamma_m = \mathbb{1} \otimes \Gamma_m\,,\\
\text{(odd,odd)}: &\quad \hat\Gamma_\mu = \sigma_1 \otimes \gamma_\mu \otimes \mathbb{1}\,, &&\hat\Gamma_m = \sigma_2 \otimes \mathbb{1} \otimes \Gamma_m\,,
\end{aligned}
\tag{1.145}
$$

where γ denotes the chirality matrix in an (even) lower-dimensional spacetime, and Γ denotes the chirality matrix in an (even) internal space. The 2×2 matrices σ_1 and σ_2 are just two of the standard Pauli matrices.

1.7.2 Supersymmetry transformation rules

To do a complete job of discussing the fermions, and the supersymmetry transformation rules, would be a very complicated task. In practice, we can make a number of simplifications. First of all, it is customary to draw a veil over the higher-order terms–the quartic fermion terms–in the Lagrangian, and focus only on the quadratic terms. The quartic terms do matter, of course; the theory would not be supersymmetric without them, but they do make life enormously more complicated, and for many purposes one can suppress them, with the expectation that in a more careful and correct treatment they will work out properly too.

Accordingly, we shall suppress the higher-order fermion terms in what follows. In the eleven-dimensional theory itself, the supersymmetry transformation laws are then given by

$$
\begin{aligned}
\delta\hat e_M{}^A &= \mathrm{i}\,\bar{\hat\epsilon}\,\hat\Gamma^A\,\hat\psi_M\,,\\
\delta\hat A_{MNP} &= \tfrac32\bar{\hat\epsilon}\,\hat\Gamma_{[MN}\,\hat\psi_{P]}\,,
\end{aligned}
\tag{1.146}
$$

$$\delta\hat{\psi}_M = \widetilde{D}_M\,\hat{\epsilon} \equiv \hat{D}_M\,\hat{\epsilon} - \tfrac{1}{288}\,\hat{F}_{N_1\cdots N_4}\,\hat{\Gamma}_M{}^{N_1\cdots N_4}\,\hat{\epsilon} + \tfrac{1}{36}\hat{F}_{MN_1\cdots N_3}\,\hat{\Gamma}^{N_1\cdots N_4}\,\hat{\epsilon}\,,$$

where $\hat{\epsilon}$ is the local supersymmetry parameter. The derivative $\hat{D}_M$ is the fully Lorentz-covariant derivative, defined by

$$\hat{D}_M \equiv \partial_M + \tfrac{1}{4}\hat{\omega}_M^{AB}\,\hat{\Gamma}_{AB}\,\hat{\epsilon}\,, \qquad (1.147)$$

where $\hat{\omega}^{AB} = \hat{\omega}_M^{AB}\,dX^M$ is the spin connection. The "supercovariant derivative" $\widetilde{D}_M$ defined in (1.146) is a useful quantity because in terms of this the eleven-dimensional gravitino equation of motion takes the simple form

$$\hat{\Gamma}^{MNP}\,\widetilde{D}_N\,\hat{\psi}_N = 0\,. \qquad (1.148)$$

All the above formulae can be obtained from the complete expressions given in [3].

It should be evident that it is now just a mechanical exercise to implement all the dimensional reduction procedures for the bosonic and fermionic fields that we have discussed previously, in order to derive the equations of motion and supersymmetry transformation rules in all the toroidally-reduced theories. Of course saying that it is a mechanical exercise does not at all mean that it is a simple process! But there are no particularly difficult conceptual issues involved in implementing the reductions. Technically, one of the most complicated points is concerned with precisely how to make field redefinitions so that the fermions one ends up with the lower dimension all have canonical kinetic terms, and to ensure that they are defined so as to be nicely diagonalised with respect to their kinetic terms.

In practice, we are often interested in looking at *solutions* of the supergravity equations, and 99 times out of 100 these solutions will themselves be purely bosonic. Consequently, if we want to look at supersymmetry variations of the solutions, we will commonly only need to worry about the transformation law that gives how the bosons vary into the fermions. This would be the case, for example, if we wanted to test whether a given bosonic solution preserved any of the supersymmetries. Thus we are typically only interested in the last of the three transformation rules given in (1.146), for $\delta\hat{\psi}_M$, and its toroidal dimensional reduction.

1.8 General remarks about coset Lagrangians

As we have already remarked, the scalar cosets that we encountered in the toroidal compactifications of eleven-dimensional supergravity are somewhat special, in the sense that the numerator groups (i.e. the global symmetry groups themselves) are all maximally non-compact. In addition, our way of parameterising the cosets involved making a specific "gauge choice," which in our case was achieved by choosing the coset representative $\mathcal{V}$ to be in the Borel gauge. One can perfectly well, in principle, make some other gauge choice. Alternatively, one is not obliged to make any choice of gauge at all. One could simply exponentiate the entire Lie algebra of the global symmetry group G. This would give too many fields, of course, since the dimension of the coset G/K is $\dim(G) - \dim(K)$, and so there should be this number of scalar fields, rather than the $\dim(G)$ fields that one would get if no gauge choice were made. The resolution is a simple one, and it is essentially something that we have already seen: two points $\mathcal{V}_1$ and $\mathcal{V}_2$ on the coset manifold G/K that are related by left-multiplication by an element of K, i.e. $\mathcal{V}_1 = \mathcal{O}\,\mathcal{V}_2$, are actually the same point. Thus if one constructs $\mathcal{V}$ by exponentiating the entire algebra, then there will be local "gauge" symmetries associated with the entire group K that remove the surplus degrees of freedom. Our way of constructing the scalar cosets in the supergravity theories exploited the fact that in those cases it was actually very simple to use these local gauge symmetries explicitly, to fix a gauge in which the redundant fields were simply set to zero.

We shall not delve here into the details of how one handles the construction of coset Lagrangians in general, for example in cases where the local K invariance is left unfixed. We shall, however, make some general remarks about how to handle a wider class of cosets in the gauge-fixed formalism, namely in those cases where the numerator group G is not maximally non-compact. To illustrate the point, let us consider the family of examples of cosets

$$M_{p,q} = \frac{O(p,q)}{O(p)\times O(q)}\ , \tag{1.149}$$

where $O(p,q)$ is the group of pseudo-orthogonal matrices that leaves invariant the indefinite-signature diagonal matrix $\eta = \mathrm{diag}\,(1,1,\ldots,1,-1,-1,\ldots,-1)$, where there are p plus signs and q minus signs. Thus $O(p,q)$ matrices Λ satisfy

$$\Lambda^T\,\eta\,\Lambda = \eta\ . \tag{1.150}$$

For a given value of $n = p+q$, the algebras $O(p,q)$ are all just different forms of the same underlying algebra, which would be $D_{n/2}$ in the Dynkin classification if n were even, and $B_{(n-1)/2}$ if n were odd. However, the partition into compact and non-compact generators is different for different partitions of $n = p+q$. In fact, the denominator groups $O(p)\times O(q)$ are the maximal compact subgroups in each case, telling us that of the total of $\frac{1}{2}(p+q)(p+q-1)$ generators of $O(p,q)$, there are $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)$ compact generators, with the rest being non-compact. Evidently, then, the dimensions of the cosets are different depending on the partition of $n = p + q$; simple subtraction gives us

$$\dim (M_{p,q}) = \tfrac{1}{2}(p+q)(p+q-1) - \tfrac{1}{2}p(p-1) - \tfrac{1}{2}q(q-1) = p\,q\ . \tag{1.151}$$

When $n = p+q$ is even, the rank of $O(p,q)$ is $\frac{1}{2}n$, and ones finds that the dimension $p\,q$ of the coset space is equal to the dimension of the Borel subalgebra, which is $\frac{1}{2}n + \frac{1}{2}(\frac{1}{2}n(n-1) - n/2) = \frac{1}{4}n^2$, only if $p = q$. Thus when $n = p+q$ is even, only the cosets of the form $O(p,p)/(O(p)\times O(p))$ are maximally non-compact. (We encountered such a coset in $D = 6$, where the scalar Lagrangian was $O(5,5)/(O(5)\times O(5))$.) A similar analysis for the case $n = p+q$ odd shows that only the case $O(p,p+1)/(O(p)\times O(p+1))$ (or, equivalently, $O(p+1,p)/(O(p+1)\times O(p)))$ is maximally non-compact. These are the cases where, for a given n, the dimension of $M_{p,q}$ is largest.

Clearly, if we consider a coset of the form (1.149) that is not maximally non-compact, then if we are to construct a coset representative $\mathcal{V}$ in a gauge-fixed form, we must exponentiate only an appropriate subset of the Borel generators of $O(p,q)$. The general theory of how to do this was worked out by Alekseevski, in the 1970's. It again makes use of the Iwasawa decomposition, but this is now a little more complicated when the group G is not maximally non-compact. The decomposition asserts that there is a unique factorisation of a group element g as

$$g = g_K\, g_A\, g_N\ , \tag{1.152}$$

where g_K is in the maximal compact subgroup K of G and g_A is in the maximal non-compact Abelian subgroup of G. The factor g_N is in a nilpotent subgroup of G, which is defined as follows. It is generated by that subset of the positive-root generators that are strictly positive with respect to the maximal non-compact Abelian subalgebra (whose exponentiation gives g_A).

Now, if the group G were maximally non-compact, then *all* the Cartan subalgebra generators would be non-compact, and hence *all* the positive-root generators would be included in the nilpotent subalgebra. We would then be back to the previous statement of the Iwasawa decomposition for maximally non-compact groups, where we exponentiated the entire Borel subalgebra.

Here, however, we are by contrast considering a case where only a subset of the Cartan generators are non-compact. Accordingly, only a subset of the positive-root generators pass the test of having strictly positive weights with respect to this subset of the Cartan generators. In this more general situation, the subalgebra of the Borel algebra, comprising the non-compact Cartan generators A and the positive-root generators N that have positive weights under A, is known as the *Solvable Lie Algebra* of the group G. A lot of work has been done on this topic recently; see, for example, [4, 5].

We can now build a coset representative $\mathcal{V}$ by exponentiating the non-compact Cartan generators, and the nilpotent subalgebra generators. By the modified Iwasawa decomposition (1.152), it follows that a global transformation consisting of a right-multiplication by an element of G can be compensated by a local field-dependent left-multiplication by an appropriate element of the maximal compact subgroup, thereby giving a $\mathcal{V}'$ in the same "nilpotent" gauge, corresponding to a G-transformed point in the coset G/K. Thus we again have a procedure for constructing the scalar Lagrangian for the coset, in this more general situation where G is not maximally non-compact.

Let us close this discussion with an illustrative example. There is string theory in $D = 10$, known as the heterotic string, whose low-energy effective Lagrangian is different from the ten-dimensional theory that comes by S^1 reduction from eleven-dimensional supergravity. For our present purposes, it suffices to say that the Lagrangian in $D = 10$ can be taken to have the general form

$$\mathcal{L}_{10} = R\,{*1} - \tfrac{1}{2}{*d\phi_1} \wedge d\phi_1 - \tfrac{1}{2}e^{\phi_1}\,{*F_{(3)}} \wedge F_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\phi_1} \sum_{I=1}^{N} {*G^I_{(2)}} \wedge G^I_{(2)}\,, \tag{1.153}$$

where $G^i_{(2)} = dB^i_{(1)}$ are a set of N 2-form field strengths, and

$$F_{(3)} = dA_{(2)} + \tfrac{1}{2}B^I_{(1)} \wedge dB^I_{(1)}\,. \tag{1.154}$$

(Actually, in the heterotic string itself $N = 16$, and the 16 gauge fields $B^I_{(1)}$ are just in the $U(1)^{16}$ Cartan subgroup of a 496-dimensional Yang-Mills group, which can be $E_8 \times E_8$ or $SO(32)$. But for our purposes it suffices to consider the Abelian subgroup fields, and also we can generalise the discussion by allowing N to be arbitrary.)

Clearly there is a global $O(N)$ symmetry in $D = 10$, under which the N gauge fields are rotated amongst each other. If one performs a Kaluza-Klein dimensional reduction of the theory on T^n, then it turns out that the resulting theory in $D = 10 - n$ has an $O(n, n+N)$ global symmetry, and that the scalar manifold is the coset

$$\frac{O(n, n+N)}{O(n) \times O(n+N)}\,. \tag{1.155}$$

These cosets are of precisely the type that we discussed above, which can be parameterised by means of an exponentiation of their solvable Lie algebras. To keep things simple, let us consider the case $n = 1$. Thus we shall reduce (1.153) on a circle, and show that the scalar sector in $D=9$ has an $O(1,N+1)/O(N+1)$ coset structure. (Actually, there will be another $\mathbb{R}$ factor too, associated with an extra scalar that decouples from the rest.)

Let us denote the dilaton of the $d=10$ to $D=9$ reduction by ϕ_2. After performing the reduction, using the standard rules that we established previously, we find, after making a convenient rotation of the dilatons, that the nine-dimensional Lagrangian is

$$\begin{aligned}\mathcal{L}_9 &= R{*1} - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 {*d\varphi}\wedge d\varphi - \tfrac12 e^{\sqrt2\,\varphi}\sum_I {*dB^I_{(0)}}\wedge dB^I_{(0)} - \tfrac12 e^{-\sqrt{\frac87}\,\phi}\,{*F_{(3)}}\wedge F_{(3)}\\ &\quad -\tfrac12 e^{-\sqrt{\frac27}\,\phi}\Big(e^{\sqrt2\,\varphi}\,{*F_{(2)}}\wedge F_{(2)} + e^{-\sqrt2\,\varphi}\,{*\mathcal{F}_{(2)}}\wedge \mathcal{F}_{(2)} + \sum_I {*G^I_{(2)}}\wedge G^I_{(2)}\Big)\,, \qquad (1.156)\end{aligned}$$

where $\mathcal{F}_{(2)}$ is the Kaluza-Klein gauge field, and $F_{(2)}$ and $G^i_{(1)} = dB^I_{(0)}$ are the dimensional reductions of $F_{(3)}$ and $G^i_{(2)}$ respectively. The various field strengths are given in terms of potentials by

$$\begin{aligned}&F_{(3)} = dA_{(2)} + \tfrac12 B^I_{(1)}\, dB^I_{(1)} - \tfrac12 \mathcal{A}_{(1)}\, dA_{(1)} - \tfrac12 A_{(1)}\, d\mathcal{A}_{(1)}\ ,\\ &\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}\ , \qquad G^I_{(2)} = m dB^I_{(1)} + dB^I_{(0)}\,\mathcal{A}_{(1)}\ , \qquad (1.157)\\ &F_{(2)} = dA_{(1)} + B^I_{(0)}\, dB^I_{(1)} + \tfrac12 B^I_{(0)}\, B^I_{(0)}\, d\mathcal{A}_{(1)}\ .\end{aligned}$$

(A field redefinition has been made here, to move the derivative off the axionic scalars $B^I_{(0)}$; this is analogous to the one we did in the nine-dimensional theory coming from the T^2 reduction of eleven-dimensional supergravity.) Note that we are omitting the wedge symbols here, to avoid some clumsiness in the appearance of the equations.

Let us just focus on the scalar part of the Lagrangian, namely

$$\mathcal{L} = -\tfrac12 {*d\phi}\wedge d\phi - \tfrac12 {*d\varphi}\wedge d\varphi - \tfrac12 e^{\sqrt2\,\varphi}\sum_I {*dB^I_{(0)}}\wedge dB^I_{(0)}\ . \qquad (1.158)$$

We may first observe that the dilaton ϕ is decoupled from the rest of the scalar Lagrangian; it just contributes a global $\mathbb{R}$ symmetry of constant shift transformations $\phi \longrightarrow \phi + c$. We shall ignore ϕ from now on. The rest of the scalar manifold can be described as follows. First, introduce a Cartan generator H, and positive-root generators E_I, with the commutation relations

$$[H,H]=0\ , \qquad [H,E_I] = \sqrt2\, E_I\ , \qquad [E_I,E_J]=0\ . \qquad (1.159)$$

We define the coset representative $\mathcal{V}$ as

$$\mathcal{V} = e^{\frac12\varphi\, H}\, e^{B^I_{(0)}\, E_I}\ . \qquad (1.160)$$

It is easily seen that

$$d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac12 d\varphi\, H + dB^I_{(0)}\, E_I\ . \qquad (1.161)$$

Now, we wish to argue that H and E_I generate a subalgebra of $O(1,N+1)$. In, fact, we want to argue that they generate the solvable Lie algebra of $O(1,N+1)$. The orthogonal algebras $O(p,q)$ divide into two cases, namely the D_n series when $p+q=2n$, and the B_n series when $p+q=2n+1$. The positive roots are given in terms of an orthonormal basis e_i as follows:

$$\begin{aligned} D_n: &\quad e_i \pm e_j\ , \qquad i<j\le n\ , \\ B_n: &\quad e_i \pm e_j\ , \qquad i<j\le n\ , \qquad \text{and} \qquad e_i\ , \end{aligned} \tag{1.162}$$

where $e_i \cdot e_j = \delta_{ij}$. It is sometimes convenient to take e_i to be the n-component vector $e_i = (0,0,\ldots,0,1,0,\ldots,0)$, where the "1" component occurs at the i'th position. The Cartan subalgebra generators, specified in a basis-independent fashion, are h_{e_i}, which satisfy $[h_{e_i}, E_{e_j\pm e_k}] = (\delta_{ij}\pm\delta_{ik})\, E_{e_j\pm e_k}$, *etc.* Of these, $\min(p,q)$ are non-compact, with the remainder being compact. It is convenient to take the non-compact ones to be h_{e_i} with $1\le i \le \min(p,q)$.

Returning now to our algebra (1.159), we find that the generators H and E_I can be expressed in terms of the $O(1,N+1)$ basis as follows:

$$\begin{aligned} H &= \sqrt{2}\, h_{e_1}\ , \\ E_{2k-1} &= E_{e_1-e_{2k}}\ , \qquad E_{2k} = E_{e_1+e_{2k}} \qquad 1\le k\le [\tfrac12+\tfrac14 N]\ , \\ E_{1+\frac12 N} &= E_{e_1}\ , \qquad \text{if } N \text{ is even}\ . \end{aligned} \tag{1.163}$$

It is easily seen that h_{e_1} and $E_{e_1\pm e_i}$, together with E_{e_1} in the case of N even, are precisely the generators of the solvable Lie algebra of $O(1,N+1)$. In other words, h_{e_1} is the non-compact Cartan generator of $O(1,N+1)$, while the other generators in (1.163) are precisely the subset of positive-root $O(1,N+1)$ generators that have strictly positive weights under h_{e_1}. Thus it follows from the general discussion at the beginning of this section that the scalar Lagrangian for the $D=9$ theory is described by the coset[4] $(O(1,N+1)/O(N+1))\times \mathbb{R}$. (Recall that there is the additional decoupled scalar field ϕ with an $\mathbb{R}$ shift symmetry.)

