689-605: Mathematical Methods in Theoretical Physics

Some suggested topics to revise for Final Exam, Friday 12'th Dec, 12:30pm - 2:30pm

General Remarks

* **REMEMBER THAT MOST OF THE PROBLEMS CAN BE SOLVED IN A FEW LINES! If a problem seems to be getting very complicated , then pause to see if you have made a mistake!**

* **It is much easier to give partial credit if you have presented a clear and logical (and legible!) argument.**

Differential Equations

* Separation of variables. Use of boundary conditions to determine signs of separation constants. (E.g. If you are wanting to make $\phi(x,y,z)$ in Cartesian coodinates vanish at $x=0$ and at $x=a$, then the separated solutions in the x diection will be combinations of sin and cos, rather than sinh and cosh.)

* Wronskian. Using Wronskian to construct the second solution of a 2'nd-order ODE, given the first solution. Using Wronskian to solve the inhomogeneous equation (i.e. with source term), given the solutions of the homogeneous equation. Green functions.

* Singular points of the equation. Regular and irregular singular points. **Always** remember to check $x=\infty$ as well as finite x!

* Make sure you can **correctly** obtain the indicial equation, and the recursion relation! Mistakes at this stage can cause major trouble later in the problem!

* Be sure to know when the indicial factor x^σ is needed, and when it isn't, in series solutions $y=\sum_n a_n\, x^{n+\sigma}$. Do **NOT**, repeat **NOT**, include x^σ when it is not needed!!!

* If you are asked to prove that a given expression for the a_n solves the recursion relation, be sure to *prove* this, for example by showing the given solution implies the same recursion relation.

* Orthogonality of eigenfunctions of Sturm-Liouville operators.

Complex Analysis

* Understand how to derive the Cauchy-Riemann equations for an analytic function. Understand how to work with $\partial/\partial z$ and $\partial/\partial\bar{z}$, and how to express analyticity in terms of these derivatives.

* Make sure you understand how to calculate the leading terms in a Laurent expansion; for example, know how to expand the following functions around $z = 0$, keeping the first few terms:

$$f(z) = \frac{1}{z^2\,(1+z)}\,, \qquad f(z) = \frac{1}{z\,(e^z - 1)^3}\,, \qquad f(z) = \frac{z}{\sin^3 z}\,.$$

* Be familiar with Cauchy's theorem, which says $\oint f(z)\,dz = 0$ when $f(z)$ has no poles within the contour, and Cauchy's integral formula

$$f(z) = \frac{1}{2\pi\,\mathrm{i}} \oint_C \frac{d\zeta\, f(\zeta)}{\zeta - z}\,.$$

(What analyticity properties must $f(\zeta)$ have within the contour C?)

* Calculus of residues. Make sure you understand why

$$\oint_C \frac{dz}{z^n} \tag{1}$$

is zero for **any** integer (positive or negative) except for $n = 1$, for which the integral gives $2\pi\,\mathrm{i}$. (The contour C encloses the origin.)

* Be familiar with how to evaluate integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta\,, \qquad \int_{-\infty}^{\infty} f(x)\,dx\,, \qquad \int_0^{\infty} dx\, x^{\alpha-1}\, f(x)$$

by using the calculus of residues. ($R(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.) Make sure you understand the reason for the conditions on $f(z)$ in the second type of integral; why $z\,f(z)$ must fall off in the upper half-plane, etc. How would you modify the contour if instead $z\,f(z)$ falls off in the lower half-plane?

* Be familiar with the relation between analyticity of a function, the existence of a Taylor expansion, and the radius of convergence of the Taylor expansion.

* Know how to determine $v(x, y)$, given $u(x, y)$, by performing the necessary differentiations and integrations, using the Cauchy-Riemann equations.

* Summation of infinite series using calculus of residues. For example, check how to evaluate

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}\,.$$

* Analytic continuation. Be sure to understand why it is so powerful, and how we can use it in order to derive results in some restricted region of the complex plane, and then argue that it is valid in the entire complex plane.

Vector and Tensor Analysis

* Be aware of the criterion for what is a scalar, vector, tensor (i.e. correct transformation rule under coordinate rotations.) Review the basic points that have been covered in the lectures on vectors and tensors.