Phys 617 P.S. 10 Solns.

(1) Classical spins, $U = -\vec{m}\cdot\vec{B} = -mB\cos\theta$
$\quad\quad\quad\quad\quad\quad\quad\quad\;$ & $\vec{M} \equiv \langle\vec{m}\rangle/V$

clearly $\langle m_x\rangle$ & $\langle m_y\rangle = 0$.

$$\langle m_z\rangle = \frac{\int_0^{2\pi}\int_0^{\pi}(m\cos\theta)\,e^{-\frac{(-mB\cos\theta)}{kT}}\sin\theta\,d\theta\,d\varphi}{\iint e^{-(-mB\cos\theta/kT)}\sin\theta\,d\theta\,d\varphi}$$

$$= \cancel{2\pi} m \int_{-1}^{1} du\, u\, e^{+\alpha u} \Big/ \cancel{2\pi}\int_{-1}^{1} du\, e^{+\alpha u}, \quad u=\cos\theta,\ \alpha \equiv \frac{mB}{kT}$$

$$= m\,\frac{e^{\alpha u}(\alpha u-1)\big|_{-1}^{1}/\alpha^2}{e^{\alpha u}\big|_{-1}^{1}/\alpha} = \frac{m}{\alpha}\left[\frac{e^{\alpha}(\alpha-1)-e^{-\alpha}(-\alpha-1)}{e^{\alpha}-e^{-\alpha}}\right]$$

$$= \frac{m}{\alpha}\left[\alpha\left(\frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}\right)-1\right]$$

$$= m\left(\coth\alpha - 1/\alpha\right) \qquad [1]$$

in hitemp limit, $\alpha\to 0$, $\coth\alpha \simeq \frac{1}{\alpha}+\frac{\alpha}{3}+\cdots$

$$\text{so } \chi = \frac{\partial M}{\partial B} = \frac{m}{V}\frac{\partial}{\partial B}\left[\coth\frac{mB}{kT} - \frac{kT}{mB}\right]$$

$$\text{Becomes, } \chi \cong \frac{m}{V}\frac{\partial}{\partial B}\left[\frac{mB}{3kT}\right]$$

so $\chi \simeq \frac{m^2}{3kT}$. Which is the Curie law,

$$\chi \equiv C/T, \text{ with } C = \frac{m^2}{3k_B}$$

This is same as the limit of the QM expression,
$C = \frac{S(S+1)\mu_B^2}{3k_B}$, for $S\to\infty$ as expected.

Note that the expression labeled [1] on the last page is the Langevin function, also mentioned in the handout.

Classical moments do appear in real-life situations, for example when nm-sized clusters precipitate out from solid solutions, the result can be a "superparamagnet" if the clusters are internally ferromagnetic. [for example a solid solution of Co in Cu.] Or large-spin magnetic molecules can sometimes be considered as classical superparamagnets.

(2) $i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(p + \frac{e}{c}A\right)^2 \psi$, and so

$$-i\hbar \frac{\partial}{\partial t}\psi^* = \frac{1}{2m}\left(p^* + \frac{e}{c}A\right)^2 \psi^* .$$

Given $\dot{\rho} = \frac{\partial}{\partial t}(\psi^*\psi) = \frac{-1}{2mi\hbar}\left[\psi^*(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})^2\psi - \psi(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})^2\psi^*\right]$,

$$\dot{\rho} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) + \frac{e}{mc}\left(\psi^*\vec{\nabla}\cdot\vec{A}\psi + \psi\,\vec{\nabla}\cdot\vec{A}\psi^*\right)$$

$$= \frac{\hbar}{2mi}\,\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) + \frac{e}{mc}\,\vec{\nabla}\cdot\vec{A}\,\psi^*\psi$$

where $\vec{\nabla}$ acts to the right, not on $\vec{A}$.

And comparing, $\dot{\rho} = -\vec{\nabla}\cdot\vec{J}$ (continuity equation),

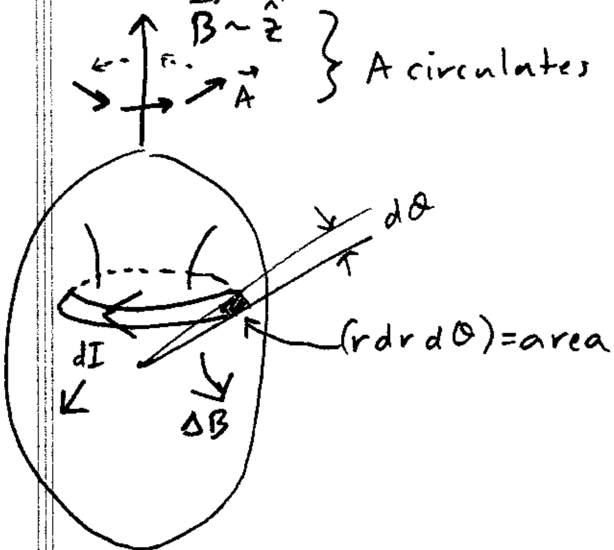
we see that this holds if we define,

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) + \frac{e}{mc}\vec{A}\,\psi^*\psi .$$

so the probability current picks up an extra term proportional to $\vec{A}$ in regions where $\vec{A} \neq 0$.

So even an s-orbital will circulate in regions where there is a circulating $\vec{A}$ (e.g. $B \neq 0$), giving diamagnetism.

(b) For a symmetric orbital oriented as shown,



} A circulates as shown

- $r\,dr\,d\theta$ is an area, when rotated about $\hat{z}$ generates a current loop, dI

- $\vec{j} = \frac{e}{mc}|\psi|^2\vec{A}$ is probability current density; multiply by $r\,dr\,d\theta$ to get probability current, and by $(-e)$ to get dI.