[4]It should be emphasised that the mere fact that one can embed the algebra (1.159) into the Lie algebra of a larger Lie group G does not, of itself, mean that the group G acts effectively on the scalar manifold. Only when (1.159) is the solvable Lie algebra of the group G can one deduce that G has an effective group action on the scalar manifold.

2 Kaluza-Klein Reduction on Spheres and Other Compact Manifolds

2.1 Introduction

Up to this point, we have considered Kaluza-Klein reductions on the circle S^1, and on the n-torus T^n, which can be viewed as a sequence of S^1 reductions. As discussed in the previous chapter, the reductions in these cases, with the associated setting to zero of all the massive Kaluza-Klein modes, are guaranteed to be consistent, since we are retaining all the singlets under the $U(1)^n$ isometry group of the n-torus, and setting all the non-singlets to zero. This guarantees consistency, since products of singlets under a group action can never generate non-singlets. Thus we can be assured that no matter how non-linear the higher-dimensional theory, the Kaluza-Klein reduction will be a consistent one.

One can extend the idea of Kaluza-Klein reduction in a number of ways. One possibility is to perform a reduction on an internal space that is a group manifold G. An example would be $G = SU(2)$, which is actually isomorphic to the 3-sphere. Higher examples, like $G = SU(3)$, typically don't have any isomorphisms to other more "well-known" manifolds. The group manifold G admits a metric that has $G \times G$ as its isometry group (assuming that G is non-abelian), since it admits a transitive action of the group G by left multiplication, and independently by right multiplication. Thus if U denotes an element of G, i.e. a point in the group manifold, then we can act with constant elements A and B of the group G to give

$$U \longrightarrow U' = A\, U\, B\,, \tag{2.1}$$

which leave invariant the so-called bi-invariant metric

$$ds^2 = \mathrm{tr}(dU\, U^{-1})^2\,. \tag{2.2}$$

The group manifold G is homogeneous, since the left (or the right) action of G is transitive.

Another type of internal manifold that one might consider is a coset space, G/H. An example is the n-dimensional sphere S^n, which is the coset space $SO(n+1)/SO(n)$. Another example would be the complex projective space CP^n, which is the coset $SU(n+1)/U(n)$. This is a complex manifold, with complex dimension n (meaning that is has real dimension $2n$). In a coset space points in the group manifold G are identified under the action of the subgroup H. Thus we view two points U_1 and U_2 as being equivalent if there exists an element h in H such that

$$U_1 = U_2\, h\,. \tag{2.3}$$

One can see that now we shall have transitively-acting isometries given by the left-action of G on the coset, but we no longer have isometries corresponding to the right action of G. Thus the coset space G/H can be equipped with a metric that is invariant under G. Since the isometries act transitively, this means that the coset space is homogeneous.

Another possibility for an internal space would be compact a manifold that is not a coset space. Its metric may still have isometries, but these will no longer act transitively, and so the space will be inhomogeneous. Finally, of course, one may consider a space for which the metric has no isometries at all.

What do we expect to get out of a Kaluza-Klein reduction on some general compact manifold M? In particular, let us suppose that M has an isometry group G. We need not yet concern ourselves with the question of whether M is a group manifold, a coset space or an inhomogeneous space. One can show, by carrying out a linearised analysis of small fluctuations around a background of the form $\mathcal{N} \times M$, where $\mathcal{N}$ denotes the lower-dimensional spacetime manifold, that the massless fields in the lower dimensional spacetime will certainly include the Yang-Mills gauge bosons of the group G, and, of course, the lower-dimensional metric. There may also be further massless fields, such as scalars. The whole issue of identifying what is massless requires a lot of care now, since the spacetime $\mathcal{N}$ may well not be Minkowski spacetime. For example, in the case of sphere reductions in supergravities, one commonly finds that there is a "vacuum solution" which is a product of anti-de Sitter spacetime and a sphere. In such a case, the notion of mass has to be defined with respect to the anti-de Sitter background, and this is quite an involved business. However, for gravity itself, and for gauge fields, we have a rather clear picture of what it means to be massless, since for these fields we have the guide of gauge invariance. So we can proceed for now without getting too involved in the definition of mass, at least for a discussion of the Yang-Mills gauge bosons.

Having noted that one will always find the Yang-Mills gauge bosons of the group G of isometries of the internal manifold, it is evident why one might in principle like to use a coset space G/H rather than a group manifold G for the Kaluza-Klein reduction. The coset space would be much more "economical," in the sense that the number of extra dimensions needed in order to obtain a given gauge group would be less. For example, to get the gauge bosons of $SO(8)$ one could use the group manifold $SO8)$ itself, which would require 28 extra dimensions. But by using the coset $SO(8)/SO(7)$, which is the seven-sphere, one would need only 7 extra dimensions.

In addition to the massless modes, one will also of course obtain infinite towers of Kaluza-

Klein massive modes, in much the same way as one does in a circle or torus reduction. In other words, at the linearised level we can imagine expanding all the higher-dimensional fields in terms of complete sets of eigenfunctions on the internal space. For example, the lower-dimensional components of the higher-dimensional metric would be expanded as

$$\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu}(x) + \sum_{i=0}^{\infty} h_{\mu\nu}^{(i)}(x)\, P^{(i)}(y)\,, \tag{2.4}$$

where $\bar{g}_{\mu\nu}$ denotes the "ground-state" lower-dimensional metric around which the expansion is being performed, $h_{\mu\nu}^{(i)}(x)$ denotes the fluctuations, and $P^{(i)}(y)$ denotes the eigenfunctions of the scalar Laplacian on the internal space, starting with the constant zero-eigenvalue function, $P^{(0)} = 1$. Similarly, the mixed components of the higher-dimensional metric would be expanded in terms of a complete set of vector eigenfunctions on the internal space:

$$\hat{g}_{\mu m} = \sum_{i=0}^{\infty} A_{\mu}^{(i)}(x)\, P_m^{(i)}(y)\,. \tag{2.5}$$

The zero-mode eigenfunctions $P_m^{(0)}$ here will be the Killing vectors K_m on the internal space. In an analogous fashion, all the other components of the higher-dimensional fields can be expanded in terms of complete sets of eigenfunctions on the internal space.

Although we have discussed a linearised analysis here, there is no reason in principle why we shouldn't apply the idea to the full theory, by just substituting all the expansions into the higher-dimensional equations of motion, or, even, the higher-dimensional Lagrangian. As long as we continue to keep all the infinite Kaluza-Klein towers nothing can possibly go wrong. After all, effectively what we would be doing is just performing a generalised Fourier expansion of the higher-dimensional theory. The general formalism for performing coset-space Kaluza-Klein reduction was elegantly described in a paper by Salam and Strathdee [6].[5]

Usually, however, in Kaluza-Klein reductions we would like to do something more, namely to set all the massive fields to zero. Unless we do this, we *are* really just describing the higher-dimensional theory in a rather clumsy way, in terms of infinite sums over generalised Fourier modes. And it is at this point that we will typically run into trouble. Nothing can go wrong if we restrict attention to the linearised level, but if we try to set the massive modes to zero and keep only the massless modes, the attempt will in fact almost always fail, if we go to the full non-linear theory. We should not give up, however,

[5] Of course if one kept all the massive and massless modes in the full theory with all its non-linearities, the result would be a dog's breakfast, and would certainly look anything but elegant!

because it turns out that the very few exceptions where it works are precisely the cases of greatest interest in string theory and M-theory!

The reason why reduction combined with truncation to the massless sector usually runs into problems for a general internal manifold is the following. Imagine first keeping all the massive fields too, so that we have a gigantically complicated, but perfectly consistent, reduction. The resulting lower-dimensional theory will involve all kinds of complicated interactions between the various fields. In particular, it will typically involve cubic interaction terms in the Lagrangian of the form $H\,L^2$, where H represents a heavy field that we want to set to zero, and L represents a light (i.e. massless) field that we want to keep. But this means that the field equation for the heavy field will be of the form

$$\Box H + m^2\,H = L^2\,, \tag{2.6}$$

where m is the mass of H. Clearly, then, it would be inconsistent to set $H=0$, since this would then force the light field L to vanish too.

The reason why such dangerous interactions are present is because in a reduction on some general internal manifold M such as a coset space, the product of zero-mode eigenfunctions on M will generate non-zero-mode eigenfunctions. Recall that this could not happen on the circle or torus, since the zero-modes were all independent of the torus coordinates, while the non-zero-modes were coordinate-dependent. Or, put more elegantly, the zero-mode eigenfunctions were singlets under the $U(1)^n$ isometry group of the n-torus, while the non-zero-mode eigenfunctions were all charged (like $e^{\mathrm{i}\,n\,y}$).

In fact by studying precisely the cubic interactions of the form $H\,L^2$, we can produce a rather simple explicit demonstration of why the Kaluza-Klein reduction combined with truncation to the massless sector will normally fail, for some generic internal manifold M such as a coset space.[6]

Our strategy will be the following. First, we shall determine what the Kaluza-Klein metric reduction ansatz giving the gauge bosons would have to be, if a reduction were possible. Having established this, we shall then show that in general the attempt to make such a reduction will fail, once we look beyond the linearised level.[7] Having seen why it fails

[6]Salam and Strathdee never made a truncation to the massless sector in their paper [6], and so their analysis was completely valid. Much confusion resulted later when others made the false assumption that one could make the truncation in general. On the other hand, it was partly *because* of overlooking this point that people stumbled upon the exceptional cases that do work.

[7]To be precise, what in general fails is the attempt to keep all the Yang-Mills gauge bosons of the isometry group of the internal manifold M, while setting the massive Kaluza-Klein fields to zero.

in general, we shall then be in a position to look for exceptional cases where a consistent reduction is in fact possible. These exceptional cases in fact depend on first of all having a very special starting point for the higher-dimensional theory, and then choosing a very particular internal space M.

To discuss the gauge bosons, it is convenient to suppress for now the scalar sector of the reduction. This, of course, is potentially a recipe for trouble; we already saw in the previous chapter that even the Kaluza-Klein reduction on S^1 will be inconsistent if the scalar dilaton field is omitted. However, the inconsistencies resulting from neglecting scalars occur in rather easily-identifiable sectors, and provided we proceed with appropriate caution, we can still learn many useful things about the structure of the Kaluza-Klein reduction, and why, in general, it will fail.

2.2 The Yang-Mills gauge bosons

We saw in the previous chapter that the $U(1)$ gauge invariance of the Kaluza-Klein vector coming from the S^1 reduction of the metric tensor had its origin in a specific subset of general coordinate transformations. Namely, it came from making a transformation of the coordinate on the circle, of the form $\delta z = -\lambda(x)$. For a Kaluza-Klein reduction on a manifold M with isometry group G, we can similarly write down the general structure for the metric reduction ansatz, and then we can see how the gauge transformations of the gauge bosons emerge from certain general coordinate transformations.

Proceeding, as discussed above, by suppressing scalar fields, the metric reduction ansatz will be

$$d\hat{s}^2 = ds^2 + g_{mn}\,(dy^m + K^{mI}\,A^I_{(1)})(dy^n + K^{nJ}\,A^J_{(1)})\,, \tag{2.7}$$

where K^{mI} are the Killing vectors of the metric g_{mn} on M, with I being the adjoint index for the isometry group G. The coordinates $\hat{x}^M$ of the higher-dimensional theory are split as $\hat{x}^M = (x^\mu, y^m)$. From (2.7) we can read off the components $\hat{g}_{MN}$ of the higher-dimensional metric, giving

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + K^{mI}\,K_m{}^J\,A^I_\mu\,A^J_\nu\,,\qquad \hat{g}_{\mu m} = K_m{}^I\,A^I_\mu\,,\qquad \hat{g}_{mn} = g_{mn}\,. \tag{2.8}$$

We emphasise that here g_{mn} is the undistorted metric on the internal manifold M (and thus it depends on y^m, but not on x^μ). The Killing vectors K^{mI} depend only on the y^m also, and $K_m{}^I \equiv g_{mn}\,K^{nI}$. The gauge bosons A^I_μ depend, of course, only on the lower-dimensional coordinates x^μ, as does the metric $g_{\mu\nu}$.

To see why (2.7) is the appropriate ansatz, let us first study the gauge transformations. These correspond to making general coordinate transformations $\delta\hat{x}^M = -\hat{\xi}^M$ of the following type:

$$\hat{\xi}^\mu = 0\,, \qquad \hat{\xi}^m = K^{mI}\,\lambda^I(x)\,. \tag{2.9}$$

We can now proceed as in section 1.2, where we derived the gauge transformation for the $U(1)$ gauge potential in the S^1 reduction. Looking first at the internal components of the metric, we get

$$\begin{aligned}
\delta\hat{g}_{mn} &= \hat{\xi}^p\,\partial_p\,\hat{g}_{mn} + \hat{g}^{pn}\,\partial_m\,\hat{\xi}^p + \hat{g}_{mp}\,\partial_n\,\hat{\xi}^p\,,\\
&= \lambda^I\,K^{pI}\,\partial_p\,g_{mn} + g_{pn}\,\partial_m\,K^{pI}\,\lambda^I + g_{mp}\,\partial_n\,K^{pI}\,\lambda^I\,,\\
&= \lambda^I\,\mathcal{L}_{K^I}(g)_{mn} = 0\,.
\end{aligned} \tag{2.10}$$

To reach the final line, we recognised that the three terms in the second line assemble into the Lie derivative of the metric with respect to K^I, and then we finally used the fact that since K^I is a Killing vector, by definition we will get zero when we use it to take the Lie derivative of the metric. Getting zero is reasonable, since the internal components of $\hat{g}_{MN}$ are just g_{mn}, which is unchanged under variation of the lower-dimensional fields that we have included in the ansatz.

Next, we look at the variation of the mixed components $\hat{g}_{\mu m}$ of the higher-dimensional metric. For these we shall have

$$\begin{aligned}
\delta\hat{g}_{\mu m} &= \hat{\xi}^p\,\partial_p\,\hat{g}_{\mu m} + \hat{g}_{pm}\,\partial_\mu\,\hat{\xi}^p + \hat{g}_{\mu p}\,\partial_m\,\hat{\xi}^p\,,\\
&= K^{pI}\,\lambda^I\,\partial_p\,K_m{}^J\,A_\mu^J + g_{pm}\,\partial_\mu\,\lambda^I\,K^{pI} + K_p{}^J\,A_\mu^J\,\partial_m\,K^{pI}\,\lambda^I\,,\\
&= \mathcal{L}^{K^I}(K^J)_m\,\lambda^I\,A_\mu^J + K_m{}^I\,\partial_\mu\,\lambda^I\,,\\
&= K_m^I\,(\partial_\mu\,\lambda^I + f^{JK}{}_I\,\lambda^J\,A_\mu^K)\,.
\end{aligned} \tag{2.11}$$

Again, we recognised that two of the terms in the second line assemble to make the Lie derivative. Then, we used the fact that the Killing vectors satisfy the Lie algebra of the isometry group G, with structure constants $f^{JK}{}_I$:

$$[K^I, K^J] = f^{IJ}{}_K\,K^K\,, \tag{2.12}$$

and $[K^I, K^J]^m = \mathcal{L}_{K^I}(K^J)^m$. On the other hand, we see from (2.8) that if the lower-dimensional fields are varied we shall have

$$\delta\hat{g}_{\mu m} = K_m{}^I\,\delta A_\mu^I\,. \tag{2.13}$$

Comparing with (2.11), we learn that

$$\delta A_\mu = \partial_\mu \lambda^I + f^{JK}{}_I \, \lambda^J \, A_\mu^K \,, \tag{2.14}$$

which is precisely the correct result for infinitesimal Yang-Mills gauge transformations. Finally, we learn nothing new by considering the variation of $\hat{g}_{\mu\nu}$ under the coordinate transformation (2.9).

By showing that we obtain the correct Yang-Mills gauge transformations for A_μ^I from the general coordinate transformations (2.9), we can be sure that the metric reduction ansatz (2.7) is the right one.[8] It is instructive now to calculate the curvature for higher-dimensional metric. To do this, we first note that the following is a convenient choice for a vielbein for (2.7):

$$\hat{e}^\mu = e^\mu \,, \qquad \hat{e}^a = e^a + K^{aI} \, A^I_{(1)} \,, \tag{2.15}$$

where e^μ is a vielbein for the lower-dimensional metric ds^2, and e^a is a vielbein for the undistorted metric g_{mn} on the internal manifold M, so $g_{mn} = e^a_m \, e^a_n$. Of course K^{aI} just means $e^a_m \, K^{mI}$. It is now a straightforward, if laborious, task to calculate the spin connection and curvature. The spin connection turns out to be

$$\begin{aligned}
\hat{\omega}_\mu &= \omega_{\mu\nu} - \tfrac{1}{2} K^{aI} \, F^I_{\mu\nu} \, \hat{e}^a \,, \\
\hat{\omega}_{\mu a} &= -\tfrac{1}{2} K^{aI} \, F^I_{\mu\nu} \, \hat{e}^\nu \,, \\
\hat{\omega}_{ab} &= \omega_{ab} + \nabla_a K^I_b \, A^I_\mu \, \hat{e}^\mu \,,
\end{aligned} \tag{2.16}$$

where all components here refer to vielbein indices. Note that $\omega_{\mu\nu}$ is the spin connection for the lower-dimensional vielbein e^μ, and ω_{ab} is the spin connection for the vielbein e^a on the undistorted internal manifold M.

We shall not present the full expressions for the curvature 2-forms here, since they are a little complicated. After reading off the Riemann tensor, and then contracting to get the Ricci tensor, one finds that the vielbein components are given by

$$\begin{aligned}
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2} K^{aI} \, K_a{}^J \, F^I_{\mu\rho} \, F^J_\nu{}^\rho \,, \\
\hat{R}_{\mu a} &= \tfrac{1}{2} K^I_a \, D_\nu \, F^{I\,\nu}_\mu \,, \\
\hat{R}_{ab} &= R_{ab} + \tfrac{1}{4} K^I_a \, K^J_b \, F^I_{\mu\nu} \, F^{J\,\mu\nu} \,,
\end{aligned} \tag{2.17}$$

where D_ν is the Yang-Mills covariant derivative. The Ricci tensors $R_{\mu\nu}$ and R_{ab} are those for the lower-dimensional metric $g_{\mu\nu}$ and the undistorted internal metric g_{mn} respectively.

[8]Again, with the caveat that we have suppressed the scalar fields that should possibly be included here!

Now, let us consider what happens if we try using this proposed Kaluza-Klein ansatz to reduce a generic higher-dimensional theory. We shall take pure Einstein gravity as our example of a generic theory, so the higher-dimensional equations of motion are simply $\hat R_{AB} = 0$. Looking at (2.17), the middle equation is very nice, because it gives us the lower-dimensional Yang-Mills equations,

$$D_\nu\, F^{I\,\nu}_\mu = 0\,, \tag{2.18}$$

as we would have hoped. The last equation in (2.17) is a bit of a disaster, since it gives

$$R_{ab} + \tfrac14 K^I_a\, K^J_b\, F^I_{\mu\nu}\, F^{J\,\mu\nu} = 0\,. \tag{2.19}$$

But we should not be too alarmed by this; it is exactly the kind of problem that we should have been expecting from the moment we decided to omit scalar fields from our ansatz. It is exactly analogous to the trouble one would encounter in the $\hat R_{55}$ component of the Ricci-flat condition in the S^1 reduction that we discussed in the previous chapter, had we neglected to include the dilatonic scalar ϕ. The point is that in the present case we are about to encounter a quite different kind of inconsistency, which would not be resolved by including the scalars. Since the *Titanic* is sinking anyway, we need not concern ourselves too much with trying to rearrange the deckchairs nicely!

The new inconsistency occurs in the sector where we would have hoped to obtain the lower-dimensional Einstein equation, with the Yang-Mills fields acting as a source. This would come from the combination $\hat R_{\mu\nu} - \frac12 \hat R\, \eta_{\mu\nu} = 0$ (recall that we are using vielbein components here, hence the $\eta_{\mu\nu}$!), and so from (2.17) we see that this gives

$$R_{\mu\nu} - \tfrac12 R\, \eta_{\mu\nu} = \tfrac12 K^{aI}\, K^J_a\, [F^I_{\mu\rho}\, F^{J\,\rho}_\nu - \tfrac14 F^I_{\rho\sigma}\, F^{J\,\rho\sigma}\, \eta_{\mu\nu}]\,. \tag{2.20}$$

The problem now is clear; everything would be fine if it were the case that

$$K^{aI}\, K^J_a = c\, \delta^{IJ} \tag{2.21}$$

for some constant c. Then, the right-hand side in (2.20) would give precisely the energy-momentum tensor for the Yang-Mills fields. However, in general if we define a matrix Y^{IJ} by

$$Y^{IJ} = K^{aI}\, K^J_a\,, \tag{2.22}$$

then there are two things that go wrong. First of all, we note that Y^{IJ} is written as a sum over n vectors, where n is the dimension of the internal space M. But the I and J indices range over $\dim(G)$ values, the dimension of the isometry group. It is perfectly possible that

$\dim(G)$ is greater than n, as, for example, in the case of a coset space, $M = G/H$, for which $n = \dim(G) - \dim(H)$. Clearly, in such a case, the matrix Y^{IJ} must be degenerate, with $(\dim(G) - n)$ zero eigenvalues. So it cannot possibly be of the form (2.21).

The second problem is that Y^{IJ} is in general a function of the coordinates y^m on the internal space M. Thus we have a mis-match between the left-hand side of (2.20), which depends only on the x^μ coordinates, and the right-hand side, which will have y^m dependence because of the y^m dependence of Y^{IJ}. This problem is at the leading order of post-linearised terms, namely it is a problem at the trilinear order in the putative lower-dimensional Lagrangian. This means that it cannot possibly be resolved by putting back those scalar fields that we previously wilfully suppressed. This is a new inconsistency problem, and nothing in general can rescue it.

How, then, might we avoid this problem, and obtain a Kaluza-Klein sphere reduction that is consistent at the full non-linear level? We shall discuss an example in the next subsection.

2.3 $SO(5)$-gauged $N = 4$ supergravity in $D = 7$ from $D = 11$

2.3.1 The seven-dimensional $SO(5)$-gauged theory

The reduction of eleven-dimensional supergravity on the 4-torus gives rise to the maximal ungauged supergravity in $D = 7$. In its bosonic sector this comprises

$$g_{\mu\nu}\,, \quad \vec{\phi}\,, \quad \mathcal{A}^i_{(1)}\,, \quad \mathcal{A}^i_{(0)j}\,, \quad A_{(3)}\,, \quad A_{(2)i}\,, \quad A_{(1)ij}\,, \quad A_{(0)ijk}\,, \tag{2.23}$$

where the index i runs over the 4 directions of T^4. Thus we see that in total there are 14 fields in the spin-0 sector, comprising the 4 dilatonic scalars $\vec{\phi}$, the 6 axions $\mathcal{A}^i_{(0)j}$ and the 4 axions $A_{(0)ijk}$. As we saw in chapter 1, these scalars parameterise the non-linear sigma model coset $SL(5, \mathbb{R})/SO(5)$. There are in total ten vectors, comprising four $\mathcal{A}^i_{(1)}$ and six $A_{(1)ij}$.

The global symmetry $SL(5, \mathbb{R})$, which we studied just in the scalar sector, in fact extends to the entire seven-dimensional theory. It turns out that one can gauge the $SO(5)$ maximal compact subgroup, thereby ending up with a theory with a local $SO(5)$ symmetry. This is achieved by using the ten abelian vector fields that we counted in the previous paragraph, and which now become the non-abelian Yang-Mills potentials of the gauge group $SO(5)$. It will be noted that by happy chance, there are exactly the right number of vector fields available to do the job!

All the fields of the seven-dimensional gauged supergravity fall into representations of the $SO(5)$ gauge group. Of course the metric is a singlet, and the ten Yang-Mills gauge potentials are in the adjoint representation of $SO(5)$. It is convenient to represent them now by $A^{ij}_{(1)}$, antisymmetric in i and j, where i ranges over 5 values corresponding to the fundamental 5-dimensional representation of $SO(5)$. The 14 scalars form the irreducible symmetric 2-index representation, and in fact it is convenient to parameterise the scalars as the symmetric unimodular $SO(5)$ tensor T_{ij}. Finally, in the ungauged theory we saw that there were four 2-form potentials and a 3-form potential. Since a 4-form field strength is dual to a 3-form field strength in $D=7$, we could have dualised from the 3-form potential to a 2-form, giving five in total. In fact these form an irreducible 5 of $SL(5,\mathbb{R})$ in the ungauged theory. In the gauged theory, we have five 3-form fields that form the irreducible 5-dimensional representation of $SO(5)$. They will be represented by $S^i_{(3)}$ now.

Without further ado, we can now present the bosonic Lagrangian for seven-dimensional $SO(5)$-gauged maximal supergravity, which was derived in [7]. It is

$$\begin{aligned}\mathcal{L}_7 \;=\;& R{*\mathbb{1}} - \tfrac14 T^{-1}_{ij}\, {*DT_{jk}}\wedge T^{-1}_{k\ell}\, DT_{\ell i} - \tfrac14\, T^{-1}_{ik}\, T^{-1}_{j\ell}\, {*F^{ij}_{(2)}}\wedge F^{k\ell}_{(2)} - \tfrac12 T_{ij}\, {*S^i_{(3)}}\wedge S^j_{(3)}\\ &+\frac{1}{2g} S^i_{(3)}\wedge H^i_{(4)} - \frac{1}{8g}\epsilon_{ij_1\cdots j_4}\, S^i_{(3)}\wedge F^{j_1j_2}_{(2)}\wedge F^{j_3j_4}_{(2)} + \frac1g \Omega_{(7)} - V{*\mathbb{1}}\,,\end{aligned}\tag{2.24}$$

where

$$H^i_{(4)} \equiv DS^i_{(3)} = dS^i_{(3)} + g\, A^{ij}_{(1)}\wedge S^j_{(3)}\,.\tag{2.25}$$

V is a potential for the scalar fields, given by

$$V = \tfrac12 g^2\Big(2T_{ij}\, T_{ij} - (T_{ii})^2\Big)\,,\tag{2.26}$$

and $\Omega_{(7)}$ is a Chern-Simons type of term built from the Yang-Mills fields, which has the property that its variation with respect to $A^{ij}_{(1)}$ gives

$$\delta\Omega_{(7)} = \tfrac34 \delta^{j_1j_2j_3j_4}_{i_1i_2k\ell}\, F^{i_1i_2}_{(2)}\wedge F^{j_1j_2}_{(2)}\wedge F^{j_3j_4}_{(2)}\wedge \delta A^{k\ell}_{(1)}\,.\tag{2.27}$$

It is given explicitly in [7]. The rest of the notation is as follows. The Yang-Mills field strengths $F^{ij}_{(2)}$ are given by

$$F^{ij}_{(2)} \equiv dA^{ij}_{(1)} + g\, A^{ik}_{(1)}\wedge A^{kj}_{(1)}\,,\tag{2.28}$$

and the symbol D denotes the Yang-Mills covariant exterior derivative:

$$DT_{ij} \equiv dT_{ij} + gA^{ik}_{(1)}\, T_{kj} + gA^{jk}_{(1)}\, T_{ik}\,.\tag{2.29}$$

Note that the $S^i_{(3)}$ are viewed as fundamental fields in the Lagrangian. The equations of motion following from (2.24) can straightforwardly be shown to be

$$\begin{aligned} D\Big(T^{-1}_{ik}T^{-1}_{j\ell}{*F}^{ij}_{(2)}\Big) &= -2g\,T^{-1}_{i[k}{*DT}_{\ell]i} - \frac{1}{2g}\,\epsilon_{i_1i_2i_3k\ell}\,F_2^{i_1i_2}\,H^{i_3}_{(4)} \\ &\quad +\frac{3}{2g}\delta^{j_1j_2j_3j_4}_{i_1i_2k\ell}\,F^{i_1i_2}_{(2)}\wedge F^{j_1j_2}_{(2)}\wedge F^{j_3j_4}_{(2)} - S^k_{(3)}\wedge S^\ell_{(3)}\,. \qquad (2.30) \\ D\Big(\,T^{-1}_{ik}{*D}(T_{kj})\Big) &= 2g^2(2T_{ik}\,T_{kj} - T_{kk}\,T_{ij})\epsilon_{(7)} + T^{-1}_{im}\,T^{-1}_{k\ell}\,{*F}^{m\ell}_{(2)}\wedge F^{kj}_{(2)} \\ &\quad +T_{jk}\,{*S}^k_{(3)}\wedge S^i_{(3)} - \tfrac15\delta_{ij}\Big[2g^2\Big(2T_{ik}T_{ik} - 2(T_{ii})^2\Big)\epsilon_{(7)} \\ &\quad +T^{-1}_{nm}T^{-1}_{k\ell}\,{*F}^{m\ell}_{(2)}\wedge F^{kn}_{(2)} + T_{k\ell}\,{*S}^k_{(3)}\wedge S^\ell_{(3)}\Big]\,, \qquad (2.31) \\ D(T_{ij}\,{*S}^j_{(3)}) &= F^{ij}_{(2)}\wedge S^j_{(3)}\,, \qquad (2.32) \\ H^i_{(4)} &= gT_{ij}\,{*S}^j_{(3)} + \frac18\epsilon_{ij_1\cdots j_4}F^{j_1j_2}_{(2)}\wedge F^{j_3j_4}_{(2)}\,, \qquad (2.33) \end{aligned}$$

It is worth pausing at this point to make an important observation about the gauging. One cannot take the limit $g\to 0$ in the Lagrangian (2.24), on account of the terms proportional to g^{-1} in the second line. We know, on the other hand, that it must be possible to recover the ungauged $D=7$ theory by turning off the gauge coupling constant. In fact the problem is associated with a pathology in taking the limit at the level of the Lagrangian, rather than in the equations of motion. This can be seen by looking instead at the seven-dimensional equations of motion, which were given earlier. The only apparent obstacle to taking the limit $g\to 0$ is in the Yang-Mills equations (2.30), but in fact this illusory. If we substitute the first-order equation (2.33) into (2.30) it gives

$$D\Big(T^{-1}_{ik}T^{-1}_{j\ell}{*F}^{ij}_{(2)}\Big) = -2gT^{-1}_{i[k}{*DT}_{\ell]i} - \tfrac12\,\epsilon_{i_1i_2i_3k\ell}\,F_2^{i_1i_2}\wedge T_{ij}\,{*S}^j_{(3)} - S^k_{(3)}\wedge S^\ell_{(3)}\,, \qquad (2.34)$$

which has a perfectly smooth $g\to 0$ limit. It is clear that equations of motion (2.33) and (2.31) and the Einstein equations of motion also have a smooth limit. (The reason why the Einstein equations have the smooth limit is because the $1/g$ terms in the Lagrangian (2.24) do not involve the metric, and thus they give no contribution.) One sometimes hears the statement made that "the seven-dimensional gauged supergravity does not have a continuous limit to the ungauged theory." This statement, as we can see from this discussion, is therefore incorrect.

We may remark that the theory admits a simple solution in which the Yang-Mills and 3-form fields vanish, the scalars are trivial (i.e. $T_{ij}=\delta_{ij}$), and the metric is seven-dimensional anti-de Sitter spacetime, AdS_7. In this background the scalar potential $V=-\frac{15}{2}g^2$, and behaves just like a cosmological constant. Thus the Einstein equation implies that

$$R_{\mu\nu} - \tfrac12 R\,g_{\mu\nu} = -\tfrac{15}{4}g^2\,g_{\mu\nu}\,, \qquad (2.35)$$

or in other words,

$$R_{\mu\nu} = -\tfrac{3}{2}g^2\, g_{\mu\nu}\,. \tag{2.36}$$

2.3.2 A first look at the S^4 reduction of $D = 11$ supergravity

The $SO(5)$-gauged theory described above was first obtained in [7], by carrying out the process of gauging the original ungauged seven-dimensional supergravity. Had it not been for all the objections raised in the previous subsection, it might have seemed natural to expect that it should be obtainable instead by a direct process of reduction from eleven-dimensional supergravity on S^4. After all, the isometry group of the 4-sphere is $SO(5)$, which is exactly what we would want.

It turns out that this is one of the cases where the discussion of the previous subsection, which was considering the situation for the possible coset-space Kaluza-Klein reduction of a generic theory, can be evaded. We shall first give a considerably simplified discussion, to indicate how the principle obstacle to performing a consistent reduction can be overcome. Later on, we shall present the complete result. This was first derived, incidentally, in [8].

Recalling that the bosonic Lagrangian for eleven-dimensional supergravity is

$$\mathcal{L}_{11} = \hat R\,{\hat *}1\!\!1 - \tfrac{1}{2}{\hat *}\hat F_{(4)}\wedge \hat F_{(4)} + \tfrac{1}{6}\hat F_{(4)}\wedge \hat F_{(4)}\wedge \hat A_{(3)}\,, \tag{2.37}$$

it follows that the equations of motion are

$$\begin{aligned} \hat R_{MN} &= \tfrac{1}{12}(\hat F^2_{MN} - \tfrac{1}{12}\hat F^2_{(4)}\,\hat g_{MN})\,,\\ d{\hat *}F_{(4)} &= \tfrac{1}{2}\hat F_{(4)}\wedge \hat F_{(4)}\,, \end{aligned} \tag{2.38}$$

where $\hat F^2_{MN}$ means $\hat F_{MPQR}\,\hat F_N{}^{PQR}$ and $\hat F^2_{(4)}$ means $\hat F_{PQRS}\,\hat F^{PQRS}$. It is easy to see that this admits the solution $\mathrm{AdS}_7\times S^4$, where we split the index $M=(\mu,m)$, with μ running over 7-dimensional spacetime and m running over the remaining four internal directions, and we set

$$\hat F_{mnpq} = 3\,g\,\epsilon_{mnpq}\,. \tag{2.39}$$

This clearly satisfies the equation of motion for $\hat F_{(4)}$ in (2.38), since the right-hand side vanishes, and the left-hand side is constructed from the divergence of ϵ_{mnpq}, which is zero too. Thus we have

$$\hat F^2_{\mu\nu} = 0\,,\qquad \hat F^2_{mn} = 54g^2\, g_{mn}\,,\qquad \hat F^2_{(4)} = 216 g^2\,, \tag{2.40}$$

and so from (2.38) we get

$$\hat R_{\mu\nu} = -\tfrac{3}{2}g^2\, g_{\mu\nu}\,,\qquad \hat R_{mn} = 3g^2\, g_{mn}\,. \tag{2.41}$$

A solution is clearly then obtained by taking the seven-dimensional metric $g_{\mu\nu}$ to be AdS$_7$, and the 4-dimensional metric to be S^4. A unit 4-sphere has $R_{mn} = 3g_{mn}$, so the 4-sphere here is one of radius g^{-1}. Note that by choosing the radius like this, we have ensured that the AdS$_7$ has precisely the cosmological constant that we found for the AdS$_7$ solution of the seven-dimensional gauged supergravity, in (2.36). Thus if the 4-sphere reduction *does* give the seven-dimensional gauged supergravity, then the radius of the compactifying 4-sphere will be the inverse of the seven-dimensional Yang-Mills coupling constant g.

The "vacuum" solution of eleven-dimensional supergravity that we have just found may be written as

$$\begin{aligned} d\hat{s}_{11}^2 &= ds_7^2 + g^{-2}\, d\Omega_4^2\,, \\ \hat{F}_{(4)} &= 3g^{-4}\,\Omega_{(4)}\,, \end{aligned} \tag{2.42}$$

where $d\Omega_4^2$ is the metric on the unit 4-sphere, $\Omega_{(4)}$ is the volume form of the unit 4-sphere, and ds_7^2 is the AdS$_7$ metric. If we let e^a denote the vielbein for the unit 4-sphere, then the Kaluza-Klein metric reduction ansatz (2.7) that we discussed previously would be

$$d\hat{s}_{11}^2 = ds_7^2 + g^{-2}\,(e^a - g\,K^{aI}\,A_{(1)}^I)\,(e^a - g\,K^{aJ}\,A_{(1)}^J)\,, \tag{2.43}$$

where K^I are the 10 Killing vectors of the isometry group $SO(5)$ of the 4-sphere. As before, we are ignoring scalars for now; we shall focus on looking at the lower 7-dimensional components of the Einstein equation (2.20), which previously gave us trouble. The new feature in our present discussion is that we have another field in the higher-dimensional theory, namely $\hat{F}_{(4)}$. It is this field that saves the day.