So, dI opposes $\vec{A}$ direction & right-hand rule shows that this loop produces ΔB in the diamagnetic direction.

loop radius $r\sin\theta \Rightarrow |\mu| = |dI| \cdot \frac{\pi r^2 \sin^2\theta}{c} = \frac{e^2}{mc^2}|\psi|^2 \underbrace{A}_{B r \sin\theta/2} (r\,dr\,d\theta)\pi r^2 \sin^2\theta$

Note $\mu = IS/c$ in cgs units; the c was omitted on HW sheet.

or, $\mu = r^4\,dr\,d\theta \sin^3\theta \cdot \frac{\pi}{2}\frac{e^2}{mc^2} B|\psi|^2$

to get total, integrate over r & θ:

$$\mu_{net} = \int_0^\pi \int_0^\infty \underbrace{2\pi r^2 dr\, d\theta \sin\theta}_{\text{this is } (d\,Vol)} |\psi|^2 \frac{e^2}{mc^2}\frac{r^2 B \sin^2\theta}{4}$$

so this is $\frac{e^2 B}{4mc^2}\langle r^2\sin^2\theta\rangle$ since $\langle X\rangle = \iint d^3r\,|\psi|^2 X$

$r^2\sin^2\theta$ same as x^2+y^2

$$|\chi| = \frac{1}{V}\frac{d}{dB}\langle \mu_{net}\rangle = \frac{e^2}{4mc^2 V}\langle x^2+y^2\rangle$$ as desired.

(c) given $\psi = |\psi| e^{i\phi}$, $\vec{\nabla}\psi = |\psi|(i\vec{\nabla}\phi)e^{i\phi}$

$$= i(\vec{\nabla}\phi)\psi$$

also, $\psi^* = |\psi| e^{-i\phi} \rightarrow (\nabla\psi^*) = -i(\nabla\phi)|\psi|^*$

$$= (\nabla\psi)^*$$

so $\psi^*\nabla\psi - (\nabla\psi^*)\psi = 2i(\nabla\phi)\psi^*\psi$

$$= 2i(\nabla\phi)|\psi|^2$$

gives, $\vec{j} = \left(\frac{-i\hbar}{2m}\right)2i(\vec{\nabla}\phi)|\psi|^2 + \frac{e}{mc}\vec{A}|\psi|^2$

$$= \left(\frac{\hbar}{m}\vec{\nabla}\phi + \frac{e}{mc}\vec{A}\right)|\psi|^2$$

to get ordinary electric current density, $j_{el} = -ej$,

so $\vec{J}_{el} = -\left(\frac{\hbar e}{m}\vec{\nabla}\phi + \frac{e^2}{mc}\vec{A}\right)|\psi|^2$

However, this was derived assuming particles are electrons. [also eqn (7) for j assumed electrons since we replaced $\left(-\frac{q}{c}A\right)$ by $\left(\frac{+e}{c}\vec{A}\right)$ in $\mathcal{H}$.]

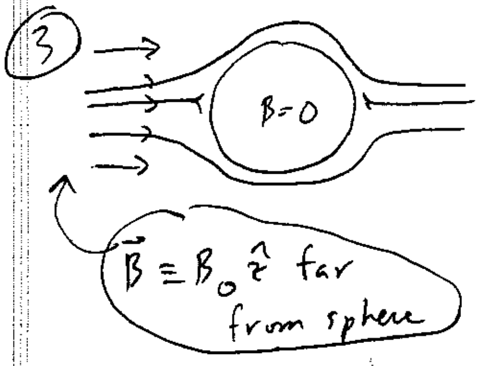
Since $|\psi|^2$ in GL theory is the density of <u>pairs</u>, we need to replace <u>$e \rightarrow 2e$</u> and <u>$m \rightarrow 2m$</u> in the above.

This yields,

$$\vec{j}_{el} = -\left(\frac{\hbar e}{m}\vec{\nabla}\phi + \frac{2e^2}{mc}\vec{A}\right)|\psi|^2$$

as desired. (Where e & m refer to the single-electron charge & mass.)

(3)



(a) since $\nabla \cdot B = 0$, we can take,

$B \equiv \nabla \varphi$ outside the sphere &

solve $\Rightarrow \varphi = B_0 z + \frac{\alpha}{r^2}\cos\theta$

(as in standard E&M texts.)

So $\quad B = B_0 \hat{z} - \hat{r}\,\frac{2\alpha}{r^3}\cos\theta - \hat{\theta}\,\frac{\alpha}{r^3}\sin\theta$

$$B_{ext} = H_{ext} = \hat{r}\cos\theta\left(B_0 - \frac{2\alpha}{r^3}\right) + \hat{\theta}\sin\theta\left(-B_0 - \frac{\alpha}{r^3}\right)$$

Inside, $B = 0$ is given, and $H \equiv H_{in}\hat{z}$,

where $\quad H_{in}\hat{z} = H_{in}(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$

Boundary conditions, $B_{\hat{r}}$ continuous $\rightarrow \frac{2\alpha}{r^3} = B_0$

$H_{\hat{\theta}}$ continuous $\rightarrow \frac{3}{2}B_0 = H_{in}$

So inside $B_{in} = 0 = \vec{H}_{in} + 4\pi\vec{M} \Rightarrow \vec{M} = -\frac{3}{8\pi}B_0\hat{z}$

(b) at $\theta = \pi/2$, from above we see $B = -\hat{\theta}\left(\frac{3}{2}B_0\right)$

$$= \frac{3}{2}B_0\,(+\hat{z})$$

so as stated in the problem, sphere experiences $\left(\frac{3}{2}\right)$ times the applied field on its boundary at $\theta = \pi/2$. This reduces the field that can be applied to a spherical superconductor to $(2/3)$ times the conventional "critical field".