One can show already from an analysis of small fluctuations around the AdS$_7 \times S^4$ "vacuum" that in order to get a proper diagonalisation of the kinetic terms for the seven-dimensional fields, it is necessary to include terms involving the Yang-Mills fields in the ansatz for the 4-form $\hat{F}_{(4)}$, as well having them appear in their standard way in (2.43). We shall not derive this here, since it is now superseded by the full non-linear result that we shall present later. It can be found, for example, in [9]. Quoting the result, it turns out that at the linearised level the ansatz for the 4-form field strength should be augmented by terms involving the $SO(5)$ Yang-Mills field strengths $F_{(2)}^I$, as follows:

$$\hat{F}_{(4)} = 3g^{-3}\,\Omega_{(4)} + \tfrac{1}{4} g^{-1}\,F_{(2)}^I \wedge L_{(2)}{}^I\,. \tag{2.44}$$

Here, $L_{(2)}^I$ denotes the 2-forms on the 4-sphere that are obtained by taking the antisymmetric derivative of the Killing vectors. Of course precisely because they are Killing vectors, the derivatives $\nabla_a K^I_b$ are already automatically antisymmetric, so we have

$$L^I_{ab} \equiv \nabla_a K^I_b\,, \qquad \text{and so} \qquad L^I_{(2)} = dK^I\,, \tag{2.45}$$

where $K^I \equiv K^I_a\, e^a$ are the Killing vectors written as 1-forms. This means that we have

$$\hat F_{abcd} = 3g\,\epsilon_{abcd}\,, \qquad \hat F_{\mu\nu ab} = g^{-1}\, F^I_{\mu\nu}\, L^I_{ab}\,. \tag{2.46}$$

Now plug everything into the 7-dimensional components of the eleven-dimensional Einstein equation

$$\hat R_{MN} - \tfrac12 \hat R\,\hat g_{MN} = \tfrac1{12}(\hat F^2_{MN} - \tfrac18 \hat F^2_{(4)}\,\hat g_{MN})\,. \tag{2.47}$$

Using (2.17) and (2.46), we therefore find that vielbein components in the lower 7 dimensions give

$$R_{\mu\nu} - \tfrac12 R\,\eta_{\mu\nu} = \tfrac12 Y^{IJ}\,[F^I_{\mu\rho}\,F^{J\,\rho}_\nu - \tfrac14 F^I_{\rho\sigma}\,F^{J\,\rho\sigma}\,\eta_{\mu\nu}] - \tfrac{15}{4} g^2\,\eta_{\mu\nu}\,, \tag{2.48}$$

where the quantities Y^{IJ} are given by

$$Y^{IJ} = K^{aI}\,K^J_a + \tfrac12 g^{-2}\,L^{abI}\,L^J_{ab}\,. \tag{2.49}$$

A remarkable thing has happened here. First of all, recall our discussion of the dimensional reduction of a generic theory, for which only the first term in (2.49) was present. Expressed in the specific context of a 4-sphere reduction, the index a runs over 4 values, while the Yang-Mills index I runs over 10 values. Thus, we would have argued, in (2.22) the matrix Y^{IJ} must have $10-4=6$ zero eigenvalues, and so it could not possibly give us the δ^{IJ} that we would have hoped for. However, in our new expression (2.49) we have precisely 6 more quantities being summed over, in the second term, since $L^I_{ab} \equiv \nabla_a K^I_b$ is antisymmetric in a and b. So (2.49) is the sum over 10 quantities, and we are in with a chance!

So far, this is just numerology. The even more remarkable fact is that one can easily show that the Killing vectors on the 4-sphere precisely do satisfy the relation

$$K^{aI}\,K^J_a + \tfrac12 g^{-2}\,L^{abI}\,L^J_{ab} = \delta_{IJ}\,. \tag{2.50}$$

(Of course there is an issue of constant normalisation factors here. More precisely, what one can show is that by normalising the Killing vectors appropriately, (2.50) is satisfied.) The key points to note here are that not only is Y^{IJ} defined in (2.49) non-degenerate on S^4, but it is completely independent of the coordinates of S^4! This can be proven quite easily by writing S^4 as the unit sphere in $\mathbb{R}^5$, with coordinates μ^i that satisfy $\mu^i\,\mu^i = 1$, and then expressing the Killing vectors in terms of these coordinates:

$$K^{ij} = \mu^i\,\frac{\partial}{\partial\mu^j} - \mu^j\,\frac{\partial}{\partial\mu^i}\,. \tag{2.51}$$

Another significant fact is that if one considers any other compact 4-dimensional Einstein space, which might *a priori* be viewed as an equally good candidate for giving a Kaluza-Klein reduction to $D = 7$, its Killing vectors cannot, in general, satisfy (2.50). For example, if one considers the 8 Killing vectors of the $SU(3)$ isometry group of the complex projective space CP^2, then one finds that Y^{IJ} defined in (2.49) depends on the coordinates of CP^2, and so (2.48) would not make sense in that case.

What we are finding here can be expressed group theoretically as follows. *A priori*, the quantity Y^{IJ} defined in (2.49) is in the reducible representation that one obtains by taking the symmetric product of two adjoint representations of the isometry group G of the 4-dimensional internal space. This reducible representation will certainly include the singlet, but it could have other terms too. For example, for the 4-sphere with $G = SO(5)$, we have

$$(\underline{10} \times \underline{10})_{\rm sym} = \underline{1} + \underline{5} + \underline{14} + \underline{35}\,. \tag{2.52}$$

Now in this case Y^{IJ} turns out to be constant, which means that all terms except the singlet in this decomposition have cancelled. In particular, had we looked at just the first term in (2.49) in isolation, or at just the second term, we would have obtained a non-constant result, consisting of a combination of the singlet and at least one of the other representations in (2.52). So there is a "conspiracy" between the two terms that leads to a cancellation of the non-singlet representations. By contrast, in the analogous discussion for the $SU(3)$ Killing vectors of CP^2, it turns out that there is no conspiracy, and so non-singlet terms from the symmetric product of $\underline{8} \times \underline{8}$ in $SU(3)$ survive.

Note that if non-singlets survive in Y^{IJ}, then (2.48) is telling us that we should really have included *massive* spin-2 fields as well as the massless graviton (the metric ds_7^2) in the Kaluza-Klein reduction. Roughly speaking, at the linearised level, it is saying that we should have expanded the 7-dimensional components of the eleven dimension metric not just as $\hat g_{\mu\nu}(x,y) = \bar g_{\mu\nu}(x) + h_{\mu\nu}(x)$, where x denotes the 7-dimensional coordinates and y denotes the 4-dimensional internal coordinates, but rather as

$$\hat g_{\mu\nu}(x,y) = \bar g_{\mu\nu}(x) + \sum_{i=0}^{\infty} h_{\mu\nu}^{(i)}(x)\, P^{(i)}(y)\,, \tag{2.53}$$

where $P^{(i)}(y)$ denotes a complete set of scalar harmonics on the internal space, with $P^{(0)}(y) = 1$ corresponding to the massless graviton $h_{\mu\nu}^{(0)}(x)$. The non-singlet part on the right-hand side of (2.48) would then match against non-singlet terms from the expansion (2.53), which in this linearised discussion would be appearing in linearised "Einstein tensors" for the higher gravity modes $h^{(i)}_{\mu\nu(x)}$. Of course once one has even a single massive spin-2 field, it is inevitable that one needs an entire infinite tower of them, since a finite number of massive spin-2 fields cannot couple consistently to gravity. The fact that Y^{IJ} in (2.48) is turning out to be purely an $SO(5)$ singlet for the S^4 reduction means that we are able to get away with never introducing the massive gravitons in the first place.

To summarise, we have looked at a *necessary* condition for the consistency of a Kaluza-Klein reduction ansatz, namely that the quantity Y^{IJ} appearing in the lower-dimensional Einstein equation (2.48) must be *independent* of the coordinates of the internal compactifying space. We saw previously that this is not satisfied for a coset-space reduction of a generic theory. However, what we have now seen is that in the reduction of eleven-dimensional supergravity on a 4-dimensional internal space, the form of the matrix Y^{IJ}, given in (2.49), is such that this necessary condition for consistency *is* satisfied in one exceptional case, namely when the internal space is the 4-sphere. There is a conspiracy going on here, between eleven-dimensional supergravity and the 4-sphere!

We should emphasise that the above discussion has certainly not, of itself, proved that the 4-sphere reduction of eleven-dimensional supergravity is consistent. Rather, it has shown that it circumvents an obstruction that would have been enough to prevent a consistent reduction from being possible for any randomly-chosen theory and internal manifold. In fact, as we shall see later, the S^4 reduction of eleven-dimensional supergravity is one of only a very few examples where coset-space Kaluza-Klein reduction can work. Before discussing that further, let us complete the job for the S^4 reduction, and give the complete result.

2.3.3 Complete reduction of $D = 11$ supergravity on S^4

The complete result for the the Kaluza-Klein reduction of eleven-dimensional supergravity on S^4 was obtained in [8]. The strategy there involved looking at the fermionic sector of the theory, and in particular the supersymmetry transformation rules. By imposing the requirement that the eleven-dimensional transformation rules should consistently yield seven-dimensional transformation rules, the form of the Kaluza-Klein ansatz for all the fields, bosonic as well as fermionic, was derived. Having obtained consistency in the supersymmetry transformation rules, it was argued [8] that this implied that the eleven-dimensional field equations would necessarily consistently reduce to seven-dimensional ones.

We shall proceed rather differently, and focus instead just on the bosonic sector. Our criterion for the consistency of the reduction will be the direct one, of insisting that the eleven-dimensional equations of motion consistently yield seven-dimensional ones, with all on the 4-sphere coordinates matching in all the equations, so that it factors out and gives sensible purely seven-dimensional equations. This was done in [10]. We saw one example of this consistent matching already, in (2.48), where it was essential that Y^{IJ} had to be independent of the 4-sphere coordinates.

There are pros and cons to the two approaches to proving the consistency of the Kaluza-Klein reduction ansatz. In fact, if the truth be told, from a rigorously mathematical point of view the consistency of S^4 reduction has not yet been completely proven by either method. In the supersymmetry transformation rule approach of [8], only the fermionic terms of quadratic order were retained in the Lagrangian; the infamous quartic-fermion terms were dropped. Of course without them the theory is not supersymmetric, so by omitting them one is definitely not doing a complete job of proving the consistency of the reduction. On the other hand, all the experience over the decades has been that if one takes care of the quadratic terms, the quartic terms will "take care of themselves," and one would have to be a masochist if one were to include them. But still, the logical point remains that the proof is not quite a complete one if these terms are omitted. On the other hand, even by omitting them one learns what the complete and exact ansatz for the bosonic fields would have to be, if the reduction were indeed a consistent one. What is lacking is that final piece of absolute certainty that the reduction *is* actually consistent. One other residual question concerns the issue of whether a proof that the supersymmetry transformation rules reduce consistently also constitutes a proof that the equations of motion must reduce consistently too. (The latter is, by definition, what one means by a consistent Kaluza-Klein reduction.) It probably does, and it certainly seems highly plausible. As far as I am aware, however, the link between the two concepts has never been spelt out in complete and unequivocal detail. Having said all this, it should also be emphasised that these are really high-order "quibbles," and in ordinary parlance one can effectively view the discussion in [8] as definitive.

The advantage of the direct approach of checking the consistency of the reduction of the equations of motion is that if this is done, then by definition one has proved the consistency, period. In practice, there may be limits to what one can explicitly calculate, just because the calculations become too involved. In the present context of the S^4 reduction from $D = 11$, the consistency of the reduction of the bosonic equations of motion was almost completely checked in [10], but certain simplifications and specialisations were made when checking the eleven-dimensional Einstein equation. With what was checked there is really no room for any doubt that it works fully, but again, strictly speaking, there remains a slight *lacuna* from a strictly rigorous point of view.

After all the quibbles and cautions, let us now present the result. The unit 4-sphere can be described by introducing five coordinates μ^i on flat Euclidean $\mathbb{R}^5$, that are subject to the constraint

$$\mu^i\,\mu^i = 1\,. \tag{2.54}$$

The metric on the unit 4-sphere is then given by

$$d\Omega_4^2 = d\mu^i\,d\mu^i\,. \tag{2.55}$$

These μ^i coordinates, subject to the constraint (2.54), are used extensively in the Kaluza-Klein reduction ansatz. It is given by

$$d\hat{s}_{11}^2 = \Delta^{1/3}\,ds_7^2 + \frac{1}{g^2}\Delta^{-2/3}\,T_{ij}^{-1}\,D\mu^i\,D\mu^j\,, \tag{2.56}$$

$$\begin{aligned}\hat{F}_{(4)} &= \frac{1}{4!}\,\epsilon_{i_1\cdots i_5}\Big[-\frac{1}{g^3}U\,\Delta^{-2}\mu^{i_1}\,D\mu^{i_2}\wedge\cdots\wedge D\mu^{i_5}\\ &\quad+\frac{4}{g^3}\Delta^{-2}\,T^{i_1 m}\,DT^{i_2 n}\,\mu^m\,\mu^n\,D\mu^{i_3}\wedge\cdots\wedge D\mu^{i_5}\\ &\quad+\frac{6}{g^2}\Delta^{-1}F_{(2)}^{i_1 i_2}\wedge D\mu^{i_3}\wedge D\mu^{i_4}\,T^{i_5 j}\,\mu^j\Big] - T_{ij}\,{*S_{(3)}^i}\,\mu^j + \frac{1}{g}\,S_{(3)}^i\wedge D\mu^i\,,\end{aligned} \tag{2.57}$$

where

$$\begin{aligned}&U \equiv 2T_{ij}\,T_{jk}\,\mu^i\,\mu^k - \Delta\,T_{ii}\,, \qquad \Delta \equiv T_{ij}\,\mu^i\,\mu^j\,,\\ &D\mu^i \equiv d\mu^i + gA_{(1)}^{ij}\,\mu^j\,.\end{aligned} \tag{2.58}$$

The 7-dimensional fields ds_7^2, $A_{(1)}^{ij}$, T_{ij} and $S_{(3)}^i$ were all described in the earlier section, where we presented the bosonic Lagrangian for the seven-dimensional $SO(5)$-gauged theory. Note that $*$ here is the Hodge dual in the seven-dimensional metric ds_7^2, and it must be carefully distinguished from Hodge dualisation in the eleven-dimensional metric $d\hat{s}_{11}^2$, which we are denoting by $\hat{*}$.

Before discussing this reduction ansatz in detail, let us just note that it does indeed look similar to something we have seen previously, if we temporarily (and illegally!) set the five 3-forms $S_{(3)}^i$ to zero and take the 14 scalars to be trivial, $T_{ij} = \delta_{ij}$. The ansatz then takes the form

$$d\hat{s}_{11}^2 = ds_7^2 + \frac{1}{g^2}\,(d\mu^i + g\,A_{(1)}^{ik}\,\mu^k)\,(d\mu^j + g\,A_{(1)}^{j\ell}\,\mu^\ell)\,, \tag{2.59}$$

$$\hat{F}_{(4)} = \frac{3}{4!}\,\epsilon_{i_1\cdots i_5}\,\frac{1}{g^3}\,\mu^{i_1}D\mu^{i_2}\wedge\cdots\wedge D\mu^{i_5} + \frac{1}{4g^2}\epsilon_{i_1\cdots i_5}\,F_{(2)}^{i_1 i_2}\wedge D\mu^{i_3}\wedge D\mu^{i_4}\,\mu^{i_5}\,, \tag{2.60}$$

and without too much trouble one can establish the relation to the approximate ansatz that we discussed in section 2.3.2,

$$\begin{aligned}
d\hat{s}_{11}^2 &= ds_7^2 + g^{-2}\,(e^a - g\,K^{aI}\,A^I_{(1)})\,(e^a - g\,K^{aJ}\,A^J_{(1)})\,,\\
\hat{F}_{(4)} &= 3\,g^{-3}\,\Omega_{(4)} + \tfrac{1}{4} g^{-2}\,F^I_{(2)} \wedge dK^I\,.
\end{aligned} \tag{2.61}$$

Of course even without the inclusion of the scalars and 3-forms, the ansatz in (2.59) and (2.60) is more complete than (2.61), but they agree in the leading orders, and indeed purely on the basis of gauge-invariance and agreeing with the leading-order terms, the structure of (2.59) and (2.60) is uniquely determined. However, degrees of completeness are rather academic, until one includes the scalars and 3-forms, since without them the ansatz violates the eleven-dimensional equations of motion. And, in terms of complexity, if one omits the scalars then, as the saying goes, "You ain't seen nothing yet!"

Now, let us go back to the complete ansatz (2.56) and (2.57). The claim is that if we substitute these into the eleven-dimensional equations of motion (2.38), and the Bianchi identity $d\hat{F}_{(4)} = 0$, then we will obtain a fully consistent reduction that yields precisely the equations of motion for the bosonic fields of seven-dimensional $SO(5)$-gauged supergravity, as given in section 2.3.1. Checking this is a considerable task; the Kaluza-Klein reduction on S^4 is *enormously* more complicated than a Kaluza-Klein reduction on S^1 or a torus! In particular, the "miracles" that must take place in order for all the dependence on the S^4 coordinates μ^i to match in the various eleven-dimensional equations of motion are, to say the least, quite remarkable. We shall just sketch the calculations here.

Consider first the Bianchi identity $d\hat{F}_{(4)} = 0$. Substituting (2.57) into this, we (eventually) obtain the following seven-dimensional equations

$$D(T_{ij} * S^j_{(3)}) = F^{ij}_{(2)} \wedge S^j_{(3)}\,, \tag{2.62}$$

$$H^i_{(4)} = g T_{ij} * S^j_{(3)} + \frac{1}{8} \epsilon_{ij_1\cdots j_4} F^{j_1 j_2}_{(2)} \wedge F^{j_3 j_4}_{(2)}\,, \tag{2.63}$$

where we define

$$H^i_{(4)} \equiv D S^i_{(3)} = dS^i_{(3)} + g\,A^{ij}_{(1)} \wedge S^j_{(3)}\,. \tag{2.64}$$

These are precisely some of the equations of motion of seven-dimensional $SO(5)$-gauged supergravity that we saw in section 2.3.1.

Next, we substitute the ansatz into the $D = 11$ field equation $d\hat{*}\hat{F}_{(4)} = \frac{1}{2} \hat{F}_{(4)} \wedge \hat{F}_{(4)}$. To do this, we need the eleven-dimensional Hodge dual $\hat{*}\hat{F}_{(4)}$. After much calculation, one finds that this is given by

$$\hat{*}\hat{F}_{(4)} = -gU\,\epsilon_{(7)} - \frac{1}{g} T^{-1}_{ij} * DT^{ik} \mu_k \wedge D\mu^j + \frac{1}{2g^2} T^{-1}_{ik} T^{-1}_{j\ell} * F^{ij}_{(2)} \wedge D\mu^k \wedge D\mu^\ell \tag{2.65}$$

$$+\frac{1}{g^4}\Delta^{-1}\,T_{ij}\,S^i_{(3)}\,\mu^j\wedge W-\frac{1}{6g^3}\,\Delta^{-1}\,\epsilon_{ij\ell_1\ell_2\ell_3}{*S^m_{(3)}}\,T_{im}\,T_{jk}\,\mu^k\wedge D\mu^{\ell_1}\wedge D\mu^{\ell_2}\wedge D\mu^{\ell_3}\,,$$

where

$$W\equiv\tfrac{1}{24}\,\epsilon_{i_1\cdots i_5}\,\mu^{i_1}\,D\mu^{i_2}\wedge\cdots\wedge D\mu^{i_5}\,. \tag{2.66}$$

The field equation for $\hat F_{(4)}$ then implies

$$\begin{aligned}
D\Big(T^{-1}_{ik}T^{-1}_{j\ell}{*F^{ij}_{(2)}}\Big) &= -2g\,T^{-1}_{i[k}{*DT_{\ell]i}}-\frac{1}{2g}\,\epsilon_{i_1i_2i_3k\ell}\,F_2^{i_1i_2}\,H^{i_3}_{(4)}\\
&\quad+\frac{3}{2g}\delta^{j_1j_2j_3j_4}_{i_1i_2k\ell}\,F^{i_1i_2}_{(2)}\wedge F^{j_1j_2}_{(2)}\wedge F^{j_3j_4}_{(2)}-S^k_{(3)}\wedge S^\ell_{(3)}\,. &(2.67)\\
D\Big(\,T^{-1}_{ik}{*D(T_{kj})}\Big) &= 2g^2(2T_{ik}\,T_{kj}-T_{kk}\,T_{ij})\epsilon_{(7)}+T^{-1}_{im}\,T^{-1}_{k\ell}\,{*F^{m\ell}_{(2)}}\wedge F^{kj}_{(2)}\\
&\quad+T_{jk}\,{*S^k_{(3)}}\wedge S^i_{(3)}-\tfrac15\delta_{ij}\Big[2g^2\Big(2T_{ik}T_{ik}-2(T_{ii})^2\Big)\epsilon_{(7)}\\
&\quad+T^{-1}_{nm}T^{-1}_{k\ell}\,{*F^{m\ell}_{(2)}}\wedge F^{kn}_{(2)}+T_{k\ell}\,{*S^k_{(3)}}\wedge S^\ell_{(3)}\Big]\,. &(2.68)
\end{aligned}$$

These are the Yang-Mills and scalar equations of motion of seven-dimensional $SO(5)$-gauged supergravity, which we also saw in section 2.3.1.

Finally, one should calculate the Ricci-tensor for the metric ansatz (2.56), and check the eleven-dimensional Einstein equation in (2.38). As mentioned above, this has not been performed completely, although many highly non-trivial consistency checks have been made. There is no doubt, though, that it will work. In summary, substituting the ansatz (2.56) and (2.57) into the equations of motion of eleven-dimensional supergravity, one consistently obtains the equations of motion of the bosonic sector of seven-dimensional $SO(5)$-gauged supergravity, which all follow from the Lagrangian (2.24).

We have repeatedly emphasised that the ability to perform this consistent Kaluza-Klein coset-space reduction is quite exceptional, and that it depends on special properties both of the original higher-dimensional theory, and of the compactifying space. In the next section, we shall explore this in more detail, and see just how exceptional are the cases where a consistent coset-space reduction can be performed.

2.4 Group-theoretic considerations

2.4.1 A criterion for consistency

The consistency of a Kaluza-Klein reduction on a circle, torus or group manifold G (keeping only the gauge bosons of G) could be understood straightforwardly by group-theoretic arguments, since one is keeping all the singlets under a transitively-acting group, and setting to zero all the non-singlets. Thus there is never any danger of non-linear terms in the retained fields acting as sources for the non-singlet fields that have been set to zero.

We have no such group-theoretic explanation for the consistency of the S^4 reduction of $D = 11$ supergravity. For example, the bilinears in the $SO(5)$ Yang-Mills gauge bosons might, *a priori*, have acted as sources for massive spin-2 fields, and it is only because of a "miracle" that the non-singlet part of the symmetric product of two adjoint representations of $SO(5)$, which could in principle have occurred in Y^{IJ} in (2.48), happened to give zero.

Although we are not in a position to explain group-theoretically why the S^4 reduction of eleven-dimensional supergravity works, we *can* give group-theoretic arguments for why coset-space reductions only have any chance of working in very exceptional circumstances. We already saw one type of argument along these lines, when we saw that the reduction of $D = 11$ supergravity would fail unless the quantity Y^{IJ} defined in (2.49) was constant; i.e. a singlet under the isometry group.

It is appropriate now to give a more general discussion. The idea can be explained by again considering the S^4 reduction from $D = 11$. We remarked that the $SO(5)$-gauged supergravity in $D = 7$ that results from the S^4 reduction was in fact first constructed, many years ago, by instead gauging the *ungauged* $D = 7$ supergravity that one gets from a 4-torus reduction of $D = 11$ supergravity. The crucial point was that the global symmetry group of the ungauged theory is $SL(5, \mathbb{R})$, the scalars are in the coset $SL(5, \mathbb{R})/SO(5)$, and so the $SO(5)$ subgroup of the global symmetry group could be gauged. This example shows us that we can formulate the following *necessary* criterion for whether a consistent Kaluza-Klein reduction of a theory on S^n might be possible:

If a consistent Kaluza-Klein reduction of a theory on S^n is to be possible, then a *necessary* condition is that if the theory is instead reduced on T^n, then the global symmetry group $\mathcal{G}$ of the resulting lower-dimensional theory must have a maximal compact subgroup $\mathcal{H}$ that is at least large enough to contain $SO(n+1)$.

We emphasise here that by a consistent Kaluza-Klein reduction on S^n, we mean one giving only a finite number of lower-dimensional fields, which include all the gauge bosons of the $SO(n+1)$ isometry group.

The point about the above criterion is that if we suppose that we have a consistent Kaluza-Klein reduction on S^n then we can always take the (smooth) limit where the radius of the sphere tends to infinity, which has the effect of turning off the gauging. In this limit we effectively have the same theory as we would have obtained from a reduction instead on the n-torus. To be able to reverse the process, and "regauge" the theory, it must therefore be that the T^n-reduced theory has a large enough global symmetry group to contain the isometry group of the n-sphere.

We now have another way to see why Kaluza-Klein sphere reductions will almost always fail to be consistent. In chapter 1 we studied the global symmetry groups of toroidally-reduced theories. In particular, we saw that a generic theory including gravity will give, after reduction on T^n, a theory with $SL(n,\mathrm{IR})$ as its global symmetry group. This is commonly enlarged to $GL(n,\mathrm{IR})$, if the higher-dimensional theory has an overall global scaling symmetry. Either way, the maximal compact subgroup is $SO(n)$, and this is certainly smaller than the $SO(n+1)$ isometry group of the n-sphere. So generically, our new necessary criterion for the existence of a consistent sphere reduction will not be satisfied. The only way to circumvent this is to start with a theory whose T^n reduction has an *enhanced* global symmetry group that is sufficiently larger than $GL(n,\mathrm{IR})$ that it can contain $SO(n+1)$. In the next section, we shall study when this can happen.

2.4.2 Global symmetry enhancements

We saw in chapter 1 that the global symmetry of a theory reduced on T^n can be studied by focusing on the scalar sector of the lower-dimensional theory. From the gravity sector alone, the higher-dimensional metric yields, after a reduction on T^n, a set of n dilatonic scalars $\vec{\phi}$, and a set of $\frac{1}{2}n(n-1)$ axionic scalars $\mathcal{A}^i_{(0)j}$. The lower-dimensional scalar Lagrangian is

$$\mathcal{L} = -\tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\, {*\mathcal{F}^i_{(1)j}} \wedge \mathcal{F}^i_{(1)j}\,, \tag{2.69}$$

where

$$\mathcal{F}^i_{(1)j} = \gamma^k{}_j\, d\mathcal{A}^i_{(0)k}\,, \qquad \gamma^k{}_j \equiv [(1+\mathcal{A}_{(0)})^{-1}]^k{}_j = \delta^k_j - \mathcal{A}^k_{(0)j} + \mathcal{A}^k_{(0)\ell}\,\mathcal{A}^\ell_{(0)j} + \cdots\,. \tag{2.70}$$

The constant "dilaton vectors" $\vec{b}_{ij}$ form the positive roots of $SL(n,\mathrm{IR})$, and $\vec{b}_{i,i+1}$ are the simple roots. We introduce Cartan generators $\vec{H}$ and positive-roots generators $E_i{}^j$ for $SL(n,\mathrm{IR})$, and define

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}\,\Big(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}\,E_i{}^j}\Big)\,, \tag{2.71}$$

where the ordering is anti-lexigraphical, i.e. $\cdots(24)(23)\cdots(14)(13)(12)$. Then the scalar Lagrangian (2.69) can be written as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}({*d\mathcal{M}^{-1}} \wedge d\mathcal{M})\,, \tag{2.72}$$

where $\mathcal{M} = \mathcal{V}^T\,\mathcal{V}$, which shows that it has the $SL(n,\mathrm{IR})$ global symmetry.

To get an enhanced global symmetry in the lower-dimensional theory, we must clearly have more scalar fields. The idea then will be that these describe a larger coset manifold, with a larger symmetry group. At this stage the discussion clearly becomes highly theory-specific, and so we shall have to focus our discussion on some particular class of higher-dimensional theories. The experience with the 4-sphere reduction of $D = 11$ supergravity suggests that a natural class of higher-dimensional theory to consider would be one comprising gravity plus a p-form field strength.

Let us begin, therefore, with a D-dimensional theory of gravity and a p-form field strength:

$$\mathcal{L}_D = \hat{R}\,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}\hat{F}_p \wedge \hat{F}_p\,. \tag{2.73}$$

We now reduce this on T^n, and study the form of the scalar sector. The higher-dimensional metric will give a contribution precisely of the form (2.69). The antisymmetric tensor will give $n!/(p!\,(n-p)!)$ axions, $A_{(0)i_1\cdots i_{p-1}}$. Their dilaton vectors can be easily calculated, using the same techniques that we used in chapter 1. To avoid a profusion of indices, let us just denote the new axions by χ_α, with dilaton vectors $\vec{a}_\alpha$. Thus the total scalar Lagrangian in $(D-n)$ dimensions will have the form

$$\mathcal{L} = -\tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\, {*\mathcal{F}^i_{(1)j}} \wedge \mathcal{F}^i_{(1)j} - \tfrac{1}{2}\sum_{\alpha} e^{\vec{a}_\alpha\cdot\vec{\phi}}\, {*G_{(1)\alpha}} \wedge G_{(1)\alpha}\,, \tag{2.74}$$

where $G_{(1)\alpha} = d\chi_\alpha + \cdots$, and the ellipses represent the various "transgression" terms of the kind that we saw in chapter 1.

The dilaton vectors $\vec{a}_\alpha$ will be found to be the weight vectors of some representation of $SL(n,\mathbb{R})$. In general, the global symmetry group of the scalar Lagrangian (2.74) will just be $GL(n,\mathbb{R})$. If an enhancement of the symmetry group is to occur, it must be that the positive-root vectors $\vec{b}_{ij}$ and weight vectors $\vec{a}_\alpha$ of $SL(n,\mathbb{R})$ "conspire" to become the positive-root vectors of the larger symmetry group.

The additional simple root vectors would have to come from $\vec{a}_\alpha$, since the $\vec{b}_{ij}$ are already supplying the full set of simple roots $\vec{b}_{i,i+1}$ for $SL(n,\mathbb{R})$. We can now invoke a basic result from the classification of simple Lie algebras, that the ratio of the lengths of any two simple roots can only take a small number of possible values, namely

$$\frac{1}{\sqrt{3}}\,,\qquad \frac{1}{\sqrt{2}}\,,\qquad 1\,,\qquad \sqrt{2}\,,\qquad \sqrt{3}\,. \tag{2.75}$$

If the ratio is 1 for all simple roots then the algebra is simply-laced.

The upshot of this observation is that if we are to get a suitable enhanced global symmetry group (i.e. a larger simple group, which can have $SO(n+1)$ as a subgroup), then the lengths of the $\vec{a}_\alpha$ dilaton vectors must be commensurate with the lengths of the $\vec{b}_{ij}$ dilaton vectors. It is a simple matter to calculate these lengths, using the Kaluza-Klein formulae that we derived in chapter 1. It turns out that all the $\vec{b}_{ij}$ have the same length $|\vec{b}|$ as each other, and all the $\vec{a}_\alpha$ dilaton vectors have the same length $|\vec{a}|$ as each other. These two lengths are given by

$$|\vec{b}|^2 = 4\,, \qquad |\vec{a}|^2 = \frac{2(p-1)(D-p-1)}{D-2}\,. \tag{2.76}$$

Consider the case where we might get a simply-laced enhanced symmetry group; this would require $|\vec{b}|^2 = |\vec{a}|^2$. In fact this is the only case that in the end turns out to be relevant. Recall that we are discussing a *necessary* condition for being able to construct a consistent Kaluza-Klein sphere reduction. It turns out after a much more elaborate analysis that the cases in (2.75) corresponding to $|\vec{b}|^2 \ne |\vec{a}|^2$ eventually seem not to allow consistent sphere reductions. Rather than getting bogged down in this analysis here, let us just focus our attention on the one case, $|\vec{b}|^2 = |\vec{a}|^2$, that *can* in the end give theories which allow consistent sphere reductions. From (2.76) we then find

$$D = p + 3 + \frac{4}{p-3}\,. \tag{2.77}$$

Since D and p must be integers, this immediately tells us that $p \le 7$, and then an enumeration of all the integer solutions gives the following:

$$(D, p) = (11, 4)\,, \quad (11, 7)\,, \quad (10, 5)\,. \tag{2.78}$$

The first two cases listed here are equivalent, since a 7-form field strength in $D = 11$ can be dualised to a 4-form. So we have deduced that there are only two examples of theories comprising gravity plus a p-form field strength that could possibly admit consistent Kaluza-Klein sphere reductions! One of these is an eleven-dimensional theory with a 4-form field strength, and the other is a ten-dimensional theory with a 5-form field strength. These ingredients sound rather familiar, of course; we seem to be seeing the emergence of eleven-dimensional supergravity and ten-dimensional type IIB supergravity, coming out from these purely bosonic considerations of the consistency of Kaluza-Klein sphere reductions!

In fact the connection with the supergravities is even stronger. So far, we have only considered a *necessary* condition for getting a global symmetry enhancement, namely that the lengths of the dilaton vectors for the scalars coming from the metric and the p-form field should be commensurate. When one checks the global symmetries in more detail, using the methods described in detail in chapter 1, it turns out that in the $D = 11$ case, the symmetry enhancement occurs only if there is an extra term added to the basic Lagrangian (2.73).

This is the "Chern-Simons" $F_{(4)} \wedge F_{(4)} \wedge A_{(3)}$ term, and it must have *exactly* the coefficient that arises in $D = 11$ supergravity. So we reach the conclusion that the only way that the $(D, p) = (11, 4)$ theory stands a chance of allowing a consistent sphere reduction is if it is *precisely* the bosonic sector of $D = 11$ supergravity,

$$\mathcal{L}_{11} = \hat{R}\,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}\hat{F}_{(4)} \wedge \hat{F}_{(4)} + \tfrac{1}{6}\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}\,. \tag{2.79}$$

Similarly, when one checks in detail for the case of $(D, p) = (10, 5)$, one finds that the proposed global symmetry enhancements for toroidal reductions actually do occur, but only if the 5-form field is self-dual (or anti-self-dual). Thus again we see that this necessary criterion for being able to make a consistent Kaluza-Klein sphere reduction has singled out a theory that is precisely contained within one of the most important of the supergravities.

To summarise, we have seen that only for two distinct cases can a D-dimensional theory of gravity plus a p-form field strength have any chance of allowing a consistent Kaluza-Klein sphere reduction, namely $D = 11$ with a 4-form field and the Chern-Simons term, and $D = 10$ with a self-dual 5-form. As it turns out, these cases where the necessary condition is satisfied do in fact all allow consistent sphere reductions. Specifically, we can make a consistent reduction on S^4 or S^7 from $D = 11$ (we saw the S^4 example previously), and on S^5 from $D = 10$.

2.4.3 Sphere reduction of gravity plus p-form plus dilaton

Before moving on to other things, we may consider a slight generalisation of the previous discussion. Since the restrictions that were implied by the requirement of having commensurate lengths for the dilaton vectors from gravity and the p-form were so strong, we might try to relax them somewhat by allowing a dilaton already in the higher-dimensional theory. Thus we may consider starting in D dimensions with a theory of gravity, p-form and dilaton, with the Lagrangian

$$\mathcal{L}_D = \hat{R}\,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\varphi \wedge d\varphi - \tfrac{1}{2}e^{c\,\varphi}\,\hat{*}\hat{F}_p \wedge \hat{F}_\pi\,. \tag{2.80}$$

The point now is that we can choose the dilaton coupling c at will, thereby changing the length of the dilaton vectors $\vec{a}_\alpha$ in the lower dimension. Specifically, the previous formula (2.76) will now clearly be changed to

$$|\vec{b}|^2 = 4\,, \qquad |\vec{a}|^2 = c^2 + \frac{2(p-1)(D-p-1)}{D-2}\,. \tag{2.81}$$

If we again demand that the lengths of the $\vec{b}$ and $\vec{a}$ dilaton vectors be equal, then we can express this as the following equation for c^2:

$$\tfrac{1}{2}(D-2)\,c^2 = -(p-3)(D-p-3)+4\,. \tag{2.82}$$

It is now easy to see that since c must be real, we will only get any further possibilities by having $p \le 3$. (To see this, recall that without loss of generality, we may assume (because of Hodge duality) that $p \le \frac{1}{2}D$.) In fact, two new classes of possibility open up, with D being allowed to be arbitrary in each case, namely

$$\underline{p=3}: \qquad c^2 = \frac{8}{D-2}\,,$$

$$\underline{p=2}: \qquad c^2 = \frac{2(D-1)}{D-2}\,. \tag{2.83}$$

The first case here, where we have gravity plus a 3-form field strength plus a dilaton in the higher dimension, actually corresponds precisely to the low-energy effective action for the bosonic string in D dimensions. The second case, with gravity, a 2-form field strength and a dilaton, is precisely the theory that one gets by reducing pure gravity in $(D+1)$ dimensions on S^1. (This can easily be verified, using results from chapter 1.) We shall not dwell on the details further here, but simply remark that in fact consistent sphere reductions can be performed for both classes of theory. Specifically, one can consistently reduce the $(D,3)$ theories on either S^3 or S^{D-3}, and one can consistently reduce the $(D,2)$ theories on S^2.

To close this part of the discussion, let us summarise the situation concerning the enhancement of global symmetry groups, for all the cases that in the end turn out to work. Thus we shall list the "naive" $GL(n,\mathbb{R})$ global symmetry, and its $SO(n)$ maximal compact subgroup, and then the actual enhanced global symmetry group that one finds, for each of the relevant cases.

Dim	Torus	Naive G/H	Enhanced G/H	Sphere	Isometry Gp
$D=11$	T^4	$GL(4,\mathbb{R})/SO(4)$	$SL(5,\mathbb{R})/SO(5)$	S^4	$SO(5)$
$D=11$	T^7	$GL(7,\mathbb{R})/SO(7)$	$E_7/SU(8)$	S^7	$SO(8)$
$D=10$	T^5	$GL(5,\mathbb{R})/SO(5)$	$SL(6,\mathbb{R})/SO(6)$	S^5	$SO(6)$
D	T^3	$GL(3,\mathbb{R})/SO(3)$	$GL(4,\mathbb{R})/SO(4)$	S^3	$SO(4)$
D	T^{D-3}	$GL(D-3,\mathbb{R})/SO(D-3)$	$\frac{SO(D-2,D-2)}{SO(D-2)\times SO(D-2)}$	S^{D-3}	$SO(D-2)$
D	T^2	$GL(2,\mathbb{R})/SO(2)$	$GL(3,\mathbb{R})/SO(3)$	S^2	$SO(3)$

Table 2: The global symmetry enhancements for the various relevant toroidal reductions. The last line refers to the theory of gravity, 2-form and dilaton, and the previous two lines refer to the theory of gravity, 3-form and dilaton.

We see, therefore, that in all these cases the hoped-for global symmetry enhancement for the toroidal reductions has indeed taken place. In each of these cases the actual, enhanced, global symmetry group for the reduction on T^n is large enough to contain the isometry group of the sphere S^n.

We have seen that a *necessary* condition for being able to perform consistent Kaluza-Klein sphere reductions in these cases has been satisfied, but it should be emphasised that this is certainly not, of itself, a proof that consistent reductions are actually possible. In fact at this point we know of no way of proving that the reductions can actually be performed other than by trying explicitly to construct them.

We already saw in section 2.3.3 that the authors of [8] have done all the hard work for the case $D = 11$ supergravity reduced on S^4, and they showed that a fully non-linear consistent reduction really does work in this case. It should be noted that in this example the reduction ansatz requires the inclusion not only of the seven-dimensional metric ds_7^2 and the ten $SO(5)$ Yang-Mills potentials $A_{(1)}^{ij}$, but also the fourteen scalars described by the unimodular symmetric matrix T_{ij}, and the five 3-forms $S_{(3)}^i$. And when we say that these other fields are required, we do mean *required.* This can be seen by looking at the seven-dimensional equations of motion (2.30)–(2.33). The equations of motion (2.31) for the scalars show that the Yang-Mills fields act as sources for them, so we *must* include the scalars. Similarly, the equations of motion (2.33) for $H_{(4)}^i \equiv S_{(3)}^i$ show that the Yang-Mills fields act as sources for these fields too.

The next simplest case to discuss is the 5-sphere reduction of ten-dimensional gravity coupled to a self-dual 5-form. This is a subset of the full type IIB supergravity, and furthermore it is itself a consistent truncation of type IIB supergravity. (It is the truncation to the $SL(2,\mathbb{R})$-singlet sector, in fact.) In fact the S^5 reduction of this truncated theory is quite nice, in that one only needs to include the five-dimensional metric ds_5^2, the $SO(6)$ Yang-Mills potentials $A_{(1)}^{ij}$ and the 20 scalars described by the unimodular symmetric tensor T_{ij} in this case. The details of this fully non-linear consistent S^5 reduction were worked out in [11]. Since it is fairly presentable and complete, we shall give the results here.

The equations of motion for ten-dimensional gravity $d\hat{s}_{10}^2$ coupled to the self-dual 5-form

$\hat{H}_{(5)}$ are

$$\begin{aligned} \hat{R}_{MN} &= \frac{1}{96}\,\hat{H}_{MPQRS}\,\hat{H}_N{}^{PQRS}\,, \\ d\hat{H}_{(5)} &= 0\,, \qquad \hat{*}\hat{H}_{(5)} = \hat{H}_{(5)}\,. \end{aligned} \tag{2.84}$$

The full S^5 Kaluza-Klein reduction ansatz is found to be

$$d\hat{s}_{10}^2 = \Delta^{1/2}\,ds_5^2 + g^{-2}\,\Delta^{-1/2}\,T_{ij}^{-1}\,D\mu^i\,D\mu^j\,, \tag{2.85}$$

$$\hat{H}_{(5)} = \hat{G}_{(5)} + \hat{*}\hat{G}_{(5)}\,, \tag{2.86}$$

$$\begin{aligned} \hat{G}_{(5)} &= -g\,U\,\epsilon_{(5)} + g^{-1}\,(T_{ij}^{-1}\,{*D}\,T_{jk})\wedge(\mu^k\,D\mu^i) \\ &\quad -\tfrac{1}{2}g^{-2}\,T_{ik}^{-1}\,T_{j\ell}^{-1}\,{*F_{(2)}}^{ij}\wedge D\mu^k\wedge D\mu^\ell\,, \end{aligned} \tag{2.87}$$

where the μ^i here are six Cartesian coordinates on $\mathbb{R}^6$, subject to the constraint $\mu^i\,\mu^i = 1$,

$$\begin{aligned} &U \equiv 2T_{ij}\,T_{jk}\,\mu^i\,\mu^k - \Delta\,T_{ii}\,, \qquad \Delta \equiv T_{ij}\,\mu^i\,\mu^j\,, \\ &F_{(2)}^{ij} = dA_{(1)}^{ij} + g\,A_{(1)}^{ik}\wedge A_{(1)}^{kj}\,, \qquad DT_{ij} \equiv dT_{ij} + g\,A_{(1)}^{ik}\,T_{kj} + g\,A_{(1)}^{jk}\,T_{ik}\,, \\ &\mu^i\,\mu^i = 1\,, \qquad D\mu^i \equiv d\mu^i + g\,A_{(1)}^{ij}\,\mu^j\,, \end{aligned} \tag{2.88}$$

and $\epsilon_{(5)}$ is the volume form on the five-dimensional spacetime. The ten-dimensional Hodge dual $\hat{*}\hat{G}_{(5)}$ of $\hat{G}_{(5)}$ is derivable from the above expressions, but since it is a rather major task we shall present the result for that too:

$$\begin{aligned} \hat{*}\hat{G}_{(5)} &= \frac{1}{5!}\,\epsilon_{i_1\cdots i_6}\Big[g^{-4}\,U\,\Delta^{-2}\,D\mu^{i_1}\wedge\cdots\wedge D\mu^{i_5}\,\mu^{i_6} \\ &\quad -5g^{-4}\,\Delta^{-2}\,D\mu^{i_1}\wedge\cdots\wedge D\mu^{i_4}\wedge DT_{i_5 j}\,T_{i_6 k}\,\mu^j\,\mu^k \\ &\quad -10g^{-3}\,\Delta^{-1}\,F_{(2)}^{i_1 i_2}\wedge D\mu^{i_3}\wedge D\mu^{i_4}\wedge D\mu^{i_5}\,T_{i_6 j}\,\mu^j\Big]\,. \end{aligned} \tag{2.89}$$

Substituting the ansatz into the ten-dimensional equations of motion (2.84), one finds after much calculation that miracles indeed occur, and all the dependence on the S^5 coordinates μ^i exactly balances. The ten-dimensional equations of motion turn out to be satisfied if and only if the five-dimensional fields ds_5^2, $A_{(1)}^{ij}$ and T_{ij} satisfy the equations that follow from the Lagrangian

$$\begin{aligned} \mathcal{L}_5 &= R\,{*\mathbb{1}} - \tfrac{1}{4}T_{ij}^{-1}\,{*DT_{jk}}\wedge T_{k\ell}^{-1}\,DT_{\ell i} - \tfrac{1}{4}T_{ik}^{-1}\,T_{j\ell}^{-1}\,{*F_{(2)}^{ij}}\wedge F_{(2)}^{k\ell} - V\,{*\mathbb{1}} \\ &\quad -\tfrac{1}{48}\,\epsilon_{i_1\cdots i_6}\Big(F_{(2)}^{i_1 i_2}\,F_{(2)}^{i_3 i_4}\,A_{(1)}^{i_5 i_6} - g\,F_{(2)}^{i_1 i_2}\,A_{(1)}^{i_3 i_4}\,A_{(1)}^{i_5 j}\,A_{(1)}^{j i_6} + \tfrac{2}{5}g^2\,A_{(1)}^{i_1 i_2}\,A_{(1)}^{i_3 j}\,A_{(1)}^{j i_4}\,A_{(1)}^{i_5 k}\,A_{(1)}^{k i_6}\Big)\,, \end{aligned} \tag{2.90}$$

where the potential V for the scalar fields is given by

$$V = \tfrac{1}{2}g^2\,\Big(2T_{ij}\,T_{ij} - (T_{ii})^2\Big)\,. \tag{2.91}$$

Since in these notes we have just reported what happens when one substitutes one of these Kaluza-Klein sphere-reduction ansätze into the higher-dimensional equations of motion, without actually carrying it out before the reader's eyes, it is perhaps worth commenting on what is involved. (Better yet, the reader is invited to try the calculations for himself or herself!) One finds that 99% of the complexity of the calculations, if not more, is caused by the presence of the scalar fields T_{ij}. Without the scalars, the calculations would be enormously simplified. They would also, of course, not work, since it is inconsistent to set the scalars to zero! Incidentally, another interesting contrast is that when one is first trying to figure out what the correct ansatz should be, it is the determination of the ansatz for the antisymmetric tensor that occupies the overwhelming majority of one's attention. The metric ansatz in these S^4 and S^5 examples is relatively simple, without too much room for maneoveur, but the determination of the antisymmetric tensor ansatz is much less under control. Again, the real struggle comes from having to cope with the scalar fields.

It will be seen that the structure of the S^5 reduction ansatz is quite similar to the S^4 reduction ansatz from $D = 11$, given in (2.56) and (2.57). A difference is that while in the S^4 reduction it was necessary to include also the 3-form fields $S^i_{(3)}$, here in the S^5 reduction one needs only gravity, Yang-Mills and the scalars T_{ij}. Essentially, this difference results from the fact that in the S^4 reduction to $D = 7$, the Yang-Mills bilinears $\epsilon_{ik_1\cdots k_4}\, F^{k_1k_2}_{(2)} \wedge F^{k_3k_4}_{(2)}$ act as sources for $DS^i_{(3)}$, i.e.

$$DS^i_{(3)} = \tfrac{1}{8}\epsilon_{ik_1\cdots k_4}\, F^{k_1k_2}_{(2)} \wedge F^{k_3k_4}_{(2)} + \cdots \tag{2.92}$$

whereas in the reduction on S^5 to $D = 5$ the structure of these source terms is now $\epsilon_{ijk_1\cdots k_4}\, F^{k_1k_2}_{(2)} \wedge F^{k_3k_4}_{(2)}$, which acts as "sources" in the Yang-Mills equations themselves:

$$D{*}F^{ij}_{(2)} = \tfrac{1}{8}\epsilon_{ijk_1\cdots k_4}\, F^{k_1k_2}_{(2)} \wedge F^{k_3k_4}_{(2)} + \cdots \tag{2.93}$$

In fact if we now turn to the third of the pure "gravity plus p-form field" reductions, namely the S^7 reduction of $D = 11$ supergavity, we find that the analogous Yang-Mills source terms lead to an almighty complication. In this case, the $SO(8)$ Yang-Mills bilinears in $D = 4$ are of the form $\epsilon_{i_1i_2i_3i_4k_1\cdots k_4}\, F^{k_1k_2}_{(2)} \wedge F^{k_3k_4}_{(2)}$, and so these are going to act as sources for spin-0 fields,

$$D{*}D\phi^{i_1i_2i_3i_4} = \tfrac{1}{8}\epsilon_{i_1i_2i_3i_4k_1\cdots k_4}\, F^{k_1k_2}_{(2)} \wedge F^{k_3k_4}_{(2)}\,. \tag{2.94}$$

What is more, these are not our old friends the unimodular symmetric scalars T_{ij}, of which there are 35 in the S^7 reduction. The fields $\phi^{i_1i_2i_3i_4}$ are actually *another* set of 35 spin-0 fields, in a different 35-dimensional irreducible representation of $SO(8)$. This new set of 35 fields are actually pseudoscalars, and if one thought that dealing with the scalars T_{ij} was difficult, then by comparison dealing with pseudoscalars is an absolute nightmare! In fact in the metric ansatz they are still relatively under control, and in the 1984 paper [12] by de Wit and Nicolai that comes nearest to proving the consistency of the S^7 reduction, a very elegant formula for the metric ansatz is obtained, giving explicitly how the 28 Yang-Mills gauge fields, the 35 scalars and the 35 pseudoscalars enter in the metric reduction ansatz. However not even de Wit and Nicolai, who are probably the most powerful calculators in the business, were able to obtain a complete formula for the 4-form ansatz. One looks in vain for a sentence and equation in [12] that says "...and the ansatz for the 4-form is:"

A somewhat similar level of compexity, although probably a bit less severe, would arise in the 5-sphere reduction if we asked to perform the reduction on the full bosonic sector of type IIB supergravity, rather than just on the truncated $SL(2,\mathbb{R})$-singlet sector of gravity plus self-dual 5-form that we presented above. The full gauged supergravity in five dimensions has a total of 42 spin-0 fields, comprising the 20 in T_{ij} that we have already met, a pair of $SO(6)$ singlets that are just the direct reductions of the type IIB dilaton and axion, and then twenty further spin-0 fields arising as two 10-dimensional representations of $SO(6)$. These latter sets of $10+10$ fields are again the dreaded pseudoscalars.

A difference between the S^5 and the S^7 reductions is that with S^5 we had the luxury of being able to consistently truncate the original ten-dimensional theory to just gravity and the self-dual 5-form, and that eliminated the $10+10$ of pseudoscalars from the problem. By contrast, in the S^7 reduction there is no analogous consistent truncation possible, and so if one is keeping the full set of 28 $SO(8)$ Yang-Mills gauge fields then one has no option but to go for the "Full Monty," and include the 35 pseudoscalars as well as the 35 scalars.

It *is* possible to consider simplifications that still give non-trivial consistent reductions, by noting that the gauged supergravities with lesser amounts of supersymmetry are themselves consistent truncations of the maximal theories. By doing this, a number of gauged supergravities have been obtained fully and explicitly as consistent Kaluza-Klein sphere reductions, in cases where the reduction to the maximal theory is prohibitively complicated. Two such examples are the reduction of type IIB supergravity on S^5 to get $SU(2)\times U(1)$ gauged $N=4$ supergravity in $D=5$ [13], and the reduction of $D=11$ supergravity on S^7 to give $SO(4)$-gauged $N=4$ supergravity in $D=4$ [14].

2.5 Inconsistency of the $T^{1,1}$ reduction

We have seen that one cannot in general make a Kaluza-Klein reduction on an n-dimensional sphere in which only massless modes, including the $SO(n+1)$ Yang-Mills gauge fields, are retained. The very small number of exceptions, where such a consistent reduction *is* possible, include the S^4 and S^7 reductions of $D = 11$ supergravity, and the S^5 reduction of type IIB supergravity.

It is of interest also to see whether consistent reductions of the type we are interested in are possible on other internal spaces instead of spheres. The answer here seems to be even bleaker, in the sense that they do not work even in the cases of $D = 11$ and type IIB supergravities. Let us consider a case of some topical interest, namely the Kaluza-Klein reduction of type IIB supergravity on the five-dimensional Einstein space sometimes known as $T^{1,1}$, or $Q(1,1)$. This is a particular example of a class of five-dimensional spaces $Q(p,q)$, defined as follows. One starts with the four-dimensional base space $S^2 \times S^2$, and constructs the standard class of homogeneous metrics on the $U(1)$ bundle over $S^2 \times S^2$, where the $U(1)$ fibres have winding numbers p and q over the two S^2 factors. The metrics can be written as

$$ds_5^2 = c^2\,(dz + p\,\cos\theta_1\,d\phi_1 + q\,\cos\theta_2\,d\phi_2)^2 + \Lambda_1^{-1}\,(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2) + \Lambda_2^{-1}\,(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2)\,, \tag{2.95}$$

where c is a constant. One can show that for any choice of the integers p and q, thene by choosing the relations between the constants c, Λ_1 and Λ_2 appropriately, the metric can be an Einstein metric. The case $p = q = 1$ is particularly interesting, because then the Einstein metic admits two Killing spinors, and so one gets a supersymmetric five-dimensional theory if type IIB supergravity is reduced on this space. The Einstein metric in this case is given by

$$ds_5^2 = c^2\,(dz + \cos\theta_1\,d\phi_1 + \cos\theta_2\,d\phi_2)^2 + \Lambda_1^{-1}\,(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2) + \Lambda_2^{-1}\,(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2)\,, \tag{2.96}$$

satisfying

$$R_{ab} = 4m^2\,g_{ab}\,, \tag{2.97}$$

with

$$\Lambda_1 = \Lambda_2 = 6m^2\,, \qquad c = \frac{1}{3m}\,. \tag{2.98}$$

In section 2.3.2 we saw that based initially on a linearised analysis of the subsector comprising the seven-dimensional metric and Yang-Mills fields, we could derive the necessary condition for a consistent reduction from $D = 11$ that the quantity Y^{IJ} defined in (2.49) should be independent of the coordinates of the internal compactifying 4-space. This was essential in order that (2.48) should be a self-consistent equation, with a matching between the left-hand side that is clearly independent of the internal coordinates, and the right-hand side that will depend on these coordinates unless Y^{IJ} is a constant.

It turns out that the situation is precisely analogous in the reduction of type IIB supergravity to $D = 5$. Again, a linearised analysis around an $\mathrm{AdS}_5 \times M_5$ background shows that if M_5 is an Einstein space satisfying

$$R_{ab} = 4m^2\, g_{ab}\,, \tag{2.99}$$

and with isometry group G, then a Kaluza-Klein reduction that retains only the massless fields in $D = 5$, including the gauge bosons of the Yang-Mills group G, can be consistent only if the quantity

$$Y^{IJ} = K^{aI}\, K^J_a + \tfrac{1}{2} m^{-2}\, L^{abI}\, L^J_{ab} \tag{2.100}$$

is constant, where $L^I_{ab} \equiv \nabla_a\, K^I_b$, and K^I_a are the Killing vectors on M_5. Satisfying this condition is not of itself a guarantee of consistency, but violating it is a guarantee of inconsistency. Of course it turns out that if M_5 is taken to be the 5-sphere, then Y^{IJ} is independent of the 5-sphere coordinates.

It is actually a fairly simple matter to apply this test to the $T^{1,1}$ (or $Q(1,1)$) space described above. Its isometry group is $SU(2)\times SU(2)\times U(1)$, corresponding to the isometry group $SU(2)\times SU(2)$ of the $S^2\times S^2$ base, times the $U(1)$ isometry of the fibres. The Killing vector for the $U(1)$ factor is just $\partial/\partial z$. The remaining Killing vectors can all be expressed rather simply in terms of those on the $S^2 \times S^2$ base space. A general analysis for the much more extensive class of metrics on spaces called $Q^{q_1\cdots q_N}_{n_1\cdots n_N}$, defined as $U(1)$ bundles over $CP^{n_1}\times CP^{n_2}\times\cdots CP^{n_N}$, with winding numbers q_i over each CP^{n_i} factor, was carried out in [15]. (Our case is Q^{11}_{11} in this classification, since $S^2 = CP^1$.) Two facts are of great importance in allowing the problem to be explicitly solved. Firstly, the base spaces are Kähler, and in fact they are the product of Einstein-Kähler spaces. It is easy to see that on any compact Einstein-Kähler space M, with $R_{mn} = \Lambda\, g_{mn}$, the each Killing vector K^m can be written as

$$K^m = J^{mn}\, \partial_n\, \psi\,, \tag{2.101}$$

where ψ is a scalar eigenfunction on M with eigenvalue 2Λ:

$$-\Box\, \psi = 2\Lambda\, \psi\,. \tag{2.102}$$

Secondly, the fact that CP^n is homogenous, with a large symmetry group $((SU(n+1))$, means that it is easy to construct the scalar eigenfunctions. Using these facts, it is proven in [15] that none of the $SU(n_i$ Killing vectors on the bundle spaces $Q^{q_1 \cdots q_N}_{n_1 \cdots n_N}$ can satisfy the condition that Y^{IJ} in (2.100) is constant. In fact, the calculation is particularly easy for the case $Q^{11}_{11} = T^{1,1}$ of interest to us here, since the base itself is just $S^2 \times S^2$, with the Einstein metric. The simple proof for spaces including this one is handled as an additional separate discussion in [15].

To summarise, the upshot from the analysis is that none of the $SU(2) \times SU(2)$ Killing vectors on $T^{1,1}$ has the property that Y^{Ij} is constant, while on the other hand the $U(1)$ Killing vector by itself does give a constant Y^{IJ}. In other words, this proves that a consistent Kaluza-Klein reduction on $T^{1,1}$, in which the massless fields including the $SU(2) \times SU(2) \times U(1)$ Yang-Mills fields are retained, while setting the massive fields to zero, is impossible. In fact the best thta one can do is to retain just the $U(1)$ gauge field in a consistent truncation.

3 Brane-world Kaluza-Klein Reduction

3.1 Introduction

So far, we hav met to principal types of Kaluza-Klein reduction. The first, in chapter 1, was reduction on a circle or a torus, for which the calculations are relatively simple, and the consistency of the truncation to the massless sector is guaranteed by simple group theory. The second type, in chapter 2, involved reduction on a sphere, together with the truncation to the massless sector. In this case it is only in very exceptional cases that such a consistent reduction is possible at all, and we do not have a proper understanding of why it works, in those exceptional cases where it does. The complexity of these reductions is vastly greater than that for the circle and torus reductions.

In this chapter, we shall study a third category of consistent Kaluza-Klein reduction, which was only recently discovered [16]. It grew out of the recent developments in the Randall-Sundrum "brane-world," and the intriguing suggestion that one can extract an effective four-dimensional spacetime theory from a five-dimensional theory in which the fifth dimension is *non-compact*, and infinite in extent [17, 18]. This is rather remarkable, because normally one would expect that with a non-compact fifth dimension gravity would really appear to be five dimensional! We cannot simply "pretend" not to see the fifth dimensional of a Minkowskian 5-dimensional spacetime, for example, because we would have to expand all five-dimensional fields in terms of Fourier transforms on the fifth coordinate (the radius$\longrightarrow \infty$ limit of a fifth circle dimension), and this would give us a continuum of massive four-dimensional graviton states, extending all the way down to zero mass. This would turn out just to be describing five-dimensional gravity in a (highly disgusied!) way. The remarkable thing about the Randall-Sundrum brane-world picture is that although there is still a continuum of massive graviton states extending down to zero mass, and the fifth dimension is of infinite extent, the way in which these modes are distributed as a function of mass means that actually gravity looks pretty-nearly four-dimensional.

We shall not need to concern ourselves much with the details of the "Randall-Sundrum Scenario" here, because the principal focus of this chpater will be to present the new kind of consistent Kaluza-Klein reduction that was motivated by it. This "Brane-world Kaluza-Klein Reduction" has certain features in common with the sphere reductions of the previous chapter, in that the ansatz depends on the extra coordinate, and there is no obvious reason why it should be possible to make a consistent reduction. However, the situation here is considerably simpler than in the sphere reductions, and so calculationally it is much easier to see what is going on.

The basic idea is as follows. The Randall-Sundrum brane is composed of two segments of 5-dimensional Anti-de Sitter spacetime, AdS_5. The AdS_5 metric can be written as

$$d\hat{s}_5^2 = e^{-2k\,z}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2\,, \tag{3.1}$$

where z runs from a Cauchy horizon at $z=-\infty$ to the so-called "bounadry" at $z=+\infty$. AdS_5 is being written here in "horospherical" or "Poincaré coordinates, as a nesting of 4-dimensional Minkowski spactimes, with metric $ds_4^2 = \eta_{\mu\nu}\,dx^\mu\,dx^\nu$. If one calculates the curvature, which is very simple here, one finds that it is of maximally symmetric form,

$$\hat{R}_{ABCD} = k^2\,(\eta_{AC}\,\eta_{BD} - \eta_{AD}\,\eta_{BC})\,, \tag{3.2}$$

and the Ricci tensor is therefore

$$R_{AB} = -4k^2\,\eta_{AB}\,. \tag{3.3}$$

(We use vielbein components here for simplicity.) Thus the spacetime has the constant negative curvature characteristic of AdS_5. Actually the whole construction generalises straightforwardly to higher dimensions, with AdS_D described in terms of with Minkowski$_{D-1}$ level surfaces at constant z, so from now on we shall consider the case of the general dimension.

The Randall-Sundrum brane is obtained, located at $z=0$, by taking the sector of (3.1) (generalised to D dimensions) for $z\le 0$ and joining it on to a Z_2 reflection of itself, thus:

$$d\hat{s}_D^2 = e^{-2k\,|z|}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2\,, \tag{3.4}$$

Calculating the curvature now, we find that since the glueing process has introduced a discontinuity in the gradient of the metric, there are now delta-functions in the curvature. In particular, the Ricci tensor is given now by

$$\begin{aligned} \hat{R}_{ab} &= -(D-1)k^2\,\eta_{ab} + 2k\,\delta(z)\,\eta_{ab}\,, \\ \hat{R}_{zz} &= -(D-1)\,k^2 + 2k\,(D-1)\,\delta(z)\,, \end{aligned} \tag{3.5}$$

The basic idea of the brane-world Kaluza-Klein reduction can be seen in a rather trivial example, where we attempt only to get pure gravity in $(D-1)$ dimensions, starting from gravity with a negative cosmological constant in D dimensions. All we have to do is to generalise (3.4) to

$$d\hat{s}_D^2 = e^{-2k\,|z|}\,ds_{D-1}^2 + dz^2\,, \tag{3.6}$$

where the $(D-1)$-dimensional metric is as yet unspecified, excpet that it depends only on the coordinates of the $(D-1)$ dimensions. If we now calculate the D-dimensional Ricci tensor for this metric, now viewed as a Kaluza-Klein reduction ansatz, we get

$$\begin{aligned} \hat{R}_{ab} &= e^{2k\,|z|}\,R_{ab} - (D-1)\,k^2\,\eta_{ab} + 2k\,\delta(z)\,\eta_{ab}\,, \\ \hat{R}_{zz} &= -(D-1)\,k^2 + 2k\,(D-1)\,\delta(z)\,, \end{aligned} \tag{3.7}$$

where we have decomposed the D-dimensional vielbein index $A=(a,z)$. Leaving aside the delta-function terms, which ultimately will be assumed to be supplied by singular brane sources, we see that if the D-dimensional metric satisfies the Einstein equation with a negative cosmological constant,

$$\hat{R}_{AB} = -(D-1)\,k^2\,\eta_{AB}\,, \tag{3.8}$$

then the $(D-1)$-dimensional metric ds_{D-1}^2 satisfies the $(D-1)$-dimensional pure Einstein equation with no cosmological constant:

$$R_{ab} = 0\,. \tag{3.9}$$

This is therefore a consistent Kaluza-Klein reduction.

This example, in the case $D=5$, can be extended to include a gravitino too, if one starts with the appropriate gauged supergravity in $D=5$. (It needs to be gauged supergravity so that we have the necessary negative cosmological constant.) In fact if we start with minimal gauged supergravity in $D=5$ (called $N=2$ supergravity, in the scheme where the possible supersymmetries in $D=5$ are $N=2,4,6,8$), then we can end up with $N=1$ ungauged supergravity in $D = 4$, using the reduction scheme described above. If we just look at the bosonic sector, the gauged $N = 2$ theory in $D = 5$ has the Lagrangian

$$\mathcal{L}_5 = \hat{R}\,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}\hat{F}_{(2)} \wedge \hat{F}_{(2)} - \tfrac{1}{3\sqrt{3}}\hat{F}_{(2)} \wedge \hat{F}_{(2)} \wedge \hat{A}_{(1)} - 12g^2\,\hat{*}\mathbb{1}\,, \tag{3.10}$$

where $\hat{A}_{(1)}$ is the "graviphoton" of the $N = 2$ supermuliplet, and g is the gauge coupling constant. (The $N = 2$ gravitini in $D = 5$ carry charge $\pm g$ with respect to the graviphoton.) The Kaluza-Klein reduction scheme in the bosonic sector is then precisely as above for the metric, together with setting the graviphoton to zero:

$$\begin{aligned} d\hat{s}_5^2 &= e^{-2k\,|z|}\,ds_4^2 + dz^2\,, \\ \hat{F}_{(2)} &= 0\,. \end{aligned} \tag{3.11}$$

Substituting into the equations of motion following from (3.10), one gets the equations of motion of the bosonic sector of ungauged $N = 1$ supergravity in four dimensions, namely

$$R_{ab} = 0\,. \tag{3.12}$$

Notice that we do not get any four-dimensional field out of the original 5-dimensional graviphoton $\hat{A}_{(1)}$. Much was made of this in some of the literature, but actually it is a very reasonable result. It is well known that the basic p-brane solutions, including domain walls of the form (3.4), break half of the supersymmetry of the supergravity in which they are a solution. It is thus very reasonable to expect to see just *half* the supersymmetry of the higher-dimensional theory, when one looks for lower-dimensional fields localised on the brane. In this case, for example, we are seeing $N = 1$ ungauged supergravity localised on the 4-dimensional brane, starting from $N = 2$ gauged supergravity in the 5-dimensional bulk.

If we want to see more interesting fields on the 4-dimensional brane, we should start with larger theories, with more supersymmetry, in the 5-dimensional bulk. For example, if we start with $N = 4$ gauged supergravity in five dimensions, then we should end up with $N = 2$ ungauged supergravity in four dimensions. The bosonic sector of this theory comprises the Einstein-Maxwell system, so now we can expect to get a photon as well as gravity localised on the brane. This is interesting for many reasons, including the fact that we can now study BPS black-hole solutions on the brane.

In the next section, we shall see just how the reduction to $N = 2$ supergravity works.

3.2 $N=2$ supergravity in $D=4$ from gauged $N=2$ supergravity in $D=5$

Here, we show that we can obtain ungauged four-dimensional Maxwell-Einstein ($N=2$) supergravity as a consistent Kaluza-Klein reduction of gauged five-dimensional $N=4$ supergravity, within a Randall-Sundrum type of framework. The bosonic sector of the five-dimensional theory comprises the metric, a dilatonic scalar ϕ, the $SU(2)$ Yang-Mills potentials $A^i_{(1)}$, a $U(1)$ gauge potential $B_{(1)}$, and two 2-form potentials $A^\alpha_{(2)}$ which transform as a charged doublet under the $U(1)$. The Lagrangian [19], expressed in the language of differential forms that we shall use here, is given by [13]

$$\begin{aligned}
\mathcal{L}_5 &= R\,\tilde{*}\mathbb{1} - \tfrac{1}{2}\tilde{*}d\phi\wedge d\phi - \tfrac{1}{2}X^4\,\tilde{*}G_{(2)}\wedge G_{(2)} - \tfrac{1}{2}X^{-2}\,(\tilde{*}F^i_{(2)}\wedge F^i_{(2)} + \tilde{*}A^\alpha_{(2)}\wedge A^\alpha_{(2)})\\
&\quad +\frac{1}{2g}\epsilon_{\alpha\beta}\,A^\alpha_{(2)}\wedge dA^\beta_{(2)} - \tfrac{1}{2}A^\alpha_{(2)}\wedge A^\alpha_{(2)}\wedge B_{(1)} - \tfrac{1}{2}F^i_{(2)}\wedge F^i_{(2)}\wedge B_{(1)}\\
&\quad +4g^2\,(X^2+2X^{-1})\,\tilde{*}\mathbb{1}\,, \qquad (3.13)
\end{aligned}$$

where $X=e^{-\frac{1}{\sqrt{6}}\phi}$, $F^i_{(2)} = dA^i_{(1)} + \frac{1}{\sqrt{2}}g\,\epsilon^{ijk}\,A^j_{(1)}\wedge A^k_{(1)}$ and $G_{(2)} = dB_{(1)}$, and $\tilde{*}$ denotes the five-dimensional Hodge dual. It is useful to adopt a complex notation for the two 2-form potentials, by defining

$$A_{(2)} \equiv A^1_{(2)} + \mathrm{i}\,A^2_{(2)}\,. \qquad (3.14)$$

Our Kaluza-Klein reduction ansatz involves setting the fields ϕ, $A^i_{(1)}$ and $B_{(1)}$ to zero, with the remaining metric and 2-form potentials given by

$$\begin{aligned}
ds_5^2 &= e^{-2k\,|z|}\,ds_4^2 + dz^2\,,\\
A_{(2)} &= \tfrac{1}{\sqrt{2}}\,e^{-k\,|z|}\,(F_{(2)} - \mathrm{i}\,{*F_{(2)}})\,, \qquad (3.15)
\end{aligned}$$

where ds_4^2 is the metric and $F_{(2)}$ is the Maxwell field of the four-dimensional $N=2$ supergravity, and $*$ denotes the Hodge dual in the four-dimensional metric.

To show that this ansatz gives a consistent reduction to four dimensions, we note from (3.13) that the five-dimensional equations of motion are [13]

$$\begin{aligned}
d(X^{-1}\,\tilde{*}\,dX) &= \tfrac{1}{3}X^4\,\tilde{*}\,G_{(2)}\wedge G_{(2)} - \tfrac{1}{6}X^{-2}\,(\tilde{*}\,F^i_{(2)}\wedge F^i_{(2)} + \tilde{*}\,\bar{A}_{(2)}\wedge A_{(2)})\\
&\quad -\tfrac{4}{3}g^2\,(X^2 - X^{-1})\,\tilde{*}\,\mathbb{1},\\
d(X^4\,\tilde{*}\,G_{(2)}) &= -\tfrac{1}{2}F^i_{(2)}\wedge F^i_{(2)} - \tfrac{1}{2}\bar{A}_{(2)}\wedge A_{(2)},\\
d(X^{-2}\,\tilde{*}\,F^i_{(2)}) &= \sqrt{2}\,g\,\epsilon^{ijk}\,X^{-2}\,\tilde{*}\,F^j_{(2)}\wedge A^k_{(1)} - F^i_{(2)}\wedge G_{(2)},\\
X^2\,\tilde{*}\,F_{(3)} &= -\mathrm{i}\,g\,A_{(2)}\,,\\
R_{MN} &= 3X^{-2}\,\partial_M X\,\partial_N X - \tfrac{4}{3}g^2\,(X^2+2X^{-1})\,g_{MN}\\
&\quad +\tfrac{1}{2}X^4\,(G_M{}^P G_{NP} - \tfrac{1}{6}g_{MN}\,G^2_{(2)}) + \tfrac{1}{2}X^{-2}\,(F^i_M{}^P F^i_{NP} - \tfrac{1}{6}g_{MN}\,(F^i_{(2)})^2)
\end{aligned}$$

$$+\tfrac{1}{2}X^{-2}\,(\bar{A}_{(M}{}^{P}\,A_{N)P} - \tfrac{1}{6}g_{MN}\,|A_{(2)}|^2)\,, \tag{3.16}$$

where

$$F_{(3)} = DA_{(2)} \equiv dA_{(2)} - \mathrm{i}\,g\,B_{(1)} \wedge A_{(2)}\,. \tag{3.17}$$

It follows from (3.15) that

$$F_{(3)} = -\tfrac{1}{\sqrt{2}}\,k\,\epsilon(z)\,e^{-k\,|z|}\,(F_{(2)} - \mathrm{i}\,{*F_{(2)}}) \wedge dz + \tfrac{1}{\sqrt{2}}\,e^{-k\,|z|}\,(dF_{(2)} - \mathrm{i}\,d{*F_{(2)}})\,, \tag{3.18}$$

where $\epsilon(z) = \pm 1$ according to whether $z > 0$ or $z < 0$. Thus the equation of motion for F_3 implies first of all that

$$dF_{(2)} = 0\,, \qquad d{*F_{(2)}} = 0\,, \tag{3.19}$$

and so then, after taking the Hodge dual of the remaining terms in (3.18), we find from (3.16) that

$$-\tfrac{1}{\sqrt{2}}\,k\,\epsilon(z)\,e^{-k\,|z|}\,({*F_{(2)}} + \mathrm{i}\,F_{(2)}) = -\tfrac{1}{\sqrt{2}}\,\mathrm{i}\,g\,e^{-k\,|z|}\,(F_{(2)} - \mathrm{i}\,{*F_{(2)}})\,, \tag{3.20}$$

which is identically satisfied provided that

$$g = \begin{cases} +k\,, & z > 0\,, \\ -k\,, & z < 0\,. \end{cases} \tag{3.21}$$

Since k is always positive (to ensure the trapping of gravity), this means that the Yang-Mills gauge coupling constant g has opposite signs on the two sides of the domain wall. This implies that the Randall-Sundrum scenario cannot arise strictly within the standard five-dimensional gauged supergravity, where g is a fixed parameter. It has a completely natural explanation from a ten-dimensional viewpoint, where g arises as a constant of integration in the solution for an antisymmetric tensor, and the imposed Z_2 symmetry in fact *requires* that the sign must change across the wall. For convenience, however, we shall commonly treat the coupling constant g of the gauged supergravity as if its sign can be freely chosen to be opposite on opposite sides of the domain wall, with the understanding that this can be justified from the higher-dimensional viewpoint.

The equations of motion for X and $G_{(2)}$ are satisfied since for our ansatz

$$\bar{A}_{(2)} \wedge A_{(2)} = 0 \tag{3.22}$$

and $\tilde{*}A_{(2)} = \mathrm{i}\,A_{(2)}$. The only remaining non-trivial equation in (3.16) is the Einstein equation. Substituting (3.7) with $D = 5$ into the five-dimensional Einstein equations, we find that the "internal" (zz) component is identically satisfied, whilst the lower-dimensional components imply $k^2 = g^2$ (consistent with (3.21)), and

$$R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} = \tfrac{1}{2}(F_{\mu\rho}\,F_{\nu}{}^{\rho} - \tfrac{1}{4}F^2\,g_{\mu\nu})\,, \tag{3.23}$$

where $R_{\mu\nu}$ is the four-dimensional Ricci tensor. Thus we have shown that the ansatz (3.15), when substituted into the equations of motion for the five-dimensional $N = 2$ gauged supergravity, gives rise to the equations of motion (3.19) and (3.23) of four-dimensional Einstein-Maxwell supergravity.

The fact that the Kaluza-Klein reduction that we have performed here gives a consistent reduction of the five-dimensional equations of motion to $D = 4$ is somewhat non-trivial, bearing in mind that the five-dimensional fields in (3.15) are required to depend on the coordinate z of the fifth dimension. The manner in which the z-dependence matches in the five-dimensional field equations so that consistent four-dimensional equations of motion emerge is rather analogous to the situation in a non-trivial Kaluza-Klein sphere reduction, although in the present case the required "conspiracies" are rather more easily seen.

One indication of the localisation of gravity in the usual Randall-Sundrum model is the occurrence of the exponential factor in the metric $ds_5^2 = e^{-2k\,|z|}\, dx^\mu\, dx_\mu + dz^2$, which falls off as one moves away from the wall. It is therefore satisfactory that we have found that this same exponential fall-off occurs for the complete reduction ansatz (3.15), which we derived purely on the basis of the requirement of consistency of the embedding. In fact the very consistency of the brane-world Kaluza-Klein reduction immediately guarantees that the localisation of gravity on the brane will extend to the entire supergravity multiplet. This can be seen from the fact that the consistency implies that if the reduction ansatz is substituted into the higher-dimensional Lagrangian, it will give a result that has just a homogeneous factor of $e^{-2k\,|z|}$ multiplying the z-independent lower-dimensional Lagrangian. Thus the integration over z converges for the whole Lagrangian, exactly as it did for the Einstein-Hilbert term.

Of course the $N = 4$ gauged five-dimensional supergravity that was our starting point here can itself be obtained from a 5-sphere reduction of type IIB supergravity, and so the entire discussion can be reinterpreted back in $D = 10$. Because it would involve setting up quite a lot more formalism we shall not present the reults here; they are discussed in detail in [16]. In the next section, we shall present an analogous discussion for another example of a brane-world Kaluza-Klein reduction, in a case where we have already extensively studied the associated sphere reduction in chapter 2 of these lectures.

The idea that we have exhibited here for the consistent Kaluza-Klein reduction of gauged $N = 4$ supergravity in $D = 5$ to ungauged $N = 2$ supergravity in $D = 4$ can be generalised to many other cases. In general, a gauged supergravity in D dimensions turns out to allow a consistent brane-world Kaluza-Klein reduction to ungauged supergravity in $(D-1)$

dimensions, with one half of the original supersymmetry [16]. The principal cases that have been worked out, in [16] and [20], are summarised in the following Table:

D	D-dimensional Theory	$(D-1)$-dimensional Theory from Brane-world Reduction
10	Massive IIA	$D=9,\ N=1$
8	$SU(2)$-gauged $N=2$	$D=7,\ N=2$
7	$SO(5)$-gauged $N=4$	$D=6,\ N=(2,0)$
6	$SU(2)$-gauged $N=2$	$D=5,\ N=2$
5	$SO(6)$ gauged $N=8$	$D=4,\ N=4$

Table 3: The ungauged supergravities in $(D-1)$ dimensions obtained by brane-world Kaluza-Klein reductions.

We shall present one further example here, which is quite intriguing because it shows how a *chiral* supergravity arises from a brane-world Kaluza-Klein reduction of a non-chiral one. The example we shall give is one of those worked out in [20]; the brane-world reduction of $SO(5)$-gauged $N=4$ supergravity in $D=7$ to give ungauged $N=(2,0)$ chiral supergravity in $D=6$.

3.3 $(2,0)$ supergravity in $D=6$ from $SO(5)$-gauged supergravity in $D=7$

We already discussed the $SO(5)$-gauged seven-dimensional supergravity in section 2.3.1. Let us just repeat the key details here. The bosonic Lagrangian for maximal $SO(5)$-gauged supergravity in $D=7$ can be written as

$$\begin{aligned}\mathcal{L}_7 \;=\;& \hat R\,\hat{*}\mathbb{1} - \tfrac14 T^{-1}_{ij}\,{*DT_{jk}}\wedge T^{-1}_{k\ell}\,DT_{\ell i} - -\tfrac14\,T^{-1}_{ik}\,T^{-1}_{j\ell}\,\hat{*}\hat F^{ij}_{(2)}\wedge \hat F^{k\ell}_{(2)} - -\tfrac12 T_{ij}\,\hat{*}\hat S^i_{(3)}\wedge \hat S^j_{(3)}\\ &+\frac{1}{2g}\hat S^i_{(3)}\wedge \hat H^i_{(4)} - \frac{1}{8g}\epsilon_{ij_1\cdots j_4}\,\hat S^i_{(3)}\wedge \hat F^{j_1j_2}_{(2)}\wedge \hat F^{j_3j_4}_{(2)} + \frac1g\Omega_{(7)} - V\,\hat{*}\mathbb{1}\,, \end{aligned}\tag{3.24}$$

where

$$\hat H^i_{(4)} \equiv D\hat S^i_{(3)} = d\hat S^i_{(3)} + g\,\hat A^{ij}_{(1)}\wedge \hat S^j_{(3)}\,. \tag{3.25}$$

The potential V is given by

$$V=\tfrac12 g^2\Big(2T_{ij}\,T_{ij} - (T_{ii})^2\Big)\,, \tag{3.26}$$

and $\Omega_{(7)}$ is a Chern-Simons type of term built from the Yang-Mills fields, which has the property that its variation with respect to $\hat A^{ij}_{(1)}$ gives

$$\delta\Omega_{(7)} = \tfrac34 \delta^{j_1j_2j_3j_4}_{i_1i_2k\ell}\,\hat F^{i_1i_2}_{(2)}\wedge \hat F^{j_1j_2}_{(2)}\wedge \hat F^{j_3j_4}_{(2)}\wedge\delta \hat A^{k\ell}_{(1)}\,. \tag{3.27}$$

Let us now set the $SO(5)$ Yang-Mills potentials $A^{ij}_{(1)}$ to zero, and take the scalars to be trivial also, $T_{ij} = \delta_{ij}$. This is not in general a consistent truncation, since the remaining fields $\hat{S}^i_{(3)}$ would act as sources for the Yang-Mills and scalar fields that have been set to zero. If we impose that these source terms vanish, i.e.

$$\hat{S}^i_{(3)} \wedge \hat{S}^j_{(3)} = 0\,, \qquad \hat{*}\hat{S}^i_{(3)} \wedge \hat{S}^j_{(3)} = 0\,, \tag{3.28}$$

then the truncation will be consistent. (As we shall see below, these sources terms will indeed vanish in the brane-world reduction that we shall be considering.) The remaining equations of motion following from (3.24) are then

$$\begin{aligned} &d\hat{*}\hat{S}^i_{(3)} = 0\,, \qquad d\hat{S}^i_{(3)} = g\,\hat{*}\hat{S}^i_{(3)}\,, \\ &\hat{R}_{AB} = \tfrac{1}{4}(\hat{S}^i_{ACD}\,\hat{S}^{i\ CD}_{B} - \tfrac{2}{15}\,(S^i_{(3)})^2\,\hat{g}_{AB}) - \tfrac{3}{2}g^2\,\hat{g}_{AB}\,. \end{aligned} \tag{3.29}$$

We find that the following Kaluza-Klein Ansatz for the seven-dimensional fields yields a consistent reduction to six dimensions:

$$\begin{aligned} &d\hat{s}_7^2 = e^{-2k\,|z|}\,ds_6^2 + dz^2\,, \\ &\hat{S}^i_{(3)} = e^{-2k\,|z|}\,F^i_{(3)}\,, \qquad \hat{A}^{ij}_{(1)} = 0\,, \qquad T_{ij} = \delta_{ij}\,, \end{aligned} \tag{3.30}$$

where the constant k is related to the gauge coupling constant g by

$$g = \begin{cases} -2k\,, & z > 0\,, \\ +2k\,, & z < 0\,. \end{cases} \tag{3.31}$$

Substituting this Ansatz into the field equations of seven-dimensional $SO(5)$-gauged supergravity, we find that all the equations are consistently satisfied provided that the six-dimensional fields ds_6^2 and $F^i_{(3)}$ satisfy the equations of motion of six-dimensional ungauged $N = (2,0)$ chiral supergravity, namely

$$F^i_{(3)} = *F^i_3\,, \qquad dF^i_{(3)} = 0\,, \qquad R_{\mu\nu} = \tfrac{1}{4}F^i_{\mu\rho\sigma}\,F^{i\rho\sigma}_{\nu}\,. \tag{3.32}$$

Note that the self-duality of the 3-forms ensures that the constraints (3.28) are indeed satisfied, since $F^i_{(3)} \wedge F^j_{(3)} = 0$ for any pair of self-dual 3-forms. Of course the self-duality of the $F^i_{(3)}$ fields also implies one cannot write a covariant Lagrangian for this theory.

It is intriguing that the consistency of the Kaluza-Klein reduction here depends crucially on the fact that the fields in the six-dimensional theory are restricted to those of the chiral $N = (2,0)$ supergravity. Thus consistency has forced us to obtain a *chiral* theory in $D = 6$, even though we started (of course) with a non-chiral theory in $D = 7$. This is an interesting new feature in these brane-world reductions; usually, one would have said that Kaluza-Klein reductions could not generate chiral theories from non-chiral starting points.

Since we have already discussed the exact embedding of seven-dimensional maximal $SO(5)$-gauged supergravity in $D=11$, via the S^4 reduction, it is now a simple matter to lift the above Ansatz to an embedding in eleven-dimensional supergravity. Using the S^4 reduction Ansatz of [8], which we presented in section 2.3.3, we therefore obtain

$$\begin{aligned} d\hat{s}_{11}^2 &= e^{-2k\,|z|}\, ds_6^2 + dz^2 + g^{-2}\, d\mu_i\, d\mu_i\,, \\ \hat{F}_{(4)} &= \frac{1}{8g^3}\,\epsilon_{i_1\cdots i_5}\,\mu_{i_1}\, d\mu_{i_2}\wedge\cdots\wedge d\mu_{i_5} - g^{-1}\, d(\mu_i\, e^{-2k\,|z|}\, F^i_{(3)})\,, \end{aligned} \tag{3.33}$$

where μ_i are coordinates on $\mathbb{R}^5$, subject to the constraint

$$\mu_i\,\mu_i = 1\,, \tag{3.34}$$

which defines the unit 4-sphere. This gives us a direct reduction from $D=11$ supergravity to chiral $N=(2,0)$ supergravity in $D=6$.

3.4 Puzzles on the horizon

There are some curious and perhaps slightly surprising features of the brane-world reductions that we have been considering in this chapter. At first sight it looks very appealing to have gravity in the lower dimension described in terms of the brane-world metric reduction

$$d\hat{s}^2 = e^{-2k\,|z|}\, ds^2 + dz^2\,. \tag{3.35}$$

If we take the lower-diensional metric to be close to Minkowski spacetime, $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu\, dx^\nu$, then at looks rather satisfactory that the fluctuation $h_{\mu\nu}$ is multiplied by the factor $e^{-2k\,|z|}$, which decreases exponentially as one approaches the Cauchy horizons at $z=\pm\infty$. However, this is perhaps a bit misleading, as one can see by looking at (3.35) itself. If one calculates the Riemman tensor $\hat{R}_{ABCD}$ of the D-dimensional metric $d\hat{s}^2$ in terms of the curvature of the $(D-1)$-dimensional metric ds^2, one finds that the scalar invariant built from the square of the Riemman tensor is given by

$$\hat{R}_{ABCD}\,\hat{R}^{ABCD} = e^{4k\,|z|}\, R_{abcd}\, R^{abcd} - 4k^2\, e^{2k\,|z|}\, R + 2D(D-1)\, k^4 \tag{3.36}$$

in the bulk, where R_{abcd} and R are the Riemann tensor and Ricci scalar of the reduced metric ds^2. This implies that any curvature of the lower-dimensional metric for which $R_{abcd}\, R^{abcd}$ or R is non-zero, no matter how small, will lead to curvature singularities in the higher-dimensional metric on the Cauchy horizons at $z=\pm\infty$. If an inmate in the brane-world at $z = 0$ were to let a pin drop, the resulting disturbance in the gravitational field would lead to a curvature singularity on the Cauchy horizon. The legendary butterfly in the Amazonian rain forest that flaps its wing and causes a hurricane in Florida pales into insignificance by comparison![9] The basic point here is that when a metric is scaled by a conformal factor that gets small, the curvature gets large.

These singularities were discussed in detail for a Schwarzschild black hole on the brane in [21], and for BPS Reissner-Nordström black holes on the brane, in the context of $N = 2$ supergravity on the brane, in [16]. In [22], it was argued that such curvature singularities on the horizons arise as an artefact of considering only the zero-mode of the metric tensor, and that if the massive Kaluza-Klein modes are taken into account they could actually become dominant near the horizons, and may lead to a finite curvature there. The results of [16] and [20] that we have been describing in this chapter suggest that the phenomenon of diverging curvature on the Cauchy horizons in the brane-world reductions may be more severe. Specifically these results show that the brane-world reductions correspond to exact fully non-linear consistent embeddings in which the massive Kaluza-Klein modes can be consistently decoupled. This implies that there certainly exist exact solutions on the brane-world where massive Kaluza-Klein modes do not enter the picture, even at the non-linear level. For these solutions, the curvature will inevitably diverge at the horizons. It becomes necessary, therefore, either to live with these singularities or else to find a principle, perhaps based on the imposition of appropriate boundary conditions, for rejecting the solutions of this type.

These, then, are puzzles that arise out of the brane-world reductions. Notwithstanding this, it is intriguing that exact consistent Kaluza-Klein reductions are possible within the brane-world scenario.

References

[1] H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities*, Nucl. Phys. **B465** (1996) 127, hep-th/9512012.

[2] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities*, Nucl. Phys. **B523** (1998) 73, hep-th/9710119.

[9] A more topical example might be the flapping chad on a ballot paper that brings an army of lawyers to the state.

[3] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. **B76** (1978) 409.

[4] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, *R-R scalars, U-duality and solvable Lie algebras*, Nucl. Phys. **B496** (1997) 617, hep-th/9611014.

[5] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré, R. Minasian, M. Trigiante, *Solvable Lie algebras in type IIA, type IIB and M theories*, Nucl. Phys. **B493** (1997) 249, hep-th/9612202.

[6] Abdus Salam and J. Strathdee, *On Kaluza-Klein theory*,

[7] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged maximally extended supergravity in seven dimensions*, Phys. Lett. **B143** (1984) 103.

[8] H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistency of the $AdS_7 \times S_4$ reduction and the origin of self-duality in odd dimensions*, Nucl. Phys. **B581** (2000) 179, hep-th/9911238.

[9] H.J. Kim, L.J. Romans, and P. van Nieuwenhuizen, *Mass spectrum of ten dimensional $N = 2$ supergravity on S^5*, Phys. Rev. **D32** (1985) 389.

[10] M. Cvetic, H. Lü, C.N. Pope, A. Sadrzadeh and T.A. Tran, *S^3 and S^4 reductions of type IIA supergravity*, hep-th/0005137.

[11] M. Cvetic, H. Lü, C.N. Pope, A. Sadrzadeh and T.A. Tran, *Consistent $SO(6)$ reduction of type IIB supergravity on S^5*, Nucl. Phys. **B586** (2000) 275, hep-th/0003103,

[12] B. de Wit and H. Nicolai, *The consistency of the S^7 truncation in $D = 11$ supergravity*, Nucl. Phys. B281 (1987) 211.

[13] H. Lü, C.N. Pope and T.A. Tran, *Five-dimensional $N = 4$ $SU(2) \times U(1)$ gauged supergravity from type IIB*, Phys. Lett. **B475** (2000) 261, hep-th/9909203.

[14] M. CvetiVc, H, Lü and C.N. Pope, *Four-dimensional $N = 4$, $SO(4)$ gauged supergravity from $D = 11$*, Nucl. Phys. **B574** (2000) 761, hep-th/9910252.

[15] P. Hoxha, R.R Martinez-Acosta and C.N. Pope, *Kaluza-Klein consistency, Killing vectors and Kähler spaces*, Class.Quant.Grav. **17** (2000) 4207, hep-th/0005172.

[16] H. Lü and C.N. Pope, *Branes on the brane*, hep-th/0008050.

[17] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, Phys. Rev. Lett. **83** (1999) 3370, hep-th/9905221.

[18] L. Randall and R. Sundrum, *An alternative to compactification*, Phys. Rev. Lett. **83** (1999) 4690, hep-th/9906064.

[19] M. Günaydin, L.J. Romans and N.P. Warner, *Compact and non-compact gauged supergravity theories in five dimensions*, Nucl. Phys. **B272** (1986) 598.

[20] M. Cvetic, H. Lü and C.N. Pope, *Brane-world Kaluza-Klein reductions and branes on the brane*, hep-th/0009183.

[21] A. Chamblin, S.W. Hawking and H.S. Reall, *Brane world black holes*, Phys. Rev. **D61** (2000) 065007, hep-th/9909205.

[22] S.B. Giddings, E. Katz and L. Randall, *Linearised gravity in brane backgrounds*, JHEP 0003:023,2000, hep-th/0002091.